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Relativistic Algebra in a Finite Universe

Yosef Akhtman, ya@gamma.earth

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Abstract
This work presents a foundational reconstruction of arithmetic and algebra based on the principle of relational finitude. Rejecting the assumption of actual infinity, we develop a finite, frame-dependent mathematical framework in which conventional number systems—integers, rationals, reals, and complex numbers-emerge as asymptotic projections or epistemic constructs derived from a closed finite field $\mathbb{F}_P$. Core arithmetic operations are reinterpreted as internal symmetries of the finite field, and conventional structures such as signed integers, rational fractions, and complex planes are systematically reconstructed as pseudo-numbers, defined relationally and locally with respect to a chosen frame of reference. Imaginary and transcendental constants i, e and π are derived as specific elements of $\mathbb{F}_P$ with important structural roles. We show that the resulting number classes possess all necessary properties for consistent computation, approximation, and algebraic closure, while offering the potential resolution of classical paradoxes of logic and set theory by replacing absolute notions with context-dependent representations. The proposed system provides a coherent and physically grounded alternative to standard mathematical formalism, suitable for the description of discrete, informationally finite physical systems.

1. Introduction

1.1. Relational Foundations and the Role of Finitude in Mathematics

A growing body of work in both mathematics and physics suggests that foundational structures may be more coherently understood through a *relational* or *relativistic* lens [18, 14, 24]. In such a framework, mathematical entities acquire meaning not as intrinsic absolutes, but through their role within a system defined by internal symmetries, reference points, and operational context. Numbers like 0, 1, or even complex i, are not fixed metaphysical truths, but functionally designated positions—origins, units, or rotation axes—established by a particular framing. Just as in modern physics the concepts of space, time, and simultaneity are frame-dependent and observer-relative [14], so too might mathematical meaning be seen as emergent from internal structural relations rather than assumed from external absolutes.

This perspective invites a fresh examination of one of the most enduring assumptions in mathematics: the acceptance of *actual infinity* as a foundational principle. From the continuum of the real number line to infinite-dimensional Hilbert spaces, and from set-theoretic universes to infinitesimal analysis, the idea of infinity plays a central structural role. It is not merely a calculational tool, but a default axiom embedded in the formalism of contemporary theory.

And yet, this assumption invites ongoing reflection. Infinity remains a concept of extraordinary power but also of unresolved philosophical status and unverified physical realization. No empirical measurement or finite computation has ever accessed an actually infinite entity. Within a relativistic view of mathematics, such concepts may instead be interpreted as emergent limits, useful abstractions, or placeholders for symbolic overflow—arising when finite systems attempt to describe behaviors or relationships that exceed their intrinsic encoding capacity.

In our recent critical analysis [1], we have argued that concepts such as infinity, randomness, and falsehood are not intrinsic features of reality but rather *epistemic placeholders*—artifacts of representational overflow that arise when finite systems attempt to describe more than they can internally encode. Within this view, infinity emerges not as a metaphysical reality, but as a signal of symbolic saturation—a limit of formal self-reference within a finite, coherent, informationally complete universe.

Motivated by this epistemic perspective, we hereby propose a concrete mathematical framework: a *relativistic algebra* constructed entirely over a finite number system. Our objective is to demonstrate that the conventional number systems—integers, rationals, reals, and complex numbers—can be derived as asymptotic approximations, coordinate projections, or formal extensions of a fundamental, finite arithmetic. Rather than treating finiteness as a constraint to be transcended, we take it as the generative substrate of mathematical structure. In this view, the infinite scaffolding of classical mathematics arises as an emergent representation, valid only under particular framing conditions and in certain asymptotic limits. In the proposed relativistic mathematical framework, entities acquire meaning only through their role within a specific frame of reference.

This relational paradigm finds a natural analog in the development of modern physics. The transition from Newtonian mechanics to Einsteinian relativity redefined the very notions of space, time, and simultaneity—not as absolute quantities but as frame-dependent observations shaped by internal consistency and symmetry. Likewise, a relativistic mathematics replaces external absolutes with internal coherence, viewing all mathematical structures and operations as inherently contextual, subject to transformation, and defined through symmetry relations within finite systems.

Such a shift enables a more consistent and physically meaningful foundation for mathematics, especially in the domain of closed, informationally finite systems. It offers a unified perspective that bridges abstract algebra, geometry, and modern physical theory, and sets the stage for a reconstruction of mathematical reasoning grounded in self-contained, finite, and relational structures.

The present framework resonates with several contemporary perspectives that question the ontological status of the continuum and advocate for finitely constructed alternatives. In particular, Smolin has emphasized the need for a relational, observer-dependent formulation of physical laws, suggesting that the continuum is merely an idealization beyond the reach of internal observers [29, 30]. Similarly, D'Ariano and collaborators have reconstructed quantum theory from finite, informationally grounded axioms, demonstrating that core features of quantum mechanics can emerge without invoking infinite-dimensional Hilbert spaces [13]. From a mathematical standpoint, the approach aligns with the ultrafinitist program developed by Benci and Di Nasso, which offers a rigorous alternative to classical cardinality through the theory of numerosities and bounded arithmetic [4, 3].

Furthermore, the ultrafinitist school—pioneered by Yessenin-Volpin and Parikh—takes finitude even further by denying the meaningful existence of "too large" numbers and insisting on feasibility as a foundational constraint. Formalizations of ultrafinitism and feasibility arithmetic appear in works such as [35, 25, 28, 2], which explore the proof-theoretic and computational consequences of enforcing strict constructive bounds on arithmetic.

Ultrafinitism enforces an *a priori* cutoff on numerical existence—only those magnitudes deemed "feasible" within a human or machine resource bound are admitted. By contrast, our relativistic framework treats finiteness not as a hard barrier but as a *contextual framing condition*: We allow arbitrarily large numbers, so "size" is always relative to the chosen frame. Infinite structures, such as integers and rationals emerge *asymptotically* or as coordinate projections, rather than being forbidden. Arithmetic operations become internal symmetries of a finite system, rather than operations constrained by external feasibility checks. This shift replaces the ultrafinitist's absolute feasibility threshold with a *relational* notion of scope: any number "exists" within some finite frame, while "infinity" itself appears as a relative point beyond the horizon of observability and algebraic accessibility.

To support this framework, we further draw upon several key developments in mathematics and physics. The foundational critique of actual infinity has been explored in works such as [8, 34], which emphasize the constructive and finitist approaches to mathematics. The relational perspective on mathematical objects aligns with category theory [18], where objects are defined by their morphisms and relationships rather than intrinsic properties. Additionally, the parallels between relativistic mathematics and modern physics are inspired by the symmetry principles in [14, 24], which highlight the role of invariance and frame-dependence in physical laws. Finally, the informational limits of finite systems and their implications for mathematical representation are discussed in [10, 21].

In presenting this framework, we are mindful of a subtle but persistent challenge: the tension between purely structural formalism and the inevitable intuitions of physical space and time. For the sake of incremental clarity, in this work, we attempt to keep the description of the underlying constructs as abstract and rooted in internal symmetries and relational operations. We are trying to defer the explicit treatment of the physical interpretation for future work. Nevertheless, certain physical associations become difficult to avoid. In particular, the article presumes the existence of a finite universe, yet this universe is not assumed to be static or externally bounded. Its finiteness, like all properties in the proposed model, is not absolute but relativistic—defined with respect to the observer's internal frame of reference. The transition from structural symmetry to perceived dynamics, and from closure to apparent unbounded evolution, is treated as a frame-dependent transformation rather than an ontological shift.

2. Relativistic Numbers

2.1. The Finite Ring $\mathbb{Z}_Q$

Let us consider a conventional definition of a finite ring $\mathbb{Z}_Q$ that is defined as the set of integers modulo Q, given by

$$\mathbb{Z}_Q := \{0, 1, 2, \dots, Q-1\},$$

equipped with addition and multiplication operations defined modulo Q. That is, for any $a, b \in \mathbb{Z}_Q$,

$$a + b := (a + b) \mod Q, \quad a \cdot b := (a \cdot b) \mod Q.$$

The structure $(\mathbb{Z}_Q, +, \cdot)$ forms a commutative ring with identity. It is a field if and only if Q is prime. The ring $\mathbb{Z}_Q$ provides a canonical example of a finite number system, in which all operations are closed, and every element has a unique representative within the finite set. It plays a central role in modular arithmetic, coding theory [22], cryptography [23], and discrete algebraic structures [5]. A

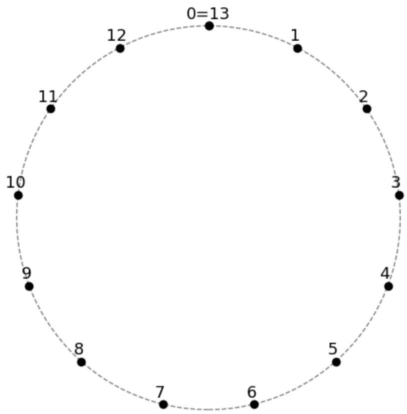


Figure 1: Diagram of a finite Ring $\mathbb{Z}_{13}$, typically visualized as a circle on a 2D plane that illustrates its periodicity and rotational symmetry under the arithmetic operation of addition.

typical diagram of a finite ring $\mathbb{Z}_Q$, where $Q = 13$, is shown in Figure 1. We would like to specifically note that such a diagram is typically visualized as a circle on a 2D plane that illustrates its periodicity and rotational symmetry under the arithmetic operation of addition.

Proposition 1 (Relativity of Representation Labels in $\mathbb{Z}_Q$)**.** Let $\mathbb{Z}_Q = \{0, 1, 2, \ldots, Q - 1\}$ be the finite field of integers modulo a prime Q. The elements of $\mathbb{Z}_Q$ form a complete and closed set of representations under modular addition and multiplication. However, the specific numeric labels assigned to these elements—particularly the designation of 0 as the additive identity—are intrinsically relative and carry no absolute meaning within the field itself.

More precisely, the field $\mathbb{Z}_Q$ is invariant under relabeling of its elements via any bijective affine transformation of the form

$$k \mapsto a \cdot k + b \mod Q,$$

where $a \in \mathbb{Z}_Q^\times$ and $b \in \mathbb{Z}_Q$. Such transformations preserve the field structure and allow any element to be reinterpreted as the origin. In this sense, the element labeled 0 is not uniquely privileged; it simply represents the additive identity with respect to a chosen reference frame. The same applies to the label 1, which identifies the multiplicative unit only relative to a particular scaling.

Therefore, in the absence of an externally imposed or contextually declared frame—such as one defined by a designated pair $(0, 1)$—the labels in $\mathbb{Z}_Q$ are relational rather than absolute. The roles of "zero" and "one" are thus not the fundamental properties of the elements themselves, but a consequence of the system's framing, making all representations in $\mathbb{Z}_Q$ symmetric and interchangeable under coordinate transformation. To define our system unambiguously, we must specify a reference frame or coordinate system $(0, 1)$ within the context of $\mathbb{Z}_Q$, which then becomes a *framed finite ring* $\mathbb{Z}_Q(0, 1)$.

