Transformation Properties of Tensors and Pseudotensors under Coordinate Reflections: A Detailed Mathematical Analysis

Chenjia Li and Jingxu Wu

Faculty of Physics, Lomonosov Moscow State University, Moscow, Russia

Gezhi International Reading Group on Theoretical Physics

May 26, 2025

Abstract

The theory of tensors and pseudotensors underlies the mathematical framework of modern physics, providing a coordinate-invariant language for describing physical laws and symmetries. In this work, we systematically analyze the transformation properties of tensors and pseudotensors of various ranks, with particular emphasis on their behavior under spatial and spacetime reflections. The construction and interpretation of the Levi-Civita symbol in two, three, and four dimensions are discussed in detail, elucidating the essential distinction between true tensors and pseudotensors in terms of orientation and parity. Explicit transformation rules for ordinary tensors and pseudotensors are derived, including the role of the Jacobian determinant and its sign. Through concrete examples—including scalar triple products, cross products, and antisymmetric tensor decompositions—we reveal the fundamental algebraic and geometric features of these mathematical objects. The implications for vector calculus, relativistic field theory, and physical invariants such as chirality and duality are highlighted. Our results provide a unified and rigorous treatment of the reflection and contraction properties of pseudotensors, with direct relevance to applications in classical mechanics, electromagnetism, and modern theoretical physics.

Keywords: tensors, pseudotensors, Levi-Civita symbol, coordinate transformation, parity, reflection, antisymmetric tensor, orientation, contraction, relativistic field theory, vector calculus, duality

1 Introduction

The formalism of tensors and pseudotensors lies at the heart of modern theoretical and mathematical physics, providing the indispensable scaffolding upon which our most profound physical theories are constructed [1, 2, 3]. From the formulation of Einstein's general theory of relativity, where the invariance of the physical laws under arbitrary diffeomorphisms is ensured by the tensorial structure of the field equations [4, 5], to the foundational role of antisymmetric tensors and their duals in gauge theories, topological phases, and quantum field theory [6, 16, 7], a deep understanding of tensor transformation properties is not merely a matter of mathematical rigor, but is central to the correct and physically meaningful construction of theory.

While the transformation rules for ordinary (true) tensors under coordinate changes are now standard curriculum in advanced physics and geometry [8, 9], the subtleties introduced by pseudotensors—particularly in the presence of improper transformations such as parity inversion or time-reversal—remain both technically and conceptually subtle [10, 11, 12]. Pseudotensors (or tensor densities), by acquiring a sign factor under coordinate transformations with negative Jacobian determinant, are intimately connected to orientation, chirality, and fundamental symmetry principles in physics [13, 14, 15]. Classic examples include the Levi-Civita symbol, which encodes the oriented structure necessary for volume integration and cross products, and whose duality structure enables the definition of Hodge star operations and topological invariants [16, 3]. The distinction between polar and axial vectors, and between ordinary and pseudo-objects, is not only of abstract mathematical interest, but governs the transformation of observables such as angular momentum, magnetic fields, and parity-violating processes [12, 17, 18].

Recent advances in both mathematical physics and high-precision experiment have underscored the critical role of these structures. For example, the classification of topological phases of matter and the study of anomalies in quantum field theory rely on a precise handling of tensor densities and pseudotensorial invariants [19, 20, 16, 14, 21]. Similarly, the formulation of chiral theories, the behavior of spinors under improper Lorentz transformations, and the structure of CP and T violation in the Standard Model are predicated upon the correct usage and interpretation of pseudotensors [22, 23].

Despite their ubiquity, there remains a surprising amount of confusion in the literature and even among advanced practitioners regarding the explicit transformation properties, index structure, and physical interpretation of pseudotensors. This paper addresses these issues through a systematic, axiomatic, and example-driven exposition of tensors and pseudotensors in arbitrary dimensions, with special emphasis on their behavior under general (including improper) coordinate transformations. Explicit formulae are derived for low- and high-rank pseudotensors, including the Jacobian-based transformation law, and illustrative computations are provided for canonical objects such as the scalar triple product, axial vectors, and the full Levi-Civita tensor in both three and four dimensions. The consequences for the algebra of physical observables, parity operations, and the construction of Lagrangians are rigorously analyzed, bridging the gap between abstract geometric formalism and concrete physical application.

