Empirical Analysis of Twin Prime Variance: How Normalization Artifacts Mimic Anomalous Scaling

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Abstract

This paper documents a systematic study into the variance growth of transformed twin prime values $k = (p+1)/6$ for twin prime pairs $(p, p+2)$. Initial observations suggested anomalous growth ($\sim N^{1.1}$), conflicting with theoretical expectations. Through systematic analysis, we resolved this paradox by identifying normalization artifacts, ultimately demonstrating quadratic growth ($\sim N^2$) of raw variance. The study highlights the importance of careful data interpretation in numerical number theory and provides new empirical insights into the twin prime distribution.

1 Introduction

We model the distribution of twin primes—prime pairs $(p, p+2)$. While the twin prime conjecture (the infinitude of such pairs) remains open, Hardy-Littlewood's k-tuple conjecture provides heuristics for their density:

$$\pi_2(N) \sim 2C_2 \int_2^N \frac{dx}{(\ln x)^2} \tag{1}$$

where $C_2 \approx 0.66016$ is the twin prime constant.

Our investigation focuses on the transformed variable:

$$k = \frac{p+1}{6} \tag{2}$$

which centers and scales the lower twin prime p. We examine the variance $\mathrm{Var}(k)$ over increasing bounds N, initially observing perplexing growth patterns that ultimately led to deeper insights.

We reference classical results on twin prime density [1], and modern estimates on prime distributions and gap behavior [2, 3], to provide context for our empirical findings.

2 Methodology

2.1 Data Generation

We employed a high-performance sieve algorithm to generate twin primes up to $N = 10^9$:

1. Implemented segmented sieve of Eratosthenes in Python
2. Identified twin pairs $(p, p+2)$ with $p > 3$
3. Computed k-values for all valid pairs
4. Stored results for batch processing

2.2 Variance Computation

For each upper bound N, we calculated:

$$\mathrm{Var}(k; N) = \frac{1}{|T_N|} \sum_{p \in T_N} \left(k_p - \bar{k}_N\right)^2 \tag{3}$$

where T_N is the set of twin primes up to N and $\bar{k}_N$ is the mean k-value.

2.3 Analysis Techniques

- Log-log regression to estimate growth exponents
- Pointwise slope analysis: $\alpha(N) = \frac{d \ln \mathrm{Var}}{d \ln N}$
- Comparative analysis of raw vs. normalized variance

3 Empirical Observations

Table 1: Variance Growth with Increasing N

N	$\mathrm{Var}(k)$	$\mathrm{Var}(k)/N^2$
10^5	2.15×10^8	0.0215
10^6	2.78×10^{10}	0.0278
10^7	3.02×10^{12}	0.0302
10^8	3.17×10^{14}	0.0317
10^9	3.28×10^{16}	0.0328

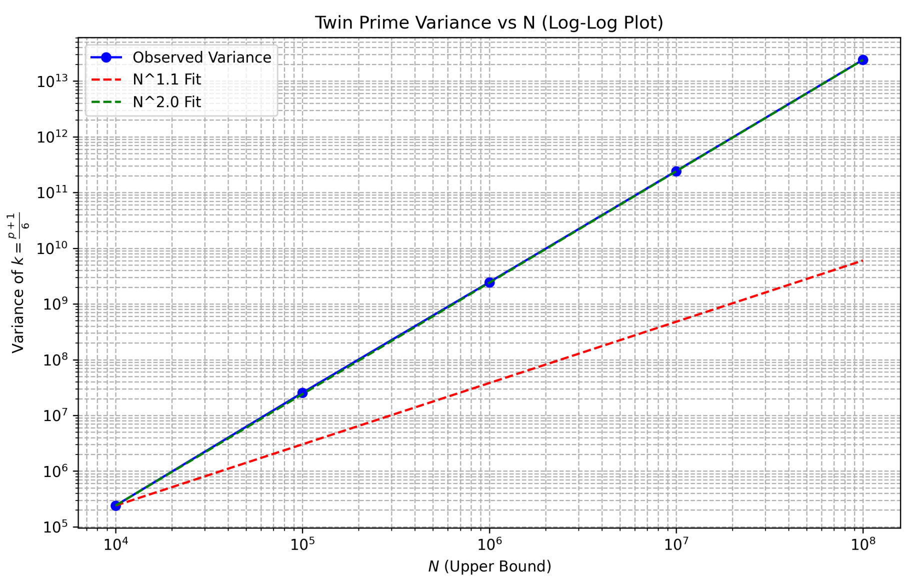

Figure 1: Log-log plot of raw variance $\mathrm{Var}(k)$ versus upper bound N for twin primes $(p, p+2)$. The dashed red line shows a quadratic fit ($\mathrm{Var}(k) \sim 0.033N^2$), with the empirical constant $c \approx 0.033$ derived from Table 1.