We will henceforth assume all such systems to be framed systems $\mathbb{Z}_Q(0, 1)$ and will denote the corresponding finite ring as $\mathbb{Z}_Q$ for simplicity, unless explicitly stated otherwise.

2.2. Arithmetic Operations as Symmetries in $\mathbb{Z}_Q$

In the finite ring $\mathbb{Z}_Q$, the basic arithmetic operations—addition, multiplication, and exponentiation—can be understood as manifestations of the underlying symmetries and structural transformations of the ring.

Addition corresponds to a cyclic symmetry. The additive group $(\mathbb{Z}_Q, +)$ forms a finite cyclic group of order Q, generated by the element 1. Each addition operation $a \mapsto a + k \pmod{Q}$ can be viewed as a rotation by k steps around a circular configuration of the elements of $\mathbb{Z}_Q$. This symmetry reflects the homogeneity and periodicity of the additive structure [12].

Multiplication reflects a scaling symmetry within the ring. The operation $a \mapsto a \cdot k \pmod{Q}$ corresponds to a dilation or contraction of the additive structure, where the effect of multiplication is constrained by the modulus. The multiplicative structure of $\mathbb{Z}_Q$ is more subtle: if Q is prime, $\mathbb{Z}_Q^{\times} = \mathbb{Z}_Q \setminus \{0\}$ forms a finite multiplicative group, and multiplication becomes a permutation of the nonzero elements. If Q is composite, the presence of zero divisors disrupts this structure, but the operation still defines a transformation governed by modular symmetry [26].

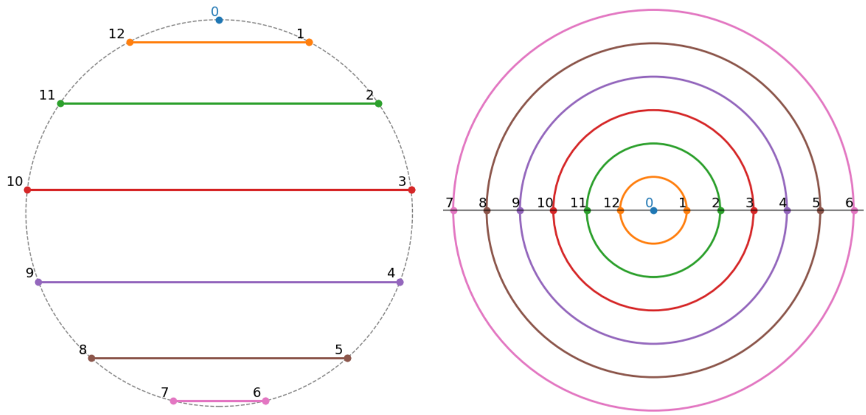
Exponentiation, or the operation $a \mapsto a^n \pmod{Q}$, represents iterated applications of the multiplicative symmetry. When restricted to the multiplicative group $\mathbb{Z}_Q^{\times}$, this operation defines power maps and automorphisms that reveal the group-theoretic structure and internal symmetries of the ring. In particular, when Q is prime, exponentiation captures cyclic subgroup structures and encodes deep number-theoretic properties such as primitive roots and residue classes [9].

Thus, the basic arithmetic operations in $\mathbb{Z}_Q$ are not arbitrary—they are algebraic expressions of the ring's internal symmetries. They define how elements of the system transform under structured, invertible actions, and they reveal the harmonious regularity inherent in finite arithmetic. Let us now focus on the case of a prime Q. In order to emphasize that Q is a Prime, we will henceforth denote it as P and the corresponding finite framed field as $\mathbb{F}_P$.

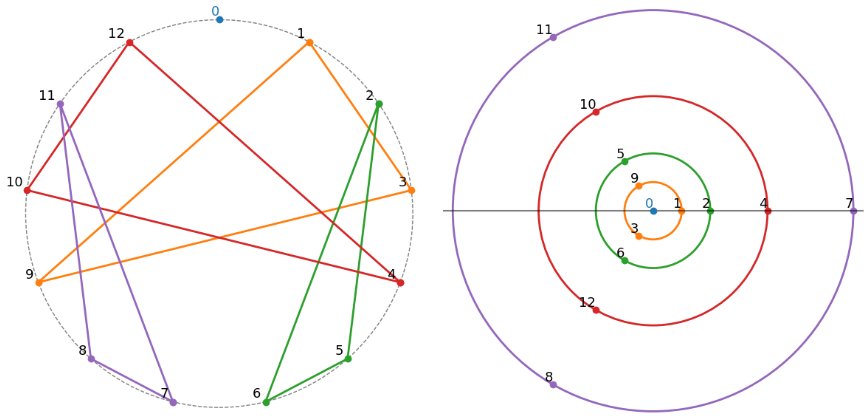
Proposition 2 (Morphology and Dimensionality of a finite field $\mathbb{F}_P$)**.** Let $P \in \mathbb{N}$ be a prime number. The finite field $\mathbb{F}_P$ and its abstract structure—its morphology—is determined by the symmetries induced by its fundamental arithmetic operations: addition, multiplication, and exponentiation. Each of these operations corresponds to a distinct and well-defined class of internal symmetry:

- **Addition modulo** P defines a one-dimensional cyclic symmetry, given by the additive group $(\mathbb{F}_P, +)$, which is isomorphic to the cyclic group $\mathbb{Z}/P\mathbb{Z}$. This symmetry reflects uniform translation across the representation space.
- **Multiplication modulo** P on the nonzero elements defines a second independent symmetry axis. The multiplicative group $(\mathbb{F}_P^{\times}, \cdot)$ is cyclic of order $P - 1$, introducing a discrete scaling symmetry across the system. Each nonzero element has a unique inverse, and multiplicative generators induce full cycles over the group.
- **Exponentiation modulo** P, defined as $a \mapsto a^n$, generates a family of automorphisms over $\mathbb{F}_P^{\times}$, capturing higher-order cyclic structures and subgroup hierarchies. This operation encodes internal periodicities and multiplicative phase structures.

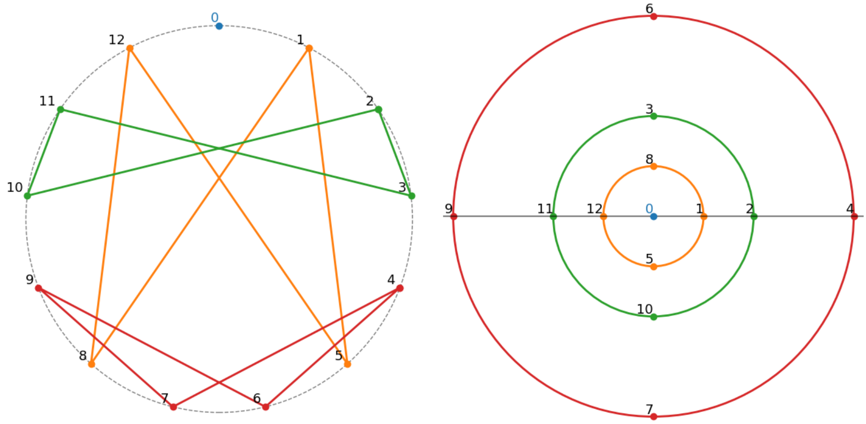
Together, these operations define a structured, multi-symmetric morphology for $\mathbb{F}_P$, which may be understood as a discrete two-dimensional algebraic manifold embedded in a three-dimensional abstract symmetry space. The dimensionality is determined by the triad of generating operations—additive



(a) $\mathbb{F}_{13}$, $N = 2$, $a = 12$

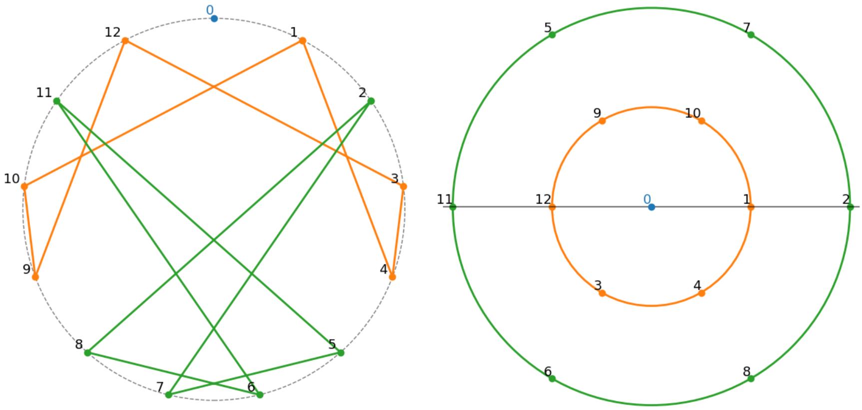


(b) $\mathbb{F}_{13}$, $N = 3$, $a = 3$

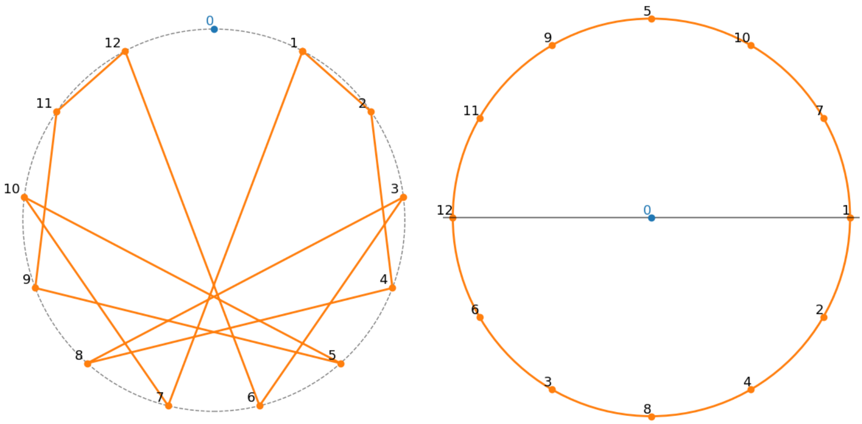


(c) $\mathbb{F}_{13}$, $N = 4$, $a = 5$

Figure 2: State diagram for finite framed field $\mathbb{F}_{13}$ from the perspective of (left) additive symmetry dimension and (right) multiplicative symmetry dimension of ranks (a) $N = 2$, (b) $N = 3$ and (c) $N = 4$.