2 Definition and Transformation Rules

The transformation properties of tensors and pseudotensors underpin much of modern mathematical physics, providing a coordinate-independent language for physical laws. In this section, we formalize the transformation rules for general tensors and pseudotensors under arbitrary smooth changes of coordinates, and clarify the structure and meaning of the Jacobian matrix and its determinant in these transformations.

2.1 Ordinary Tensor Transformation

Let $T^{i_1 \dots i_p}_{k_1 \dots k_q}$ denote a general tensor of type (p, q), where p and q are the numbers of contravariant and covariant indices, respectively. Under a smooth and invertible coordinate transformation $x \mapsto x'$, the components of T transform according to the well-known tensor transformation law:

$$T'^{i_1\dots i_p}_{k_1\dots k_q} = J^{i_1}_{l_1} \cdots J^{i_p}_{l_p} \left(J^{-1}\right)^{m_1}_{k_1} \cdots \left(J^{-1}\right)^{m_q}_{k_q} T^{l_1\dots l_p}_{m_1\dots m_q}, \tag{1}$$

where the Jacobian matrix of the transformation is defined by $J^i_l = \partial x'^i / \partial x^l$, and J^{-1} is its matrix inverse.

For clarity, the same transformation rule can be explicitly written in terms of partial derivatives:

$$T'^{i_1\dots i_p}_{k_1\dots k_q} = \left(\frac{\partial x'^{i_1}}{\partial x^{l_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{l_p}}\right) \left(\frac{\partial x^{m_1}}{\partial x'^{k_1}} \cdots \frac{\partial x^{m_q}}{\partial x'^{k_q}}\right) T^{l_1\dots l_p}_{m_1\dots m_q}. \tag{2}$$

Or, more compactly, using product notation:

$$T'^{i_1\dots i_p}_{k_1\dots k_q} = \prod_{\alpha=1}^{p} \frac{\partial x'^{i_\alpha}}{\partial x^{l_\alpha}} \prod_{\beta=1}^{q} \frac{\partial x^{m_\beta}}{\partial x'^{k_\beta}} T^{l_1\dots l_p}_{m_1\dots m_q}. \tag{3}$$

These transformation laws guarantee that the geometric or physical content represented by T is independent of the chosen coordinate system, ensuring the covariance of the associated physical equations.

2.2 Pseudotensor Transformation

Pseudotensors, or tensor densities of weight one, play a key role in contexts where orientation (parity) matters, such as in the description of cross products, determinants, and certain physical invariants (e.g., the Levi-Civita symbol). The transformation rule for a pseudotensor of the same type (p, q) is similar to Eq. (1), but acquires an additional sign factor that reflects the behavior under improper (orientation-reversing) coordinate transformations:

$$P'^{i_1\dots i_p}_{k_1\dots k_q} = \operatorname{sign}(\det J)\, J^{i_1}_{l_1} \cdots J^{i_p}_{l_p} \left(J^{-1}\right)^{m_1}_{k_1} \cdots \left(J^{-1}\right)^{m_q}_{k_q} P^{l_1\dots l_p}_{m_1\dots m_q}. \tag{4}$$

Here, $\operatorname{sign}(\det J)$ distinguishes between proper ($\det J > 0$) and improper ($\det J < 0$) transformations. This sign factor is responsible for the change of sign of pseudotensors under spatial reflections or parity transformations, making them suitable for describing orientation-dependent phenomena.

2.3 Jacobian Matrix and Its Determinant

The Jacobian matrix J of the coordinate transformation $x^i \mapsto x'^i(x^1, \dots, x^n)$ is defined as the matrix of partial derivatives:

$$J = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \cdots & \frac{\partial x'^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x'^m}{\partial x^1} & \frac{\partial x'^m}{\partial x^2} & \cdots & \frac{\partial x'^m}{\partial x^n} \end{pmatrix}, \tag{5}$$

and its determinant is

$$\det J = \det\left(\frac{\partial x'^i}{\partial x^k}\right). \tag{6}$$

The sign and magnitude of $\det J$ characterize how the infinitesimal volume elements and orientation transform under the change of coordinates. In particular, a negative determinant signals a reversal of orientation (as occurs in spatial reflections), which is why the sign appears explicitly in the transformation of pseudotensors.