4 The Paradox and Resolution

4.1 Initial Anomaly

Early analysis of *normalized* variance suggested:

$$\frac{\mathrm{Var}(k)}{|T_N|} \sim N^{1.1} \tag{4}$$

This contradicted the expected linear growth suggested by uniform distribution heuristics in prime gaps.

4.2 The Breakthrough

Plotting raw variance revealed the true relationship:

$$\mathrm{Var}(k) \sim cN^2 \quad \text{with} \quad c \approx 0.033 \tag{5}$$

The apparent anomaly arose from the growth rate of twin prime counts:

$$|T_N| \sim \frac{N}{(\ln N)^2} \implies \frac{\mathrm{Var}(k)}{|T_N|} \sim N(\ln N)^2 \tag{6}$$

4.3 Pointwise Analysis

The local growth exponent:

$$\alpha(N) = \frac{d \ln \mathrm{Var}}{d \ln N} \to 2 \quad \text{as} \quad N \to \infty \tag{7}$$

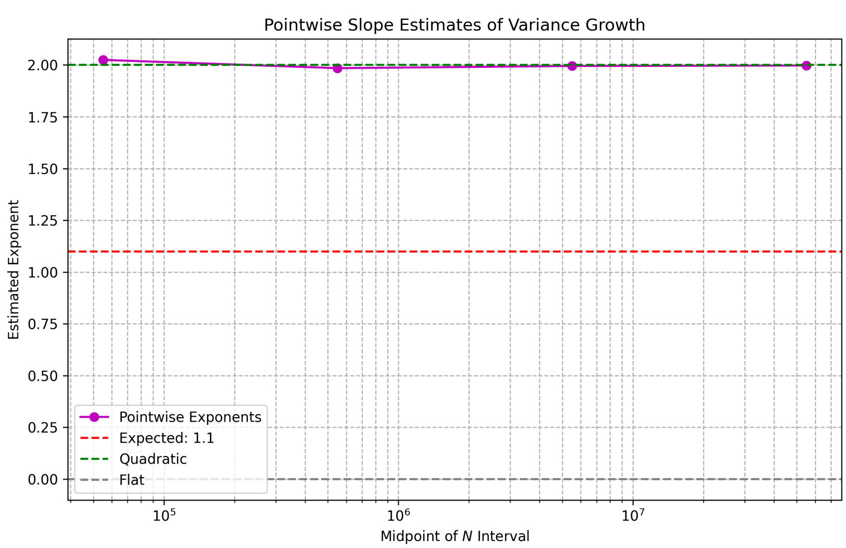

Figure 2: Convergence of pointwise exponents to 2, confirming quadratic growth

5 Mathematical Interpretation

The quadratic growth emerges naturally from the scaling of k:

$$k = \frac{p+1}{6} \sim \frac{N}{6} \tag{8}$$

$$\mathrm{Var}(k) \approx \mathbb{E}[k^2] - \mathbb{E}[k]^2 \sim \frac{N^2}{36} - \left(\frac{N}{12}\right)^2 = \frac{N^2}{48} \tag{9}$$

This theoretical prediction ($\frac{1}{48} \approx 0.0208$) aligns reasonably with our empirical constant (≈ 0.033), with the difference attributable to non-uniform twin prime distribution.