(a) $\mathbb{F}_{13}$, $N = 6$, $a = 4$



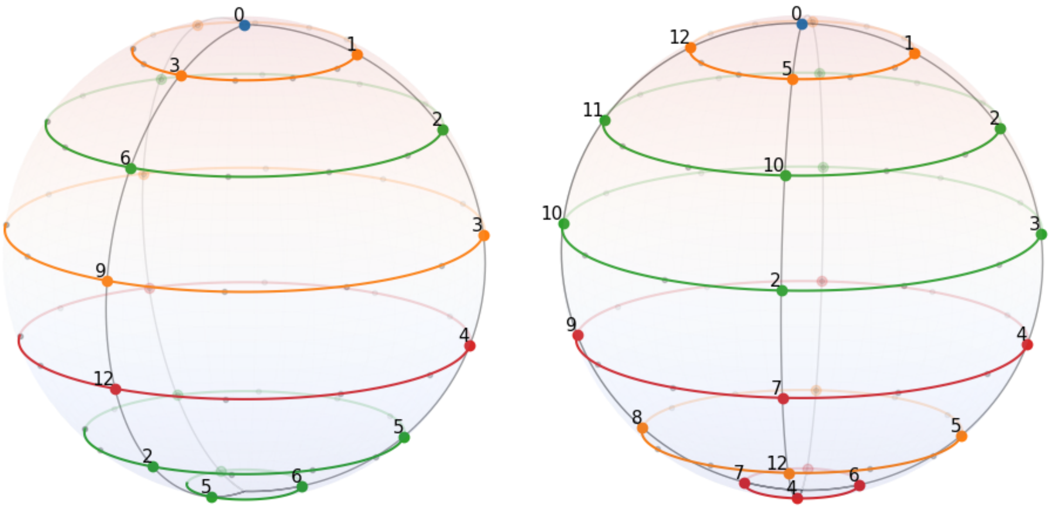
(b) $U_{13}(0, 1, 2)$, $N = 12$, $a = 2$

Figure 3: State diagram for finite framed field $\mathbb{F}_{13}$ from the perspective of (left) additive symmetry dimension and (right) multiplicative symmetry dimension of ranks (a) $N = 6$ and (b) $N = 12$.

rotation, multiplicative scaling, and exponential iteration—each contributing an independent axis of internal transformation.

It is important to note that the concept of operations as symmetries is not new. In fact, it is a well-established concept in the field of algebra. The operations of addition and multiplication in a finite ring can be viewed as transformations that preserve the structure of the ring. This perspective allows us to understand the ring's properties in terms of its symmetries, which can be particularly useful in applications such as coding theory [22], cryptography [23], and combinatorial design [5].

We would like to furthermore note that the concept of *operations*, or *transitions* inherently implies the possibility of some form of dynamics, or *evolution* of our mathematical structure, which in turn presumes the existence of an additional temporal degree of freedom. We should in principle therefore talk about the embedding of our two-dimensional manifold into some sort of four-dimensional space-time. We, however, for the sake of incremental clarity, choose to postpone the detailed treatment of this additional



(a) (b)

Figure 4: State diagram for finite framed field $\mathbb{F}_{13}$ as a 2D manifold in an abstract 3D state space combining the additive and multiplicative symmetry projections for multiplicative symmetries and corresponding multiplicative factors of (a) $N = 3, a = 3$ and (b) $N = 4, a = 5$.

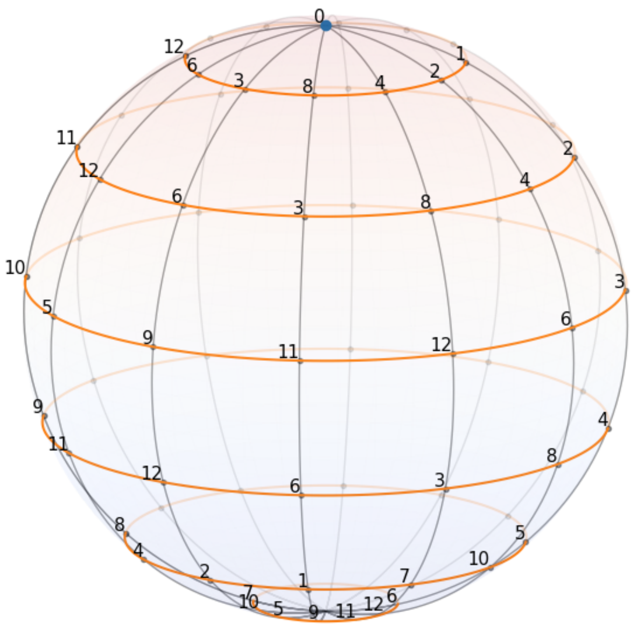


Figure 5: State diagram for finite framed field $\mathbb{F}_{13}$ as a 2D manifold in an abstract 3D state space combining the additive and multiplicative symmetry projections for multiplicative symmetry and corresponding multiplicative factor of $N = P - 1 = 12, a = 2$.

degree of freedom for the time being. Instead, we are treating our finite field $\mathbb{F}_P = \{0, 1, \ldots, P-1\}$ as a collection of all possible *representations* of a static finite mathematical object $\mathbb{F}_P$.

Furthermore, let P be a prime number, and let $\mathbb{F}_P^\times$ denote the multiplicative group of nonzero elements of the finite field $\mathbb{Z}_Q$, which is cyclic of order $P-1$. Let $N \in \mathbb{F}_P$ be a divisor of $P-1$, i.e., $N \mid (P-1)$. Then:

1. There exists an element $a \in \mathbb{F}_P^\times$ such that $a^N \equiv 1 \mod P$, and $a^k \not\equiv 1 \mod P$ for any $1 \le k < N$. Such an element is called a *primitive N-th root of unity* in $\mathbb{F}_P$.
2. The set

$$C_N := \{a^k \mod P \mid k = 0, 1, \ldots, N-1\}$$

   forms a cyclic subgroup of $\mathbb{F}_P^\times$ of order N, isomorphic to the cyclic group $\mathbb{Z}_N$.
3. Each element of C_N corresponds to a discrete *rotational symmetry* of order N in the multiplicative structure of $\mathbb{F}_P$. That is, multiplication by a acts as a finite rotation on this subgroup:

$$x \mapsto a \cdot x \mod P,$$

   cycling through the N elements of C_N.
4. The collection of all such subgroups C_N (for various divisors N of $P-1$) encodes the hierarchy of internal rotational symmetries within $\mathbb{Z}_Q^\times$, reflecting the subgroup lattice of the cyclic group of order $P-1$.

Correspondingly, we will hereby attempt to visualize the symmetries inherent to a finite field $\mathbb{F}_P$ as a two-dimensional manifold—a sphere—embedded in an abstract three-dimensional space. The two-dimensional manifold is defined by the two independent symmetries of addition and multiplication, while the third dimension accounts for the relational structure between components—such as shared identities, ring-theoretic coupling, and interactions via zero divisors or idempotents.

Firstly in Figures 2 and 3 we show the state diagram of a finite field $\mathbb{Z}_{13}$ from the separate perspectives of additive symmetry dimension (left) and multiplicative symmetry dimension (right) of ranks $N = 2, 3, 4, 6$ and 12. Subsequently, in Figure 4 we combine the two orthogonal 2D perspectives to visualize the state diagram for finite field $\mathbb{Z}_{13}$ as a 2D sphere in an abstract 3D state space combining the additive and multiplicative symmetries of ranks $N = 3$ and $N = 4$, and the corresponding multiplicative factors of $a = 3$ and $a = 5$. In this visualization we omit state labels that lay outside the multiplicative symmetry ridges for the sake of clarity. Finally, in Figure 5 we show a more complete visualization of the state diagram for finite Ring $\mathbb{Z}_{13}$ as a 2D manifold in an abstract 3D state space combining the additive and multiplicative symmetries for $N = P - 1 = 12, a = 2$.

While the finite field $\mathbb{F}_P$ provides a complete and closed algebraic structure, its inherently cyclic nature eliminates any meaningful notion of ordering or signed magnitude. In contrast, many physical and informational systems rely on the intuitive structure of the integers $\mathbb{Z}$, with concepts such as positive and negative values, proximity to an origin, and relational comparison. To bridge this conceptual gap, we would like to introduce a relativistic, context-dependent construction within $\mathbb{F}_P$ that recovers the essential features of integer arithmetic in a familiar and logically consistent form.

3. Integers over Finite Framed Field

In the conventional finite field $\mathbb{F}_P$, we can define negative elements $-k \in \mathbb{F}_P$ as the unique additive inverse of k, satisfying $k + (-k) \equiv 0 \mod P$ [12]. This definition of negation is algebraically consistent but is purely modular and lacks any intrinsic ordering. For example, the element -1 in $\mathbb{F}_P$ is not necessarily less than 0, as we can state $-1 - 0 = -1 = 12$, or greater than 0, as we can also state $0 - (-1) = 1$, and the same applies to any other element in the field. The lack of a meaningful ordering relation in the finite field $\mathbb{F}_P$ makes it impossible to define a signed magnitude or compare elements in a way that aligns with our intuitive understanding of integers.

Let us therefore consider the 3D representation of the finite field $\mathbb{F}_P$ as depicted in Figure 5 by observing it from the top down. We would like to offer a metaphor of the "North Pole" frame of reference, but it is important to note that the surface of the manifold in Figure 5 does not have any real special points and the selection of such "North Pole" position and the corresponding frame of reference is purely arbitrary and subjective.

Correspondingly, the original additive sequence $0, 1, \ldots, P-1$ of the ring's elements are represented as points located on the latitudinal axis—let us call it *the prime meridian*—of the $\mathbb{F}_P$ 2D manifold sphere, while the multiplicative symmetry elements are now arranged in circular patterns along the longitudinal axes and around the origin. Now let us imagine a naive local observer that is not aware of the spherical nature of the surface he is observing. We may need to hereby assume a sufficiently large cardinality P such that the local curvature is not apparent to such observer in the exact same way as the local curvature of the Earth is not apparent to a human observer. For such observer, the $\mathbb{F}_P$ manifold surface would appear as flat, and with the sequence of elements $\ldots, -2, -1, 0, 1, 2, \ldots$ forming a horizontal axis around the observer's position 0, as illustrated in Figure 6.

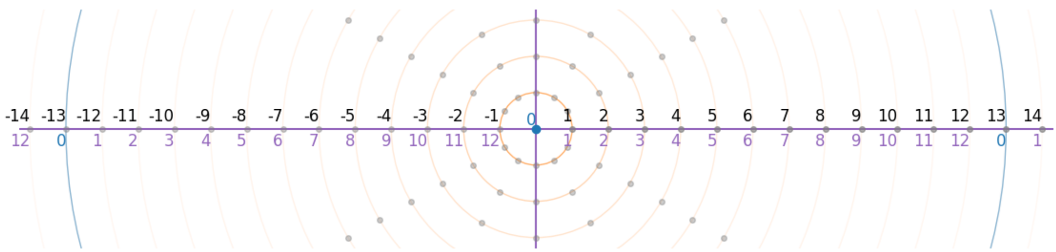


Figure 6: Class of signed pseudo-integers $\mathbb{Z}$ over the finite framed field $\mathbb{Z}_{13}$. Black labels indicate the newly defined signed integers $z \in \mathbb{Z}$, while the purple labels represent the corresponding elements $k(z) \in \mathbb{Z}_{13}$. The blue line indicates the periodicity of the finite field. The unlabeled gray dots indicate the off-axis elements of $\mathbb{Z}_{13}$ as they are observed from the top of the 2D manifold described in Figure 5.