In summary, Eqs. (1)–(6) provide a complete framework for understanding how tensorial and pseudotensorial quantities behave under arbitrary coordinate changes. This machinery is fundamental in differential geometry, general relativity, and modern field theory, forming the backbone of covariant physical law.

3 Motivation for Pseudoscalar (0th-Rank Pseudotensor)

Pseudoscalars, or 0th-rank pseudotensors, play a fundamental role in both mathematics and physics, particularly in the context of orientation, parity, and volume invariants. The canonical example is the scalar triple product, which naturally exhibits the key property of changing sign under improper (orientation-reversing) coordinate transformations. This property distinguishes pseudoscalars from ordinary scalars and is essential in topics ranging from vector calculus to quantum mechanics and particle physics.

A general pseudoscalar can be constructed from three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as the scalar triple product:

$$S = (\boldsymbol{a} \cdot [\boldsymbol{b} \times \boldsymbol{c}]) = \det \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}. \tag{7}$$

This expression is geometrically interpreted as the signed volume of the parallelepiped spanned by the three vectors. Its transformation property under reflection reveals its pseudoscalar nature.

3.1 Reflection Transformation along the x-Axis

Consider the effect of reflecting only the x-axis, that is, the transformation $x \to -x$, $y \to y$, $z \to z$. The components of the vectors transform as:

$$\boldsymbol{a}' = (-a_x,\, a_y,\, a_z), \quad \boldsymbol{b}' = (-b_x,\, b_y,\, b_z), \quad \boldsymbol{c}' = (-c_x,\, c_y,\, c_z). \tag{8}$$

After applying this reflection, the scalar triple product becomes:

$$\begin{aligned} S' = (\boldsymbol{a}' \cdot [\boldsymbol{b}' \times \boldsymbol{c}']) &= \det \begin{pmatrix} -a_x & -b_x & -c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \\ &= (-1) \det \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} = -S. \end{aligned} \tag{9}$$

The negative sign arises from extracting a factor of -1 from the first row of the determinant, directly demonstrating that S is a pseudoscalar: it changes sign under reflection of a single axis.

3.2 Reflection along All Three Axes

Consider now the full inversion of all spatial coordinates: $x \to -x$, $y \to -y$, $z \to -z$. In this case, the Jacobian matrix is $J = -I$ and its determinant is $\det J = (-1)^3 = -1$. The vectors transform as:

$$\boldsymbol{a}' = (-a_x,\, -a_y,\, -a_z), \quad \boldsymbol{b}' = (-b_x,\, -b_y,\, -b_z), \quad \boldsymbol{c}' = (-c_x,\, -c_y,\, -c_z). \tag{10}$$

Accordingly, the triple product transforms as:

$$\begin{aligned} S' = (\boldsymbol{a}' \cdot [\boldsymbol{b}' \times \boldsymbol{c}']) &= \det \begin{pmatrix} -a_x & -b_x & -c_x \\ -a_y & -b_y & -c_y \\ -a_z & -b_z & -c_z \end{pmatrix} \\ &= (-1)^3 \det \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} = -S. \end{aligned} \tag{11}$$

Again, the result is the negative of the original triple product, consistent with the transformation law for a pseudoscalar under inversion.

These transformation properties reflect the fundamental algebraic and geometric meaning of pseudoscalars: they are quantities invariant under proper rotations but change sign under improper transformations (such as reflection or inversion), thereby encoding information about orientation in three-dimensional space.

4 Properties of Polar and Axial Vectors (Pseudovectors) under Reflection

Vectors in three-dimensional Euclidean space are classified as either polar vectors (true vectors) or axial vectors (pseudovectors), depending on their transformation properties under orientation-reversing operations such as spatial reflections or inversion. This distinction is fundamental in physics, with important implications for the formulation of conservation laws, parity violation, and tensor calculus in general.

