A HEURISTIC PROOF OF DE BROGLIE WAVE FORMULA

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ABSTRACT

In this paper, we attempt an original general mathematical proof of De Broglie matter wave formula via studying the particle dynamics and wave dynamics of an electron orbiting with constant velocity around the nucleus of the atom in a Semi - Classical context, considering a plane transversal wave as the motion mode of all particles in free space including massives particles.

KEY WORDS : De Broglie Formula – Pilot Wave theory – Wave Function – Quantum Mechanics

INTRODUCTION

The De Broglie matter wave relationship between the particle's linear momentum and its wavelenght is one the fundamental formulas in Quantum mechanics [1], providing the basis not only of wave – particle duality under Lorentz invariance, but also the parameterization of the particle's wave function through its electrodynamical parameters such as energy E and momentum p, and founding of fundemental equations in Quantum theory such as Heisenberg's uncertainty principle, Shrödinger's equation, Klein – Gordon equation … For instance the simplest form of a wave function, that is a solution of Schrödinger's equation is written in the form $\psi(x,t) = e^{-\frac{i}{\hbar}(Et-kx)}$. Experimental work by C. Davisson & L. Germer (1923 – 1927) by scattering electrons on nickel crystal [2], showing diffraction lines associated a wave behavior, In the same year 1927, G.P. Thomson fired electrons towards a thin metal foil getting the same results as Davisson and Germer [3], and A. Compton's work on the scattering of charged particles after interaction with high frequency photons [4][5], as well as all the experimental success of Quantum mechanics in atomic and subatomic physics stand as direct emperical proof of the De Broglie equation.

Yet most proposed theoretical derivations of this fundemental formula have some inconsistencies, either proving the formula for the special case of a masseless photon $pc = {hc}/{\lambda}$ and then generalizing it to massive particles, or using the momentum operator for which the defining expression is already based on the De Broglie relation, or even taking the Einstein – Planck formula, not for photons but for all particles including massive particles ${hv}/{\lambda}$, and equating it to mv^2 (which is the method used originally by L. De Broglie in his 1923's paper [1]) which lacks a proper argumentation regarding the abusive use of Planck's formula $E = hf$ proved originally only for radiation quanta, and equating the total energy of the particle to 2 × kinetic energy which assumes a permanent state of equilibrium between kinetic and potential energy for massive particles.

In this short paper, we attempt a semi - classical proof of De Broglie relation considering an electron revolving around the nucleus of an atom. The derivation will be subdivided into 2 parts ; the first is dedicated to the Particle dynamics, establishing the equations of the linear momentum p and position functions based on the Euler – Lagrange equation, Newton's 2nd Law and Hamiltonian conservation. The second part concerns the Wave dynamics of the electron, where by considering a plane transversal mater wave associated to the moving electron, we establish through the well-known dispersion relation and Hamiltonian conservation the De Broglie matter wave formula $p = \hbar k$

RESULTS

1. Particle dynamics

Consider the electron in an isolated atom of atomic number Z ; interacting thus with electromagnetic fields created by the central nucleus and moving electrons, the electron's classical total energy is hence written as :

$$H = \frac{p^2}{2m} + \frac{e^2}{4\pi\varepsilon_0}\chi \tag{1}$$

Where χ is a position function determining the relative average position of the electron with respect to the other charges in the atom ; such that R is the average distance between the electron and the center of the nucleus, and r_i is the average distance between the electron e^- and another electron $e_i^-$of the electron shells of the atom

$$\chi = -\frac{Z}{R} + \sum_{i=1}^{Z-1}\frac{1}{r_i} \tag{2}$$

While the lagrangian is given as :

$$L = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0}\chi \tag{3}$$

Since the electron's total energy is considered constant, with no photon emission or absorption, hence the $dH = 0$, leading to $\partial T = -\partial V$, yields that derivative of the Lagrangian with respect to the electron's linear velocity is $2p$, Taking the derivative of Lagrangian with respect to velocity from Eq 3, one gets :

$$\frac{\partial L}{\partial v} = p - \frac{e^2}{4\pi\varepsilon_0}\frac{\partial \chi}{\partial v} = 2p \tag{4}$$

Which yields a proportionality relation between the electron's momentum p and the derivative of the potential energy's position function χ with respect to the electron's velocity:

$$p = -\frac{e^2}{4\pi\varepsilon_0}\frac{\partial \chi}{\partial v} \tag{5}$$

Taking the derivative of the momentum given in Eq 5 with respect to time, yields :

$$\frac{\partial p}{\partial t} = -\frac{e^2}{4\pi\varepsilon_0}\frac{d}{dt}\left(\frac{\partial \chi}{\partial v}\right) \tag{6}$$

By Euler – Lagrange equation with the electron's position inside the atom as a genrealized coordinate, we know that :

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) \tag{7}$$

Knowing that by definition of the kinetic energy T, the derivative with respect to position is given using the Chain rule as the time derivative of the momentum $\partial_t p$ with the momentum p being expressed as the derivative of kinetic energy with respect to the electron's linear velocity $\partial_v T$:

$$\frac{\partial T}{\partial x} = \frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) \tag{8}$$

Hence, given the defintion of the classical Lagrangian L as the difference between kinetic and potential energy, and Euler – Lagrange equation expressed in Eq 7, we have :

$$\frac{\partial V}{\partial x} = \frac{d}{dt}\left(\frac{\partial V}{\partial v}\right) \qquad (9)$$