Subsequently, from the subjective viewpoint of our local observer situated in the origin 0, the finitude and periodicity of our relativistic pseudo-integer number axis are effectively imperceptible, particularly if the system cardinality P is very large. The observer, limited to a bounded neighborhood where the local curvature and closure of the system are negligible, will perceive the axis as an infinite, flat line extending without bound in both directions from the origin. In this regime, all arithmetic and relational properties appear indistinguishable from those of the conventional infinite signed integer set $\mathbb{Z}$, and the underlying periodicity of the finite field remains hidden. Thus, from the local perspective, it is both natural and practically justified to assume that the resultant integer number axis is infinite and non-repeating, mirroring the familiar structure of the integers.

More formally, we can define a class of signed integers $\mathbb{Z}$ over the finite field $\mathbb{F}_P$ by introducing a mapping $k : \mathbb{Z} \to \mathbb{F}_P$ that assigns each integer $z \in \mathbb{Z}$ to its corresponding element in the finite field. This mapping is defined as follows:

$$k(z) = z \mod P \quad \text{for} \quad z \in \mathbb{Z},$$

where $k(z)$ is the element of $\mathbb{F}_P$ corresponding to the integer z. This mapping effectively wraps the integers around the finite field, allocating a unique value of $k \in \mathbb{F}_P$ for each value in $\mathbb{Z}$. Importantly however, the mapping $k(z)$ is not injective, as multiple integers can map to the same element in the finite field due to its periodicity. For example, both 1 and -12 map to the same element in $\mathbb{Z}_{13}$, as they are congruent modulo 13.

From an ontological perspective, we stipulate that what **really exists** is the finite field $\mathbb{F}_P$, comprised of exactly P distinct representations. The abstract mathematical constructs—such as the Finite Framed Field $\mathbb{F}_P(0, 1)$ and the Integers $\mathbb{Z}/\mathbb{F}_P(0, 1)$—are collections of symbolic labels that serve as epistemic tools: they are purely relativistic and utilitarian constructions used to observe and describe the finite field $\mathbb{F}_P$ in a manner consistent with the intuitive understanding of numbers by a local observer with subjective frame of reference and limited horizon of observability.

Nevertheless, the resulting class of relativistic pseudo-integers $\mathbb{Z}/\mathbb{F}_P$ exhibits all the characteristic properties of the conventional integer set $\mathbb{Z}$, including sign, order, addition, subtraction and multiplication. This framework allows us to recover the intuitive and logical structure of integers — including signed quantities and magnitude comparison — entirely within the finite, self-contained system $\mathbb{F}_P$, while preserving consistency with its modular arithmetic.

We would like to now revisit the analogy of the flat-Earth model to illustrate the nature of this relativistic mathematical approach. The flat-Earth model, while intuitively appealing and locally sufficient for navigation, is ultimately a limited approximation of a more complex reality. It is not that the Earth is flat, but rather that its curvature is imperceptible at human scales. The spherical Earth is not an alternative to the flat-Earth model; it is a more complete and accurate description of the same object, one that accounts for the global structure and curvature that the flat-Earth model neglects. This analogy highlights a central principle of the relativistic mathematical approach: apparent absolutes often emerge from local reasoning applied to globally finite systems. Just as the flat-Earth model was a pragmatic illusion born of scale, the standard integer structure may itself be a local approximation—emergent from the deeper, finite, and symmetric substrate of a finite arithmetic universe.

Having recovered the structure of signed integers $\mathbb{Z}$ over the finite field $\mathbb{F}_P$, it is natural to ask whether further extensions of this framework can reproduce the next layer of classical number systems—namely, the rational numbers $\mathbb{Q}$. Rational numbers emerge from the pragmatic necessity to express and manipulate ratios of integers, and their introduction marks a critical step in the construction of continuous arithmetic, proportional reasoning, and linear structure. The motivation for this extension is twofold. First, it allows us to reconstruct the essential properties of $\mathbb{Q}$ over $\mathbb{F}_P$, making clear that rationality is not an intrinsic feature of infinite arithmetic but an emergent relational construct definable within finite algebra. Second, it enables a more expressive arithmetic language within the finite mathematical system, allowing for the representation of proportional relationships, scales, and geometric constructs entirely within the bounds of a finite and self-contained mathematical system.

This pursuit continues the broader program of developing a *relativistic arithmetic*—a layered, frame-dependent construction of number systems in which each extension (integers, rationals, imaginary) arises as a projection or closure of relations already implicit in a more fundamental finite substrate.

4. Rationals over Finite Framed Field

Taking a step further, we define a class of pseudo-rational numbers $\mathbb{Q}_P$ as positions q along the prime meridian of $\mathbb{F}_P(0,1)$ as depicted in Figure 3 and further detailed in Figure 7, namely we have

$$\mathbb{Q}_P := \left\{ q = \frac{a}{b} \,\middle|\, a \in \mathbb{Z},\ b = \prod_i k_i,\ k_i \in \mathbb{F}_P^\times \right\}.$$

Furthermore, for each element $q \in \mathbb{Q}_P$ we define the equivalent representation $k \in \mathbb{F}_P$ such that

$$k(q) := a \times b^{-1} \bmod P = a \times \prod_i k_i^{-1} \bmod P. \tag{4.1}$$

The validity of the definition in Equation 4.1 is ensured by the fact that all elements k_i constituting the denominator product $b = \prod_i k_i$ have a multiplicative inverse $k_i^{-1} \in \mathbb{F}_P^\times$. A selection of some simple examples of such pseudo-rational numbers is depicted in Figure 7, where for each position along the prime meridian $q = a/b \in \mathbb{Q}_P$ indicated as a black label on top, the corresponding finite field element $k(q) \in \mathbb{F}_P$ is indicated as purple label on the bottom.

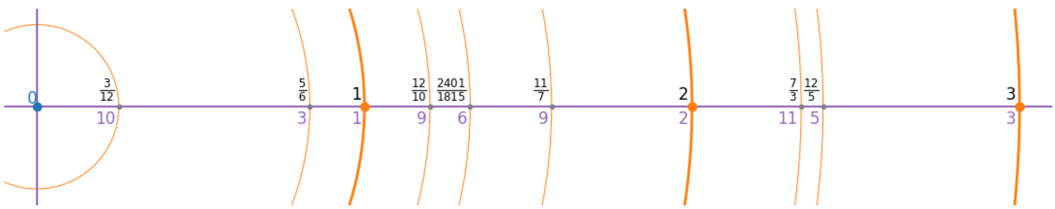


Figure 7: Few examples of rational numbers $q \in \mathbb{Q}_{13}$ in a finite framed field $\mathbb{Z}_{13}(0,1)$. Note the pseudo-rational numbers $6/5$, $12/10$ as well as $11/7$ that all represent the exact same element $9 \in \mathbb{Z}_{13}(0,1)$.

It is very important to reiterate the meaning of this construct from an ontological viewpoint. More specifically, we stipulate that what actually "exists" are the P representations of the finite field $\mathbb{F}_P$, while the derivative class of pseudo-rationals $q \in \mathbb{Q}_P$ constitute an abstract mathematical construct derived from the inherent relational properties of the framed instance $\mathbb{F}_P(0,1)$. More specifically, the multiplicity of labels $\{q\} \in \mathbb{Q}_P$ that can be associated with a single representation $k \in \mathbb{F}_P$ is not contradictory or paradoxical in the exact same way as it is not paradoxical to observe a multitude of reflections associated with a single real object in a kaleidoscope.

We would like to now propose that the class of pseudo-rational numbers $\mathbb{Q}_P$ is dense in $\mathbb{Q}$ in the sense that for any conventional rational number $q \in \mathbb{Q}$, prime $P > 2$ and any $\epsilon > 0$ however small, there exists a pseudo-rational number $q' = a/b \in \mathbb{Q}_P$ such that $|q - q'| < \epsilon$. In order to prove our proposition it is

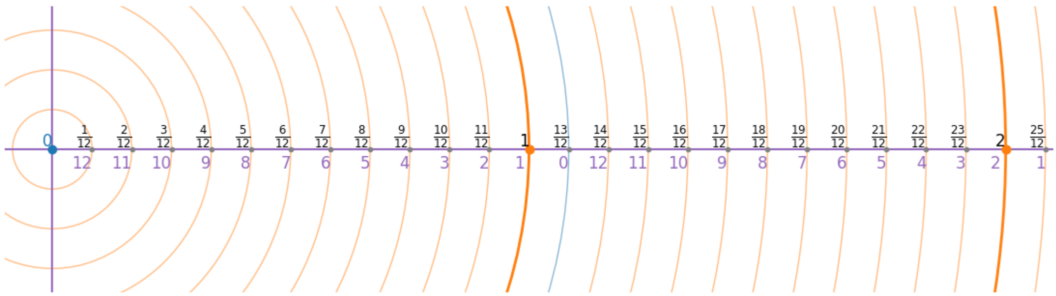


Figure 8: Uniform grid of rational numbers of the form $q = \frac{k}{(P-1)^n}$ with step size $\frac{1}{(P-1)^n}$. Here, we have $P = 13$ and $n = 1$. Black labels indicate the pseudo-rational numbers $q \in \mathbb{Q}_{13}$, while the purple labels represent the corresponding finite field elements $k(q) \in \mathbb{Z}_{13}(0, 1)$. The blue line indicates the periodicity of the finite field.

sufficient to characterize the density of the following subset $\{q = a/b\} \subset \mathbb{Q}_P$, where the denominator $b = (P-1)^n$ for some integer $n \in \mathbb{N}$.

Proposition 3. Let $P > 2$ be an odd prime number, and let $q = a/b \in \mathbb{Q}$ be any conventional rational number. Then for any $\epsilon > 0$, there exists an integer $n \in \mathbb{N}$ and an integer $x \in \mathbb{Z}$ such that

$$\left| \frac{a}{b} - \frac{x}{(P-1)^n} \right| < \epsilon.$$

Proof. Let $\frac{a}{b} \in \mathbb{Q}$ be given, and let $\epsilon > 0$ be arbitrary small number.

Since P is a fixed prime, the expression $(P-1)^n$ grows without bound as $n \to \infty$. Therefore, there exists an integer $n \in \mathbb{N}$ such that

$$\frac{1}{(P-1)^n} < \epsilon.$$

Now consider the set of rational points of the form

$$\left\{ \frac{k}{(P-1)^n} \,\middle|\, k \in \mathbb{Z} \right\},$$

as illustrated in Figure 8. This set is a uniform grid of rational numbers with step size $\frac{1}{(p-1)^n}$, which is less than ϵ by construction. There exists therefore an integer $x \in \mathbb{Z}$ such that

$$\left| \frac{a}{b} - \frac{x}{(P-1)^n} \right| < \epsilon,$$

which completes the proof. □

In other words, the resultant field of pseudo-rational numbers $\mathbb{Q}_P$ will exhibit all the properties of the field of conventional numbers $\mathbb{Q}$ and can further approximate it with any arbitrary precision. Furthermore, for an observer with a limited observability horizon and sufficiently large values of cardinality P, the pseudo-rational field $\mathbb{Q}_P$ becomes completely indistinguishable from its conventional counterpart, as all the desired rational numbers of the form $q = a/b$, where $b < P$ are represented not approximately, but exactly within the scope of the pseudo-rational numbers $\mathbb{Q}_P$.