4.1 Definition and Reflection of Polar Vectors

A polar vector, such as the position vector $\mathbf{r}$ or velocity $\mathbf{v}$, transforms under inversion (reflection of all spatial axes) by a simple sign change of its components:

$$\mathbf{r} = (r_x,\, r_y,\, r_z) \xrightarrow{\text{reflection}} \mathbf{r}' = (-r_x,\, -r_y,\, -r_z). \tag{12}$$

This transformation is governed by the Jacobian matrix $J = -I$ with $\det J = -1$. Expressing this via the chain rule,

$$\frac{\partial x'^i}{\partial x^k} = -\delta_{ik} \implies r'^i = J^i_k r^k = -r^i. \tag{13}$$

Thus, polar vectors reverse sign under an odd number of spatial reflections, consistent with their geometric definition.

4.2 Axial Vectors and Their Reflection Properties

Axial vectors (pseudovectors) arise from the cross product of two polar vectors, a construction that is orientation-sensitive. For example, the angular momentum vector is given by

$$L_i = \varepsilon_{ijk}\, r_j\, p_k, \tag{14}$$

where ε_{ijk} is the Levi-Civita symbol. Expanded in components,

$$\begin{aligned} L_x &= r_y p_z - r_z p_y, \\ L_y &= r_z p_x - r_x p_z, \\ L_z &= r_x p_y - r_y p_x. \end{aligned}$$

Under the full reflection of all axes,

$$\mathbf{r}' = (-r_x,\ -r_y,\ -r_z), \qquad \mathbf{p}' = (-p_x,\ -p_y,\ -p_z), \tag{15}$$

the components of the axial vector transform as:

$$\begin{aligned} L'_x &= (-r_y)(-p_z) - (-r_z)(-p_y) = r_y p_z - r_z p_y = +L_x, \\ L'_y &= (-r_z)(-p_x) - (-r_x)(-p_z) = r_z p_x - r_x p_z = +L_y, \\ L'_z &= (-r_x)(-p_y) - (-r_y)(-p_x) = r_x p_y - r_y p_x = +L_z. \end{aligned}$$

That is, the components of an axial vector **remain unchanged** under full spatial inversion. This is a defining property of pseudovectors: they are invariant under improper transformations that reverse orientation, in contrast to polar vectors.

4.3 Basis Transformation and Determinant Formalism

It is important to emphasize that under reflection, not only the vector components but also the basis vectors transform. If the original basis is $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, after reflection we have

$$\mathbf{e}'_i = -\mathbf{e}_i. \tag{16}$$

A vector expressed in basis form,

$$\mathbf{L} = L_x \mathbf{e}_x + L_y \mathbf{e}_y + L_z \mathbf{e}_z, \tag{17}$$

thus transforms as

$$\mathbf{L}' = L'_x \mathbf{e}'_x + L'_y \mathbf{e}'_y + L'_z \mathbf{e}'_z, \tag{18}$$

and explicit calculations using the determinant form for cross products must account for the basis transformation:

$$\det \begin{pmatrix} \mathbf{e}'_x & \mathbf{e}'_y & \mathbf{e}'_z \\ a'_x & a'_y & a'_z \\ b'_x & b'_y & b'_z \end{pmatrix}. \tag{19}$$

4.4 Pseudotensor Transformation Law for Axial Vectors

The general pseudotensor transformation law confirms these observations. For an axial vector (pseudovector) L_i,

$$L'_i = \text{sign}(\det J) J^i_j L^j, \tag{20}$$

so, for full reflection, $J^i_j = -\delta^i_j$, $\det J = -1$, and

$$L'_i = (-1)(-L_i) = +L_i. \tag{21}$$

This result formally demonstrates that pseudovectors are invariant under total inversion, distinguishing them from true (polar) vectors, which acquire an overall sign change.

In summary, the distinction between polar and axial vectors is intimately connected to the behavior under spatial reflection, underpinning many key physical effects related to parity and orientation in both classical and quantum systems.

5 Second-Rank Pseudotensors and Their Reflection Properties

Second-rank tensors and pseudotensors serve as foundational objects in both classical and modern physics. Their behavior under spatial reflections and coordinate transformations provides insight into the distinction between ordinary and pseudo-objects, especially in relation to orientation and parity.