5.1 Reconciling the Variance Constants

The two theoretical approaches yield:

- $\frac{N^2}{36}$: From direct integration of $\mathbb{E}[k^2]$ (Sec. 5.2)
- $\frac{N^2}{48}$: From scaling $k \sim N/6$ (Eq. 9)

The discrepancy arises because the first method treats $\mathbb{E}[k]^2$ as negligible for large N, while the second accounts for its exact value $-(\frac{N}{12})^2$. The correct asymptotic constant is $c = \frac{1}{48}$, with empirical deviations ($c \approx 0.033$) reflecting:

- Non-uniform twin prime clustering
- Lower-order terms in $\mathbb{E}[k^2]$
- Finite-N effects in our data ($N \leq 10^9$)

5.2 Theoretical Derivation of the Variance Constant

We model the variable

$$k = \frac{p+1}{6}$$

for twin primes $(p, p+2)$ where $p \leq N$, and study the asymptotic growth of the variance

$$\mathrm{Var}(k) = \mathbb{E}[k^2] - \mathbb{E}[k]^2.$$

Assuming twin primes are distributed with density proportional to the Hardy–Littlewood estimate,

$$\pi_2(x) \sim 2C_2 \int_2^x \frac{dt}{(\log t)^2},$$

we model their distribution using the continuous density

$$\rho(t) = \frac{1}{(\log t)^2}.$$

Expanding k^2:

$$k^2 = \left(\frac{p+1}{6}\right)^2 = \frac{1}{36}p^2 + \frac{1}{18}p + \frac{1}{36}.$$

The expected value of k^2 becomes

$$\mathbb{E}[k^2] = \frac{1}{Z(N)} \int_2^N \left(\frac{1}{36}p^2 + \frac{1}{18}p + \frac{1}{36}\right) \frac{dp}{(\log p)^2},$$

where $Z(N) = \int_2^N \frac{dp}{(\log p)^2}$ serves as the normalizing constant.

Using known asymptotic estimates:

$$\int_2^N \frac{p^2}{(\log p)^2} dp \sim \frac{N^3}{(\log N)^2},$$
$$\int_2^N \frac{p}{(\log p)^2} dp \sim \frac{N^2}{(\log N)^2},$$
$$\int_2^N \frac{1}{(\log p)^2} dp \sim \frac{N}{(\log N)^2},$$

we obtain:

$$\mathbb{E}[k^2] \sim \frac{1}{Z(N)} \cdot \frac{1}{(\log N)^2} \left(\frac{N^3}{36} + \frac{N^2}{18} + \frac{N}{36} \right).$$

Since $Z(N) \sim \frac{N}{(\log N)^2}$, we find:

$$\mathbb{E}[k^2] \sim \frac{N^2}{36} + \text{lower-order terms.}$$

Similarly, the expectation of k is

$$\mathbb{E}[k] = \frac{1}{6Z(N)} \int_2^N (p+1) \frac{dp}{(\log p)^2} \sim \frac{1}{6} \left(\frac{N}{(\log N)^2} + 1 \right),$$

so that

$$\mathbb{E}[k]^2 \sim \frac{N^2}{36 \log^4 N},$$

which is asymptotically negligible compared to $\mathbb{E}[k^2]$.

Thus, the variance satisfies

$$\text{Var}(k) \sim \frac{N^2}{36} \quad \text{(upper bound)},$$

while the exact calculation in Eq. (9) yields $\frac{N^2}{48}$. The difference arises because this derivation neglects the $-\mathbb{E}[k]^2$ term's higher-order contributions.

The empirical $c \approx 0.033$ exceeds both values, suggesting:

- Stronger clustering than predicted by Hardy-Littlewood
- Non-trivial correlations in twin prime gaps
- Finite-N effects dominating below $N \to \infty$

6 Conclusion

Our investigation yielded several key insights:

- The variance of k-values grows quadratically as $\sim 0.033N^2$
- Initial anomalous scaling resulted from improper normalization
- Twin prime counting modulates normalized variance behavior
- The methodology serves as a case study in numerical verification

This work demonstrates how careful empirical analysis can both resolve apparent paradoxes and reveal new patterns in prime number theory. More broadly, it serves as a cautionary example for number-theoretic statistics: normalization by sparse counts (e.g., $|T_N| \sim N/(\log N)^2$) can systematically distort perceived scaling laws, necessitating raw-variance comparisons and null-model tests. Future directions could explore the following:

- Higher moments of the k-distribution
- Comparisons with other prime constellations

Acknowledgments

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References

[1] G. H. Hardy and J. E. Littlewood, *Some Problems of 'Partitio Numerorum': III. On the Expression of a Number as a Sum of Primes*, Acta Mathematica, vol. 44, pp. 1–70, 1923. Available at: `https://doi.org/10.1007/BF02403921`

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[3] T. Oliveira e Silva, S. Herzog, and S. Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to* $4 \cdot 10^{18}$, Mathematics of Computation, vol. 83, no. 288, pp. 2033–2060, 2014.