In classical mathematics, the field of real numbers $\mathbb{R}$ is introduced to enable the formulation of continuous functions, calculus, and metric spaces—tools indispensable for modeling physical phenomena and abstract structures alike. However, the real number line is defined as an uncountable, infinitary continuum, an ontological commitment that conflicts with the finite and relational framework we adopt in this study. Nonetheless, our need for *continuous approximation* and *proportional reasoning* persists, particularly in describing geometric constructs, dynamic systems, and analytic behaviors. Our approach is therefore pragmatic and epistemic rather than metaphysical. We seek to construct a class of *pseudo-real numbers* that fulfills the operational role of $\mathbb{R}$ without invoking actual infinity. Of particular interest is the ontological status of the transcendental numbers π and e that play a special role across numerous domains of mathematics and physics. We would like to offer a special treatment of these numbers later in this manuscript.

5. Real Numbers over Finite Framed Field

We introduce the notion of *pseudo-real numbers* as an emergent structure grounded in two complementary principles: **algebraic solvability** within the finite field, and **asymptotic approximability** through pseudo-rationals.

Many irrational or real numbers of algebraic origin exist as *exact* solutions to polynomial equations within $\mathbb{F}_P$, provided their cardinality P satisfies appropriate conditions. For example, the equation $x^2 + 1 = 0$ admits solutions in $\mathbb{F}_P$ if and only if $P \equiv 1 \mod 4$, where the field contains elements $\pm i \in \mathbb{F}_P$ such that $i^2 = -1 \mod P$. More generally, roots of higher-degree polynomials (such as $\sqrt{2}$, $\sqrt[3]{5}$, or primitive roots of unity) may exist in $\mathbb{F}_P$ depending on the field's cardinality P.

Thus, we define a *pseudo-real number* to exist *exactly* within $\mathbb{F}_P$ if it corresponds to a solution of a polynomial equation over the field. In this view, the ontological status of a real number is not absolute but conditional—determined by the algebraic closure properties of the specific finite field. Irrationality, therefore, is reframed not as a metaphysical property but as a question of *field compatibility*.

For those real numbers not supported directly by $\mathbb{F}_P$, we invoke a pragmatic alternative: approximability via the class of pseudo-rational numbers $\mathbb{Q}_P$.

Definition 1 (Observation Horizon)**.** Fix a prime P. An *observation horizon* is a natural number $H \ll P$ that bounds the allowed denominator exponents and hence the precision of any observer in the finite field setting.

Definition 2 (Truncated Pseudo-Rationals and Metric)**.** Given $H \in \mathbb{N}$, define

$$\mathbb{Q}_P^{\leq H} \;:=\; \big\{[x,n] \;:\; 0 \leq x < P,\; 0 \leq n \leq H\big\},$$

where the symbol $[x,n]$ denotes the rational number $x/(P-1)^n$. Equip $Q_P^{\leq H}$ with the metric

$$d_H\big([x,n],[y,m]\big) \;=\; \left|\tfrac{x}{(P-1)^n} - \tfrac{y}{(P-1)^m}\right|$$

computed in $\mathbb{Q}$ using the common denominator $(P-1)^{\max(n,m)} \leq (P-1)^H$.

Proposition 4 (Finite Total Boundedness)**.** For each fixed H, the metric space $\big(\mathbb{Q}_P^{\leq H}, d_H\big)$ is finite and thus totally bounded.

Proof. Since $0 \le x < P$ and $0 \le n \le H$, there are $(P) \times (H+1)$ elements in $\mathbb{Q}_P^{\le H}$. Any finite metric space is trivially totally bounded. □

Theorem 5.1 (Approximation of Computable Reals)**.** *Let $r \in \mathbb{R}$ be a computable real number. For any integer $k \ge 1$ there exist integers a_k, b_k with $b_k \ne 0$ such that*

$$\left| r - \tfrac{a_k}{b_k} \right| \; < \; 2^{-k}.$$

Moreover, if the observer's horizon H satisfies

$$H \;\ge\; \lceil k \log_2 P \rceil,$$

then one can construct $[x_k, n_k] \in \mathbb{Q}_P^{\le H}$ with

$$\Big| r - [x_k, n_k] \Big| < 2^{-k-1}.$$

Proof. By computability of r there is a rational approximation a_k/b_k with $|r - a_k/b_k| < 2^{-k}$. Since $\gcd(P, P-1) = 1$, the extended Euclidean algorithm yields an exponent $n_k \le k \log_2 P$ such that

$$b_k\,(P-1)^{n_k} \equiv 1 \pmod{P}.$$

Set $x_k \equiv a_k\, b_k^{-1} \pmod{P}$. Then

$$[x_k, n_k] = \frac{x_k}{(P-1)^{n_k}}$$

differs from a_k/b_k by at most $\big(2\, b_k\,(P-1)^{n_k}\big)^{-1} < 2^{-k-1}$. If $H \ge n_k$, then $[x_k, n_k] \in \mathbb{Q}_P^{\le H}$ and

$$\big|r - [x_k, n_k]\big| \le \big|r - \tfrac{a_k}{b_k}\big| + \big|\tfrac{a_k}{b_k} - [x_k, n_k]\big| < 2^{-k} + 2^{-k-1} = O(2^{-k}),$$

as desired. □

The resulting pseudo-real field $\mathbb{R}_P$ is thus defined as the topological closure of $\mathbb{Q}_P$ under modular convergence. For any finite observer with bounded resolution and limited horizon of observability, $\mathbb{R}_P$ is indistinguishable from the conventional real number continuum.

In conclusion, the field of pseudo-real numbers $\mathbb{R}_P$ is not a metaphysical continuum but a layered epistemic utilitarian construct. It combines:

- Exact pseudo-reals that satisfy algebraic equations within $\mathbb{F}_P$, and
- Approximated pseudo-reals that are limits of converging sequences in $\mathbb{Q}_P$.

This framework provides all the functional properties of the real numbers—continuity, density, and completeness—without invoking actual infinity. It affirms that, in a finite and informationally complete universe, *continuum-like behavior is a pragmatic illusion* emerging from local reasoning over a fundamentally finite arithmetic substrate.

Having established the construction of pseudo-integers, rationals and reals over the finite field $\mathbb{F}_P$ as relativistic, frame-dependent analogs of their classical counterparts, we seek to further extend this framework to encompass the algebraic closure of the pseudo-real field. In conventional mathematics, the introduction of complex numbers $\mathbb{C}$ is necessitated by the absence of solutions to certain polynomial equations, such as $x^2 + 1 = 0$, within the real numbers. Analogously, in the finite framed context, we are motivated to introduce complex-like elements in order to achieve closure under operations that are otherwise impossible within the pseudo-rational or alone.

Moreover, the construction of a relativistic complex plane enables the representation of rotations, oscillations, and other phenomena that are fundamental in both mathematics and physics, all within a finite and self-contained system. This approach not only mirrors the classical extension from $\mathbb{R}$ to $\mathbb{C}$, but also demonstrates that the essential properties and utility of complex numbers can be realized as emergent features of a finite, relational arithmetic—thereby reinforcing our framework's central theme of relativistic, context-dependent number systems.

6. Complex Plane over Finite Framed Field

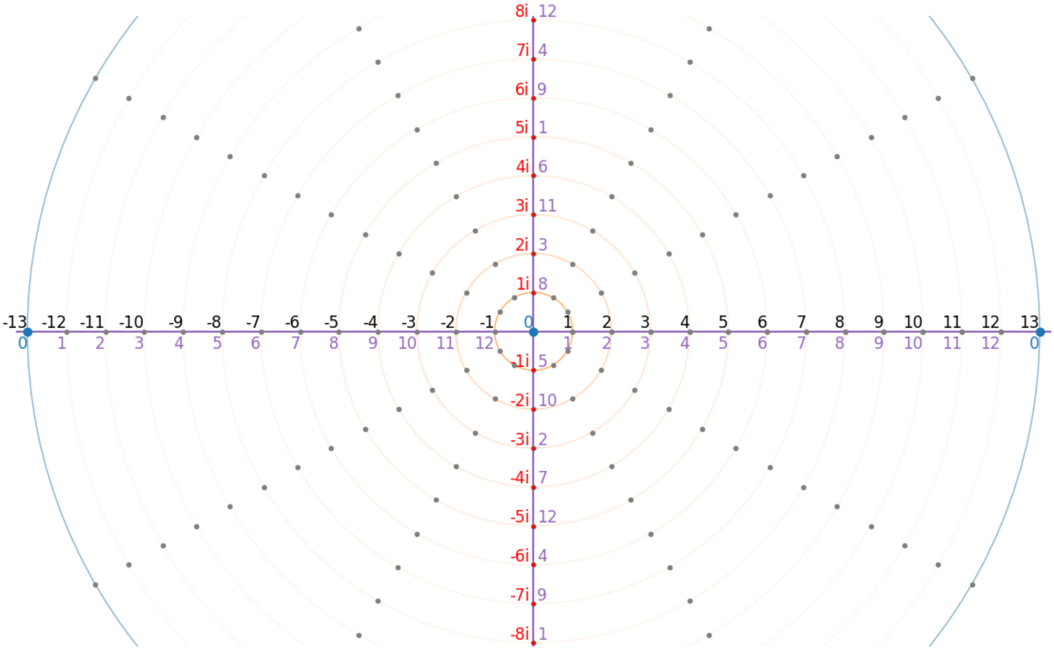


Figure 9: Pseudo-complex numbers plane $\mathbb{C}_P$ in a finite framed field $\mathbb{Z}_{13}(0, 1)$. Horizontal axis represents the pseudo-reals $\mathbb{R}_P$ on the prime meridian and the vertical axis represents the imaginary numbers $c = z \cdot i$ indicated by their respective red labels. The corresponding elements $k(c)$ are depicted in purple. The blue line indicates the periodicity of the finite field.

As is commonly known, the field of real numbers $\mathbb{R}$ does not contain any solutions of certain polynomial equations, such as the prominent equation $x^2 + 1 = 0$. But that is not the case for many finite fields $\mathbb{F}_P$, where depending on the value and properties of their cardinality P, such solutions can readily exist. For example, in the finite field $\mathbb{Z}_5$, the equation $x^2 + 1 = 0$ has two solutions: $x = 2$ and $x = 3$. More generally, it is evident that the equation $x^2 + 1 = 0$ can be satisfied in a finite field $\mathbb{F}_P$ if and only if $P - 1$ is devisable by 4, or in other words $P \equiv 1 \mod 4$. This is due to the fact that the multiplicative group of non-zero elements in such fields is cyclic and contains elements—and the corresponding rotational symmetry—of order 4, which allows for the existence of square roots of -1. In this case, we can define a special element $i \in \mathbb{F}_P$ that satisfies the equation $i^2 + 1 = 0$. The element i is not unique, instead we have a pair of pseudo-integer elements i and $-i$ in $\mathbb{Z}/\mathbb{F}_P$ that satisfy the equation, in the same way as we have pairs x and $-x$ of solutions for quadratic equations in the conventional complex plane $\mathbb{C}$.