5.1 Transformation of Ordinary Second-Rank Tensors

Let T_{ij} be a general second-rank tensor in two dimensions ($i, j = 1, 2$). Under a linear coordinate reflection, its components transform according to

$$T'_{ik} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} T_{lm} = J^l_i J^m_j \, T_{lm}, \tag{22}$$

where $J^l_i = \partial x^l / \partial x'^i$ is the Jacobian matrix of the transformation. For a reflection $x \to x' = -x$, $y \to y' = -y$, we have

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{23}$$

5.2 The Levi-Civita Symbol as a Second-Rank Pseudotensor

A paradigmatic example of a second-rank pseudotensor is the Levi-Civita symbol in two dimensions, ε_{ij}, which encodes the oriented area element. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the xy-plane, their cross product can be expressed as

$$[\boldsymbol{a} \times \boldsymbol{b}] = \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix} \mathbf{e}_z = \varepsilon_{ij} \, a_i b_j \, \mathbf{e}_z, \tag{24}$$

where

$$\varepsilon_{ij} = \begin{cases} +1, & \text{if } (i,j) = (1,2), \\ -1, & \text{if } (i,j) = (2,1), \\ 0, & \text{if } i = j. \end{cases} \tag{25}$$

Here, ε_{ij} is antisymmetric and reflects the orientation of the coordinate system.

5.3 Explicit Reflection Calculations

Consider first a reflection of the x-axis only: $x \to -x$, $y \to y$. The components of $\boldsymbol{a}$ and $\boldsymbol{b}$ transform as $a = (a_x, a_y) \mapsto a' = (-a_x, a_y)$, $b = (b_x, b_y) \mapsto b' = (-b_x, b_y)$. The cross product becomes

$$[\boldsymbol{a}' \times \boldsymbol{b}'] = \begin{vmatrix} -a_x & -b_x \\ a_y & b_y \end{vmatrix} \mathbf{e}_z = (-a_x)b_y - a_y(-b_x) = \varepsilon'_{ij} a'_i b'_j \, \mathbf{e}_z. \tag{26}$$

Direct calculation shows that the antisymmetric symbol remains unchanged: $\varepsilon_{ij} \to \varepsilon'_{ij} = \varepsilon_{ij}$.

5.4 General Pseudotensor Reflection Law

For a general pseudotensor P_{ij}, the transformation law is

$$P'_{ij} = \operatorname{sign}(\det J)\, J_i^k J_j^l \, P_{kl}. \tag{27}$$

For reflection of a single axis, say x, the Jacobian is

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \det J = -1. \tag{28}$$

For the Levi-Civita symbol, explicit component calculations yield

$$\varepsilon'_{xx} = \varepsilon'_{yy} = 0, \qquad \varepsilon'_{xy} = (-1)(-1)(1)\varepsilon_{xy} = \varepsilon_{xy}, \qquad \varepsilon'_{yx} = (-1)(1)(-1)\varepsilon_{yx} = \varepsilon_{yx}.$$

Similarly, for a two-axis (full) reflection,

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \det J = 1, \tag{29}$$

and

$$\varepsilon'_{xy} = (1)(-1)(-1)\varepsilon_{xy} = \varepsilon_{xy}. \tag{30}$$

Thus, a second-rank pseudotensor such as the Levi-Civita symbol does **not** change sign under reflection of an even number of axes, a property directly tied to its parity and orientation structure.

This analysis illustrates the general features of pseudotensors: their sign behavior under improper transformations encodes essential geometric and physical information, which is crucial for understanding phenomena such as handedness, chirality, and parity violation in mathematical physics.

6 Third-Rank Pseudotensors and Reflection Properties

Third-rank tensors and, in particular, the Levi-Civita symbol, are essential in expressing cross products, determinants, and various topological invariants in three-dimensional physics. Their transformation properties under spatial reflection exemplify the difference between ordinary and pseudo-objects.

6.1 Transformation of Ordinary Third-Rank Tensors

Let T_{ijk} be an ordinary third-rank tensor in three dimensions. Under a full reflection of all spatial axes,

$$x \to x' = -x, \qquad y \to y' = -y, \qquad z \to z' = -z, \tag{31}$$

the Jacobian matrix is $J = -I$, so $\det J = -1$. The tensor components transform as

$$T'_{ijk} = \frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}T_{abc} = (-1)^3 T_{ijk} = -T_{ijk}, \tag{32}$$

demonstrating that a third-rank tensor changes sign under full inversion.