Having the expression of the electron's potential energy as a function of the position function χ defined in Eq 2 : $V(x) = \frac{e^2}{4\pi\varepsilon_0}\chi$ We thus get :

$$\frac{\partial \chi}{\partial x} = \frac{d}{dt}\left(\frac{\partial \chi}{\partial v}\right) \qquad (10)$$

Substituting hence in Eq 6, yields :

$$\frac{\partial p}{\partial t} = -\frac{e^2}{4\pi\varepsilon_0}\frac{\partial \chi}{\partial x} \qquad (11)$$

Writing Newton's 2[Nd] Law for the electron electrodynamics under the electromagnetic forces exerted on it by the nucleus and the other electrons in the atom, we have :

$$\frac{\partial p}{\partial t} = \frac{e^2}{4\pi\varepsilon_0}\left(-\frac{Z}{R^2} + \sum_{i=1}^{Z}\frac{1}{r_i{}^2}\right) = \frac{e^2}{4\pi\varepsilon_0}\phi \qquad (12)$$

Hence we derive the relationship between the two main position functions of the electron's dynamics in its atomic shell :

$$\phi = -\frac{\partial \chi}{\partial x} \qquad (13)$$

2. Wave dynamics

While in uniform circular motion around the nucleus in its atomic orbital, we consider that the electron moves similarly to a photon in free space, in the form of a plane transversal wave propagating with a phase velocity v_φ that is equal to the linear velocity $v = \partial_t x$. resembling an electromagnetic wave propagating through space $\psi(x,t) = e^{-\frac{i}{\hbar}(Et-kx)}$. In Quantum Field theory, and experimently proved light – matter interactions such as Pair production, Compton scattering, Larmor effect … this is illustrated by the fact that there is no fundemental difference between photon dynamics and electron dynamics, for instance, one can write the principle of momentum and energy conservation (the two fundemental laws of particle dynamics along with the principle of least action) in a Light – matter interaction just like writing the same principles for collision dynamics in a two or Multi – body system [6], additionally, the principle of least action, from which all the fundemental laws of analytical mechanics for massive mechanical systems (Newton's laws, Euler – Lagrange equations, Hamilton's principle ..) are derived, is also the law from which follows Fermat's principle via the optical lagrangian $L_{optical} = n\sqrt{1 + \dot{x}^2 + \dot{y}^2}$ [7]; that is the fundemental law of photon dynamics and wave optics [8].

We know that at constant rotational velocity v, the wavenumber of the particle k is constant, where the normal acceleration a_n is given approximatly by ${v^2}/{R}$ hence one can write $v\partial k = k\partial v$, such that that the angular frequency of the particle's corresponding wave $\omega = 2\pi f = kv$ is constant since the frequency f of a plane wave does not vary only when the total energy transmitted by the wave varie via an emission, absorption or scattering. Hence, using the chain rule that the derivative of the electron's wavenumber $k = {2\pi}/{\lambda}$ with respect to time is nothing but the wave number mutiplied by the derivative of its velocity with respect to position :

$$\frac{\partial v}{\partial x} = \frac{v}{k}\frac{\partial k}{\partial x} = \frac{1}{k}\frac{\partial k}{\partial t} \qquad (14)$$

We have from the electron's classical hamiltonian formula in Eq 1 knowing that the position derivative of the hamiltonian H is zero, that :

$$\frac{\partial H}{\partial x} = p\frac{\partial v}{\partial x} + \frac{e^2}{4\pi\varepsilon_0}\frac{\partial \chi}{\partial x} = 0 \qquad (15)$$

Substituting thus the derivative of the phase velocity with respect to position $\partial_x v$ from Eq 14, in Eq 15, yields :

$$\frac{p}{k}\frac{\partial k}{\partial t}+\frac{e^2}{4\pi\varepsilon_0}\frac{\partial \chi}{\partial x}=0 \qquad (16)$$

Substituting hence for the time derivative of the electron's linear momentum p in Eq 11, we get :

$$\frac{p}{k}\frac{\partial k}{\partial t}-\frac{\partial p}{\partial t}=0 \qquad (17)$$

Yields thus that the ratio of the electron's linear momentum p over its wavenumber k is constant under a conserved total energy of the electron :

$$\frac{p}{k}=const \qquad (18)$$

To find the value of the constant $p/_k$; a simple devolopement of the linear momentum p of the electron into the derivative of its kinetic energy T with respect to its linear velocity v, and given that the angular frequency of the plane wave is given via the dispersion relation as the product of its phase velocity v and wave number $\partial\omega = k\partial v$:

$$\frac{p}{k}=\frac{\partial T}{\partial \omega} \qquad (19)$$

Hence from the Einstein – Planck formula, it's deduced that the constant is nothing but the reduced Planck constant $\hbar$, yields thus the De Broglie relation :

$$p=\hbar k \qquad (20)$$

CONCLUSION

In a semi-classical framework, by combining mathematical relations from the Particle dynamics of the electron of linear momentum p interacting with the Electromagnetic fields created by the nucleus of the atom and its other electrons in the electron shells, with wave relations from the Wave dynamics of the electron as a plane transversal wave of wave number k and phase velocity v_φ equal to the electron's linear velocity v, we have succefully established De Broglie's proportionality relation betwen the electron's momentum and wavenumber through the reduced Planck constant. Altough this does not show that the effective quantum theory of the electron is based on a plane wave model of its dynamics, however, it provides a good way of conceptualizing quatum dynamics phenomena at the subatomic scale.

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