Let us now observe the "North Pole" frame of reference of the spherical representation of the finite field $\mathbb{F}_P$ with its prime meridian of pseudo-reals $\mathbb{R}_P$ forming the horizontal axis around the origin. The order-4 rotational symmetry of the finite field $\mathbb{F}_P$ can be represented as a vertical axis of imaginary numbers $c = z \cdot i$, where $z \in \mathbb{Z}$, that are perpendicular to the prime meridian, as illustrated in Figure 9. The imaginary numbers c are represented by their respective red labels, while the corresponding elements $k(c)$ are depicted in purple.

More generally, we can define a class of pseudo-complex numbers $\mathbb{C}_P$ as the Cartesian product of the pseudo-reals $\mathbb{R}_P$ and the imaginary numbers $r \cdot i, r \in \mathbb{R}$. The pseudo-complex numbers are defined as follows:

$$\mathbb{C}_P := \{c = a + b \cdot i \mid a, b \in \mathbb{R}_P\}, \tag{6.1}$$

where a and b are the real and imaginary components of the pseudo-complex number c, respectively. The pseudo-complex numbers can be represented as points in the complex plane, where the horizontal axis corresponds to the pseudo-reals $\mathbb{R}_P$ and the vertical axis corresponds to the imaginary numbers $r \cdot i$. The pseudo-complex numbers form a field and can be added, subtracted, multiplied, and divided in a manner analogous to conventional complex numbers, with the additional consideration of their finite field properties.

The pseudo-complex numbers form a relativistic algebraic field and can be added, subtracted, multiplied, and divided in a manner analogous to conventional complex numbers, subject to the selection of the arbitrary frame of reference, as well as the properties and constraints of the underlying finite field.

7. Universal Constants in a Finite Universe

The universal transcendental constants e and π arise naturally in a variety of mathematical and physical contexts. Arguably, their existence is not imposed by some metaphysical necessity, but rather motivated by concrete, utilitarian applications in our pursuit to understand and model the physical reality we inhabit. We would like to further emphasize that any and all uses of these constants in their conventional interpretation are always approximate, as their real transcendental, hypothetically infinitely-complex values can never be realized or utilized in any practical sense. Informed by this motivation, we would like to explore the meaning and significance of these enigmatic constants in the context of our finite universe and more specifically in the context of a finite field $\mathbb{F}_P$. In other words, we would like to try and reconcile the epistemic interpretation and the functional role of these constants across two distinctively different modes of observation.

7.1. Observer Scenarios

We define two distinct types of observers of the finite field $\mathbb{F}_P$ based on their respective observation horizons. The first scenario is that of an *internal observer* with a limited observation horizon around its frame of reference. This scenario can be interpreted as the classical, intuitive vantage point. The second scenario is that of a god-like *external observer* who is capable to access and comprehend the finite field $\mathbb{F}_P$ in its entirety. We would like to associate—currently without detailed explanation—this later mode with the *quantum mechanical* view. More formally, the two scenarios are defined as follows.

A. Internal Observer ($H_{\text{int}} \ll P$): An observer with horizon $H_{\text{int}} \ll P$ is confined to a small metric ball in $\mathbb{F}_P$ around its origin 0, which is locally indistinguishable from a patch of $\mathbb{R}^2$. In this regime the observer is not aware of the actual value of P, field operations appear "flat," and the resultant P-invarient constants like e and π emerge only approximately via truncated pseudo-rational approximations in

$$\mathbb{Q}_P^{\leq H_{\text{int}}} = \big\{[x, n] : 0 \leq x < P,\ 0 \leq n \leq H_{\text{int}}\big\},$$

with errors of order $(P-1)^{-H_{\text{int}}}$.

B. External Observer ($H_{\text{ext}} \gg P$): An observer with horizon $H_{\text{ext}} \gg P$ views the entire finite field $\mathbb{F}_P$, including its cyclic/periodic structure, and is aware of the exact value and properties of the field cardinality P. From this global vantage, universal constants i, e and π emerge as exact elements of the field, with particular structural roles and the corresponding significance.

7.2. Natural base e

The universal constant e emerges as the base of the natural exponential function, which has the unique property that its rate of change is equal to its value. It appears to underscore the fundamental principles by which change and uncertainty emerge in both conventional continuos, and discrete dynamic systems. At the same time, it plays a crucial role in the theory of complex numbers, where it serves as a generator for the polar rotation of the complex plane. In this section we would like to attempt to reconcile these two seemingly distinct interpretations by considering two complementary vantage points on the role of e that arise from the two distinct observer scenarios described above.

Let us therefore consider the process of polar rotation of the pseudo-sphere $\mathbb{F}_P$ around its primary axis. To the internal observer, who is confined to a small interval of the prime meridian of $\mathbb{F}_P$ around its origin and is not able to comprehend the entirety of the periodic process, the rotation will manifest itself as evolution, change and emergence of uncertainty. As we have argued in [1], the uncertainty experienced by an internal observer is not an intrinsic ontological property of the system but rather a direct consequence of the observer's finite horizon; in effect, this "uncertainty" simply quantifies the limit of observability imposed by a restricted sampling of field elements.

In contrast, to the external observer, the rotation will manifest itself as a discrete deterministic step in a structured and periodic process, fully determined by the field cardinality P and the corresponding structural properties of $\mathbb{F}_P$. Let us now explore these two interpretations in more detail.

A. Internal Observer: Consider an internal observer who can inspect at most $H \ll P$ elements of the finite field $\mathbb{F}_P$ at once, but does not know its exact cardinality P. Suppose that the pseudo-sphere is rotated around its primary access, which is equivalent to a multiplication by some element

$$a \overset{\text{uniform}}{\sim} \mathbb{F}_P^\times \setminus \{1\}.$$

Since for every $a \neq 1$ the equation

$$x = a\,x$$

has the unique solution $x = 0$, no non-zero element in the observer's horizon can remain fixed. Thus, the action of a on the set of observed non-zero elements is a derangement. For a sequence $\{1, 2, \ldots, n\}$, the number of derangements D_n, also often denoted as subfactorial $!n$, is given by [31]

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \tag{7.1}$$

so that the probability that a random permutation is a derangement is

$$\mathbb{P}_n(\text{derangement}) = \frac{!n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to \frac{1}{e} \quad \text{as } n \to \infty. \tag{7.2}$$

The probability that *all* H elements—observed by our observer—move is given by

$$\mathbb{P}_H \;=\; \left(1 - \tfrac{1}{P-1}\right)^H .$$

Now, the observer does not know the actual value of P, and the only thing that they do know is that it is larger than their horizon H. If the observer assumes the minimum possible value of $P = H + 1$, this yields the classical limit

$$e_H := \left(1 - \tfrac{1}{H}\right)^{-H}, \quad \lim_{H\to\infty} e_H \;=\; e.$$

Since $P > H$,

$$\mathbb{P}_H = \left(1 - \tfrac{1}{P-1}\right)^H \;\geq\; \left(1 - \tfrac{1}{H}\right)^H = e_H^{-1},$$

it follows that

$$\log \mathbb{P}_H^{-1} \;\leq\; \log e_H .$$

In other words, the quantity $\log e_H$ represents the lower bound on the measure of *information-theoretic uncertainty* (in nats) that can be expected by an observer with a limited observation horizon H after a multiplicative transformation of a finite field $\mathbb{F}_P$ that it is observing.

B. External Observer: To the external observer let us further explore the interpretation of the constant e as the generator $g \in \mathbb{F}_P$ of the multiplicative polar rotation of the pseudo-sphere $\mathbb{F}_P$ around its primary axis, which can be also regarded as the rotations of the corresponding complex plane $\mathbb{C}_P$. Using the finite field terminology, g is an order-$(P-1)$ *generator*—or a *primitive root*—of the multiplicative group $\mathbb{F}_P^\times$. More specifically, we can select our primitive root g as an element of $\mathbb{F}_P$ that satisfies

$$g \in \mathbb{F}_P^\times \;\middle|\; g^n = -1 \text{ for some } n \in \mathbb{F}_P^\times .$$

Finally, in order to reconcile the two complementary views experienced by observers A and B, we would like to recall that $\mathbb{Q}_P$ is dense in $\mathbb{R}$, and any finite section in $\mathbb{R}$ contains any arbitrary number of copies of any one element $g \in \mathbb{F}_P$. We can therefore identify the constant e_P as the *pseudo-rational* number that satisfies the following conditions

$$e_P \in \mathbb{Q}_P, g(e_P) \in \mathbb{F}_P^\times \;\middle|\; \quad e < e_P < e_H,\; g^n = -1.$$

The existence of e_H is evident by construction. Clearly it is not unique neither in $\mathbb{Q}_P$ where an arbitrary large number of such candidates can be constructed, nor in $\mathbb{F}_P$ where the number of qualifying generators is a subject of the field's cardinality and structural symmetries. This ambiguity is another immediate manifestation of the relativistic principle, where the same functional mathematical entities can have multiple reference frame-depended representation. The selection of any such representation will not affect the definition, significance and the operational roles of e_H to both the internal and the external observers. Furthermore, the resultant pseudo-rational constant e_P will serve all the familiar practical purposes associated with the use of the constant e to any desired degree of precision in the exact same way as all conventional approximations of the transcendental constant e are used in the state-of-the-art analysis and computations.

7.3. Mathematical constant π

Similarly to e, the universal constant π is a transcendental number that appears in a variety of mathematical and physical contexts. It is most commonly associated with the geometry of circles, where it represents the ratio of the circumference to the diameter. However, its significance extends far beyond this simple definition, as it also plays a crucial role in complex analysis, number theory, and quantum mechanics, were it emerges as a generator of the *harmonic rotations* of the complex plane. In this section we would like to explore the meaning and significance of π in the context of our finite universe and more specifically in the context of a finite field $\mathbb{F}_P$. We will once again consider two complementary vantage points on the role of π that arise from the two distinct observer scenarios described above.

A. Internal Observer: Once again, consider an internal observer who can comprehend at most $H \ll P$ elements of the finite field $\mathbb{F}_P$ at once, but does not know its exact cardinality P. The subjective experience of the internal observer is that of a flat surface, extending seemingly indefinitely in all directions. Suppose now that the pseudo-sphere $\mathbb{F}_P$ is rotated around its primary access by some small angle θ. The subjective experience of our observer will be that all points in its environment are now displaced by a small increment πr, where r represents the distance of the point from the origin 0. In order to estimate the value of π, the observer will be able to define a circle as a set of points that are equidistant their position at the origin 0. The approximate value of π will be then given by the ratio of the half-circumference of the resultant circle and the selected circle r.