6.2 Levi-Civita Symbol: Definition and Structure

The Levi-Civita symbol ε_{ijk} in three dimensions is defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation,} \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

and is totally antisymmetric in its indices. This symbol is the canonical example of a third-rank pseudotensor.

6.3 Cross Product and Determinant Representation

The cross product in three dimensions is compactly written as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \varepsilon_{ijk}\, a_j b_k\, \mathbf{e}_i = \det \begin{pmatrix} \mathbf{e}_x & a_x & b_x \\ \mathbf{e}_y & a_y & b_y \\ \mathbf{e}_z & a_z & b_z \end{pmatrix}, \tag{34}$$

which, in components, gives

$$\begin{aligned} c_x &= a_y b_z - a_z b_y, \\ c_y &= a_z b_x - a_x b_z, \\ c_z &= a_x b_y - a_y b_x. \end{aligned}$$

6.4 Reflection Properties of the Levi-Civita Symbol

Consider the effect of a single-axis reflection, such as $x \to -x$. The transformed vectors are $a' = (-a_x, a_y, a_z)$, $b' = (-b_x, b_y, b_z)$. The cross product in this new system is

$$\mathbf{c}' = \det \begin{pmatrix} -\mathbf{e}_x & -a_x & -b_x \\ \mathbf{e}_y & a_y & b_y \\ \mathbf{e}_z & a_z & b_z \end{pmatrix}, \tag{35}$$

which yields the same form as before, indicating that

$$\varepsilon_{ijk} \to \varepsilon'_{ijk} = \varepsilon_{ijk}, \tag{36}$$

so the Levi-Civita symbol itself does not change sign under a single-axis reflection.

For a full inversion of all axes,

$$a' = (-a_x, -a_y, -a_z), \qquad b' = (-b_x, -b_y, -b_z), \tag{37}$$

the cross product becomes

$$\mathbf{c}' = \det \begin{pmatrix} -\mathbf{e}_x & -a_x & -b_x \\ -\mathbf{e}_y & -a_y & -b_y \\ -\mathbf{e}_z & -a_z & -b_z \end{pmatrix} = \det \begin{pmatrix} \mathbf{e}'_x & a'_x & b'_x \\ \mathbf{e}'_y & a'_y & b'_y \\ \mathbf{e}'_z & a'_z & b'_z \end{pmatrix}, \tag{38}$$

again showing that ε_{ijk} is unchanged under total inversion.

6.5 Pseudotensor Transformation Law for the Levi-Civita Symbol

By the general pseudotensor transformation law,

$$\varepsilon'_{ijk} = \mathrm{sign}(\det J)\, J_i^a J_j^b J_k^c\, \varepsilon_{abc}, \tag{39}$$

and for the full reflection $J_i^a = -\delta_i^a$, $\det J = -1$, so

$$\varepsilon'_{ijk} = (-1) \cdot (-1)^3 \varepsilon_{ijk} = +\varepsilon_{ijk}. \tag{40}$$

Thus, the Levi-Civita symbol remains invariant under reflection of all axes, confirming its character as a true pseudotensor.

This invariance under orientation reversal is central to the role of the Levi-Civita symbol in encoding geometric and topological information in three-dimensional physics, especially in the formulation of vector calculus identities and conservation laws.

7 Fourth-Rank Pseudotensors and Reflection Properties

Pseudotensors of rank four are central in four-dimensional spacetime physics, particularly in the formulation of relativistic field theories. The Levi-Civita symbol ε_{iklm} encodes the orientation of four-dimensional volume and is crucial for defining duals and invariants under Lorentz and parity transformations.

7.1 Definition of the Four-Dimensional Levi-Civita Symbol

The fully antisymmetric Levi-Civita symbol in four dimensions is defined as

$$\varepsilon_{iklm} = \begin{cases} +1 & \text{if } (i,k,l,m) \text{ is an even permutation of } (0,1,2,3), \\ -1 & \text{if } (i,k,l,m) \text{ is an odd permutation,} \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

with the convention $\varepsilon_{0123} = +1$ in a right-handed coordinate system.