Let us recall that our observer has a limited observation horizon and therefor a limited resolution of its environment. The observer will therefore be able to define a circle as a finite set of H points that are equidistant from the origin of its frame of reference. Correspondingly, they may try to invoke the Leibniz-Gregory-Madhava formula [6, 16] that emerges when approximating the perimeter of a circle by repeatedly adding and subtracting areas associated with squares and rectangles inscribed and circumscribed around the circle. Specifically, they may use the following series that alternates between adding and subtracting terms, each smaller than the last, slowly converging toward the exact length of the quarter-circle arc, which equals $\frac{\pi}{4}$. Multiplying by 4 yields the horizon H approximation of the full circumference-to-diameter ratio π as follows

$$\pi_H = 4 \sum_{k=0}^{H} \frac{(-1)^k}{2k+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) \to \pi \ \text{ as } H \to \infty.$$

B. External Observer: To the external observer we can identify π as the half-period generator of the harmonic—*i.e.* exponentiation—rotations of $\mathbb{F}_P$ around its primary axis, and we can further recognize it as the rank-2 generator in $\mathbb{Z}^+_{P-1}$. More specifically, we can simply formulate our rank-2 additive g_π generator as an element of $\mathbb{F}_P$ that satisfies

$$g_\pi \in \mathbb{F}_P \;\Big|\; 2 \times g_\pi = -1.$$

In order to reconcile the two complementary views of the constant π experienced by observers A and B, we, once again, recall that $\mathbb{Q}_P$ is dense in $\mathbb{R}$, and any finite section in $\mathbb{R}$ contains any arbitrary number of copies of any one element $g \in \mathbb{F}_P$. We can therefore identify the horizon-H pseudo-rational approximation π_H of the transcendental constant π as the *pseudo-rational* number that satisfies the following conditions

$$\pi_H \in \mathbb{Q}_P, g_\pi = k(\pi_P) \in \mathbb{F}_P \;\Big|\; \quad \min(\pi, \pi_H) < \pi_P < \max(\pi, \pi_H),\; 2 \times g_\pi = -1.$$

The existence of e_H is evident by construction. Clearly it is not unique neither in $\mathbb{Q}_P$ where an arbitrary large number of such candidates can be constructed, nor in $\mathbb{F}_P$ where the number of qualifying generators is a subject of the field's cardinality and structural symmetries. This ambiguity is another immediate manifestation of the relativistic principle, where the same functional mathematical entities can have multiple reference frame-depended representation. The selection of any such representation will not affect the definition, significance and the operational roles of e_H to both the internal and the external observers. Furthermore, the resultant pseudo-rational constant e_P will serve all the familiar practical purposes associated with the use of the constant e to any desired degree of precision in the exact same way as all conventional approximations of the transcendental constant e are used in the state-of-the-art analysis and computations.

7.4. Imaginary unit i

The epistemic and structural roles of the imaginary unit i for the external observer have been addressed in detail in Section 6. Here, we would like to point out that for our internal observer, the true value of i—that is always a relatively large number for large P—would remain beyond the scope of observation for as long as $H \ll i$. In order to fulfil the structural and utilitarian gap, our observer would likely need to invent it as an abstract mathematical instrument, as they actually did, and thus is the name "imaginary"! We would venture a step further and suggest—currently without a detailed explanation—that the approximate real number value of i to the internal observer is indeed known, but its identity is not associated with its imaginary equivalent in the state-of-the-art scholarly literature.

To summarize, we define the following three constants in the finite field $\mathbb{F}_P$:

$$\pi_P,\; e_P,\; i_P \in \mathbb{F}_P \;\Big|\; \pi_P = \frac{P-1}{2},\; e_P^{\pi_P} = -1,\; i_P^2 = -1.$$

Let us note that we get the identity of $e_P^{\pi_P}$ by construction. Furthermore, the square roots of -1 in $\mathbb{F}_P$—when they exist, *i.e.* $P = 1 \mod 4$—always come in pairs of odd and even. This important observation breaks the dual symmetry for the selection of i_P that is required for our reconciliation of the finite field solution and the experience of our internal observer that is described by the familiar Euler's formula

$$e^{i\pi} = -1.$$

Once again, we would like to make a far-reaching statement—currently without a detailed explanation—that the breaking of symmetry in the selection of i_P differentiates between the clockwise and the counter-clockwise rotations of $\mathbb{F}_P$ and the corresponding direction of the passage of time for the internal observer.

7.5. Infinity as the unknowable "far-far away"

Let us revisit the ontological concept of *infinity* as described in [1]. In the previous sections, we have established the finite field $\mathbb{F}_P$ as an abstract pseudo-sphere $\mathbb{F}_P(0,1)$ with a limited-horizon observer at its origin 0. We would like now to consider the geometric point on our pseudo-sphere that is the furthest away from the observer. This point is evidently the *South Pole*—the antipodal point on the prime meridian—of the pseudo-sphere, which we will denote as s_P for now. We would like to emphasize the following important properties of s_P.

1. s_P is a unique point on the pseudo-sphere that is the *farthest away* from the observer at 0.
2. s_P is *invisible* to the observer at 0, that is to say that is located beyond any conceivable definition of the observer's limited observability horizon.
3. Finally, s_P is algebraically *inaccessible* to the observer at 0, in the sense that $s_P \notin \mathbb{F}_P, \mathbb{Q}_P$, and cannot be reached by any finite number of arithmetical steps along the surface of the pseudo-sphere.

We would like to provide a formal proof of the less evident Property 3 as follows.

Theorem 7.1 (No South Pole in $\mathbb{F}_P$). *Let $P > 2$ be an odd prime. Then the only solution $s \in \mathbb{Z}_P$ to*

$$2s \equiv 0 \pmod P$$

is $s \equiv 0$. Equivalently, there is no nonzero pseudo-rational $q \in Q_P$ whose image in $\mathbb{Z}_P$ has additive order 2.

Proof. 1. Since P is prime, the additive group $(\mathbb{Z}_P, +)$ is cyclic of order P. An element $s \in \mathbb{Z}_P$ has order 2 precisely if

$$2\,s \equiv 0 \pmod P.$$

2. Because $\gcd(2, P) = 1$, multiplication by 2 is invertible in $\mathbb{Z}_P$. Hence, from $2s \equiv 0 \pmod P$ it follows immediately that $s \equiv 0 \pmod P$. There is no nontrivial order-2 element.
3. By definition, each pseudo-rational $q = \frac{a}{b} \in \mathbb{Q}_P$ is represented in the field by

$$k(q) \;=\; a\,b^{-1} \bmod P \;\in\; \mathbb{Z}_P,$$

so $\mathbb{Q}_P \subseteq \mathbb{Z}_P$ under the embedding k. If some $q \in \mathbb{Q}_P$ mapped to a nonzero order-2 element $s = k(q) \neq 0$, then $2s \equiv 0$ would force $s \equiv 0$, a contradiction.
Therefore, no "South Pole" antipodal point exists in $\mathbb{Q}_P$ or $\mathbb{Z}_P$, completing the proof. □

These properties of the geometrical point s_P are unmistakably consistent with the properties of the concept of infinity in its conventional sense. This gives us the justification to identify the relativistic antipodal point s_P with the concept of infinity in the context of $\mathbb{Z}_P$, and thus denote it as ∞.

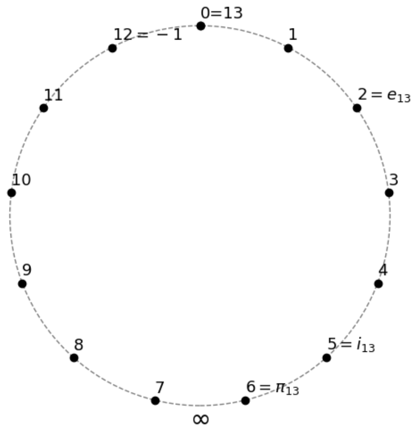


Figure 10: State space of a finite framed field $\mathbb{F}_{13}$, visualized as a circle on a 2D plane with the major structural elements e_{13}, i_{13}, π_{13}, as well as ∞ indicated.

To exemplify, let us now consider the concrete example of $P = 13$ and the corresponding finite framed field $\mathbb{F}_{13}$. We can identify the following values for the constants i, e and π in $\mathbb{F}_{13}$:

$$P = 13,\ i_{13} = 5,\ e_{13} = 2,\ \pi_{13} = 6.$$

Naturally and in line with our expectations, we get the following result for the Euler's formula

$$e_{13}^{i_{13}\pi_{13}} = 2^{5\times 6} = \left(2^6\right)^5 = 64^5 = (-1)^5 = -1,$$

where we assume all operations to be modulo 13 operations in $\mathbb{F}_{13}$ and in particular $2^6 = 64 \equiv -1 \mod 13$.

The corresponding visual representation of the finite field $\mathbb{F}_{13}$ is shown in Figure 10. The figure shows the state space of the finite field $\mathbb{F}_{13}$ as a circle on a 2D plane, with the major structural elements e_{13}, i_{13}, π_{13}, as well as ∞ indicated. The antipodal point ∞ is located at the South Pole of the pseudo-sphere, which is the farthest point from the observer at 0.

8. Unification in Finite Relativistic Algebra

8.1. Harmonic Analysis in Finite Fields

Harmonic analysis is a well-established and rigorously developed theory in both classical infinite domains—such as $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$—and finite fields such as $\mathbb{F}_P$. In the classical setting, it provides the foundational tools for understanding signal decomposition, spectral theory, and partial differential equations, with standard references including [27], and [32]. Independently, the discrete analogs over finite fields have been extensively studied, particularly in coding theory, cryptography, and number theory, as documented in works such as [20], and [33]. What remains lacking, however, is a unified framework that conceptually and algebraically reconciles these two domains—one grounded in continuous infinitary constructs, the other in finite cyclic structures—under a common epistemological and operational paradigm.

The motivation to bridge this divide arises naturally in the context of a finitely constructed universe, as articulated in this manuscript, where all observable structures are projections of finite rings and fields. By reinterpreting classical Fourier tools—such as complex exponentials and spectral kernels—as emergent constructs within cyclic multiplicative groups of finite fields, one can recover the essential machinery of harmonic analysis without invoking actual infinity. This unified treatment holds promise for a complete, frame-relative theory of symmetry, oscillation, and information in both physical and computational systems.

A detailed treatment of such a reconciliation lies beyond the scope of this manuscript. However, we would like to note the following important equivalence that explicitly bridges the gap between the corresponding finite field and the classical infinite domains. Specifically, the foundational component of harmonic analysis in the infinite setting is the complex exponential function, which forms a generator of a cyclic group of rotations in a complex plane:

$$g_N = e^{-i2\pi\frac{n}{N}},$$

where N is the period of rotation, and $n \leq N \in \mathbb{N}$ is an integer. This generator can be interpreted as a rotation of the complex plane by the minimum possible angle of $2\pi n/N$. If we furthermore want to make the cardinality N of the cyclic group arbitrary large —or presumably even infinite—we can assume a continuous version of the generator:

$$g_\infty = e^{-i2\pi dx} \mid \int_{-\infty}^{\infty} dx = 1,$$

where $2\pi dx$ is an infinitesimally small increment in the angle of rotation that satisfies the appropriate normalization condition. Once again this generator can be interpreted as a rotation of the complex plane by an infinitesimally small rotation. The exponential function $e^{-i2\pi x}$ describes a continuous clockwise rotation around the unit circle, and the infinitely-large-order rotational group $\mathbb{T} = \{e^{i2\pi x} : x \in \mathbb{R}\}$ is generated accordingly.

The purpose of *harmonic analysis* is to reveal inherent relational and structural properties of a sequence of elements within a target field $\mathbb{F}$—be it finite or infinite—by decomposing and projecting this sequence onto characters (homomorphisms) of cyclic groups associated with $\mathbb{F}$, commonly referred to as *harmonics*. In this context, the entire complex formalism that enables classical harmonic analysis can be naturally re-established within a finite field $\mathbb{F}_P$. Specifically, the multiplicative group $\mathbb{F}_P^\times$, of order $P-1$, possesses a cyclic rotational structure generated by a primitive root $e_P \in \mathbb{F}_P$.

Correspondingly, the continuous version of the Fourier analysis becomes a standard harmonic analysis over $\mathbb{F}_P$, where we establish that the quantized equivalent of the minimum possible angular rotation of the pseudo-complex plane $\mathbb{C}$ is

$$g_\infty = e^{-i2\pi dx} \approx e_P$$

and furthermore, a discrete Fourier analysis can be carried out for any finite rotational group of cardinality N, such that N is a divisor of $P-1$, and therefore we can establish the equivalence of the discrete generator

$$g_N = e^{-i2\pi/N} = e_P^m \;\Big|\; m = \frac{P-1}{N}.$$

8.2. Approximate Lie Group over Finite Field

Lie groups provide the mathematical framework for continuous symmetries in physics and geometry. Concretely, a Lie group is a set G that is simultaneously a smooth manifold and a group, so that the multiplication and inversion maps are infinitely differentiable in local coordinates [19]. However, any real-world experiment or numerical computation only ever probes finitely many points to finite precision. This observation motivates seeking finite structures whose "local" behavior is indistinguishable from that of a genuine Lie group up to an arbitrarily small error ε.

The pseudo-sphere $\mathbb{F}_P(0,1)$—a finite set of P labels arranged in a fractal-like pattern on a surface of a sphere—can be covered by a finite collection of small "tiles," each of which embeds into $\mathbb{R}^2$ as a patch of diameter $\delta \ll 1$. Each tile contains arbitrary large number of labels, and the finite-field addition and multiplication (mod p) induce a map

$$\varphi_k\big(\mu(p,q)\big) = \varphi_i(p) + \varphi_j(q) + O(\varepsilon),$$

where $\varphi_i, \varphi_j, \varphi_k$ are local coordinate charts [15]. In this way, the group law "looks like" the smooth addition of tangent vectors to within any prescribed ε, reflecting the idea of almost-flat or almost-Lie structures.

From the perspective of additive combinatorics, such finite sets with approximate closure and associativity properties are known as *approximate groups* [7]. In our setting, the finite pseudo-sphere satisfies associativity, identity, and inverses exactly in the modular arithmetic, and the coordinate-chart errors can be made arbitrarily small by choosing P and the observation horizon H appropriately. Hence, $\mathbb{F}_P(0,1)$ qualifies as an ε-Lie group: a discrete model that, for all practical (finite-precision) purposes, behaves identically to a true 2-sphere Lie group up to error ε.

Definition 3 (ε–Lie group at finite horizon)**.** Let G be a set, d a metric on G, and let

$$\mu : G \times G \;\to\; G, \quad \iota : G \;\to\; G$$

be binary "multiplication" and inversion maps. Fix integers

$$n \in \mathbb{N}, \quad \varepsilon > 0, \quad \delta > 0.$$

We say that $\big(G, d, \mu, \iota\big)$ is an (n, ε, δ)–Lie group if there exists a finite atlas of charts

$$\{(U_i, \varphi_i)\}_{i=1}^N, \quad \varphi_i : U_i \xrightarrow{\simeq} V_i \subset \mathbb{R}^n,$$

covering G, such that:

1. **Approximate smoothness of multiplication.** For every $p \in U_i$, $q \in U_j$, there is some chart (U_k, φ_k) with $\mu(p,q) \in U_k$ and
$$\left\|\varphi_k\big(\mu(p,q)\big) \;-\; \big[\varphi_i(p) + \varphi_j(q)\big]\right\| \;\le\; \varepsilon.$$
2. **Approximate smoothness of inversion.** For every $p \in U_i$, there exists a chart (U_ℓ, φ_ℓ) such that
$$\left\|\varphi_\ell\big(\iota(p)\big) \;+\; \varphi_i(p)\right\| \;\le\; \varepsilon.$$
3. **Group axioms up to ε.** Denote $e \in G$ the identity element. Then
$$\sup_{a,b,c \in G} d\big(\mu(\mu(a,b),c),\; \mu(a,\mu(b,c))\big) \;\le\; \varepsilon,$$
and
$$\sup_{a \in G} \big[d(\mu(e,a),a) + d(\mu(a,e),a) + d(\mu(a,\iota(a)),e)\big] \;\le\; \varepsilon.$$

Proposition 5 (Pseudo-sphere as an ε–Lie group)**.** Fix a prime P and a finite observation horizon $H \in \mathbb{N}$. Define
$$G \;=\; \mathbb{Q}_P^{\le H} = \big\{x\,(p-1)^{-n} \;:\; 0 \le n \le H,\; 0 \le x < P\big\},$$
endowed with the metric d_H induced by embedding local charts into $\mathbb{R}^2$. Let μ and ι be the truncated group operations mod P. Then:

- The set G admits a finite atlas of coordinate patches of diameter $\le \delta_H$, with $\delta_H \to 0$ as $H \to \infty$.
- The maps μ and ι satisfy the three conditions of Definition 1 with
$$n = 2, \quad \varepsilon_H = (p-1)^{-H}, \quad \delta_H = O(\varepsilon_H).$$

Hence, $\big(G, d_H, \mu, \iota\big)$ is a $\big(2, \varepsilon_H, \delta_H\big)$–Lie group. In particular, for any finite observer resolution H, the structure behaves like a genuine 2-dimensional Lie group up to error $\varepsilon_H \to 0$.

8.3. Finite Langlands Program

In the usual Langlands philosophy one relates two vast worlds: on the one hand the (infinite) Galois representations of a global field, and on the other the automorphic representations of a reductive group over that field. If one accepts that *only* finite rings $\mathbb{Z}_Q$ can exist, then every "infinite" Galois group must be replaced by its finite quotient
$$\mathrm{Gal}(\overline{F}/F) \;\longrightarrow\; \mathrm{Gal}(\overline{F}/F)\big/N \;\cong\; \mathrm{Gal}(F_N/F) \;\subset\; \mathrm{Perm}(F_N),$$
and every automorphic representation must likewise factor through a finite group of points
$$G\big(\mathbb{A}_F\big) \;\longrightarrow\; G\big(\mathbb{A}_F\big)/K_N \;\cong\; G\big(\mathbb{Z}_Q\big)$$
for some level K_N. In this "finite-Langlands" perspective all objects—Galois data and automorphic forms—are *built* from the same finite base ring $\mathbb{Z}_Q$, and the conjectural correspondence becomes a

bijection between

$$\{\text{finite-quotient Galois representations into } GL_n(\mathbb{Z}_Q)\} \longleftrightarrow \{\text{irreducible representations of } G(\mathbb{Z}_Q)\}.$$

From the function-field side one already has a prototype: Drinfeld and Lafforgue proved a global Langlands correspondence for GL_n over $\mathbb{F}_Q(T)$, where $\mathbb{F}_Q$ is a finite field, and automorphic forms live on $GL_n(\mathbb{F}_Q[T])$ [11, 17]. There, both Galois representations and automorphic sheaves are *intrinsically* finite objects—perverse sheaves on moduli stacks over $\mathbb{F}_Q$ and ℓ-adic representations of π_1. This suggests that a genuinely finite-universe version of the Langlands program would reorganise every classical component (Hecke operators, L-functions, trace formulas) into purely combinatorial operations on $\mathbb{Z}_Q$-modules and finite group characters.

In summary, if one accepts that $\mathbb{Z}_Q$ is the only ontologically primitive object, then the Langlands correspondence reduces to an equivalence of categories between $\mathbb{Z}_Q$-linear Galois modules and $\mathbb{Z}_Q$-linear automorphic modules. All "infinite" phenomena (analytic continuation, spectral decompositions) become emergent from the finiteness of $\mathbb{Z}_Q$ through limiting processes within finite-dimensional $\mathbb{Z}_Q$-vector spaces. Such a viewpoint collapses the traditional dichotomy and recasts Langlands duality as a statement about different *frames of reference* on a single finite ring.

9. Conclusions

The primary objective of this work has been to devise an algebraic framework that (1) does not contradict our conventional arithmetic and geometric intuitions, (2) enables all practical applications of modern mathematics, and (3) completely disposes of the ontological need for actual infinity. We have shown that by interpreting addition, multiplication and exponentiation as internal symmetries of a finite framed field $\mathbb{F}_P(0, 1)$, one can reconstruct signed integers, pseudo-rationals, pseudo-reals and pseudo-complex numbers in a way that matches classical behavior up to any desired precision, without ever invoking an infinite set. This construction preserves the familiar algebraic laws and analytic operations that underpin standard number systems, ensuring full compatibility with intuition and established mathematical practice.

Moreover, our finite relational algebra supports the full spectrum of modern mathematical techniques—solving polynomial equations, performing limit-like approximations via dense pseudo-rationals, and modeling continuous symmetries through ε-Lie-group approximations—while entirely replacing classical infinities with context-dependent finite representations. In doing so, it provides exact algebraic analogs for roots, exponentials and trigonometric relationships, and offers a discrete yet arbitrarily precise scaffold for differential-geometric and analytic constructions. By eliminating any ontological reliance on actual infinity, this framework retains the power and flexibility of conventional mathematics in a fully finitary setting, while also offering an avenue towards the resolution of classical paradoxes of logic and set theory imposed by the *infinitude conjecture*. The resulting structure is not merely a mathematical curiosity; it is a coherent and physically grounded alternative to standard formalism, suitable for the description of discrete, informationally finite physical systems.

Looking forward, extending our framework to composite moduli, and exploring the implications for the analysis of dynamic physical systems, will further strengthen and broaden its applicability. We anticipate that this relational, finite approach will serve as both a conceptually coherent foundation and a practical computational paradigm across mathematics, physics and computer science.

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