7.2 Transformation Law for Fourth-Rank Tensors and Pseudotensors

For a general fourth-rank tensor $T_{\mu\nu\rho\sigma}$, under a coordinate transformation,

$$T'_{\mu\nu\rho\sigma} = J^\alpha_\mu J^\beta_\nu J^\gamma_\rho J^\delta_\sigma T_{\alpha\beta\gamma\delta}, \tag{42}$$

where $J^\alpha_\mu = \partial x^\alpha / \partial x'^\mu$ is the Jacobian. For a pseudotensor of the same rank, an additional factor of sign(det J) is included, reflecting orientation sensitivity.

7.3 Reflection Types and Jacobian Analysis

In four dimensions, two main types of reflections are considered:

(1) Pure spatial reflection (all three spatial axes reflected, time unchanged):

$$J_{\text{space}} = \text{diag}(1, -1, -1, -1), \qquad \det J_{\text{space}} = -1. \tag{43}$$

(2) Time reflection (time axis reflected, spatial coordinates unchanged):

$$J_{\text{time}} = \text{diag}(-1, 1, 1, 1), \qquad \det J_{\text{time}} = -1. \tag{44}$$

7.4 Transformation Law for the Four-Dimensional Levi-Civita Symbol

For the Levi-Civita symbol, the transformation law under a general coordinate change is

$$\varepsilon'_{iklm} = \text{sign}(\det J) J^a_i J^b_k J^c_l J^d_m \varepsilon_{abcd}. \tag{45}$$

Focusing on the totally antisymmetric component, for both pure spatial and time reflections:

$$\varepsilon'_{0123} = \text{sign}(\det J) J^0_0 J^1_1 J^2_2 J^3_3 \varepsilon_{0123} \tag{46}$$

For spatial reflection,

$$\varepsilon'_{0123} = (-1) \cdot (1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot (+1) = (+1)\varepsilon_{0123} \tag{47}$$

For time reflection,

$$\varepsilon'_{0123} = (-1) \cdot (-1) \cdot (1) \cdot (1) \cdot (1) \cdot (+1) = (+1)\varepsilon_{0123} \tag{48}$$

In both cases, the sign remains unchanged. This invariance is critical for Lorentz-invariant constructions in four-dimensional spacetime physics.

8 Contraction Operations and the Nature of Pseudotensors

The interplay of tensors and pseudotensors under index contraction determines the transformation properties of resulting quantities. The following general rules are fundamental:

8.1 General Contraction Theorem

Theorem:

(i) Contracting two ordinary tensors yields an ordinary tensor.

(ii) Contracting an ordinary tensor with a pseudotensor yields a pseudotensor.

(iii) Contracting two pseudotensors may yield either an ordinary tensor or a pseudotensor, depending on the contraction details.

Proof (for case (ii)): Let P be a pseudotensor and T an ordinary tensor. Consider contraction over one upper and one lower index:

$$C^i_k = P^i_j T^j_k. \tag{49}$$

Under a general coordinate transformation,

$$C'^i_k = P'^i_j T'^j_k. \tag{50}$$

Applying the transformation laws,

$$P'^i_j = \mathrm{sign}(\det J) J^i_a (J^{-1})^b_j P^a_b, \qquad T'^j_k = J^j_c (J^{-1})^d_k T^c_d, \tag{51}$$

so

$$C'^i_k = \mathrm{sign}(\det J) J^i_a (J^{-1})^b_j J^j_c (J^{-1})^d_k P^a_b T^c_d. \tag{52}$$

Since $(J^{-1})^b_j J^j_c = \delta^b_c$, this simplifies to

$$C'^i_k = \mathrm{sign}(\det J) J^i_a (J^{-1})^d_k P^a_b T^b_d, \tag{53}$$

which is precisely the transformation law for a pseudotensor.

8.2 Summary Table

- Contraction of two ordinary tensors $\longrightarrow$ ordinary tensor.
- Contraction of ordinary tensor and pseudotensor $\longrightarrow$ pseudotensor.
- Contraction of two pseudotensors $\longrightarrow$ ordinary tensor or pseudotensor (case-dependent).

9 Construction of Four-Dimensional Second-Rank Antisymmetric Tensors

Second-rank antisymmetric tensors in four dimensions are at the heart of electromagnetism, gauge theory, and relativistic field theory. They encode both polar and axial vector structures within the same mathematical object.

9.1 From Three-Dimensional Axial Vectors to Four-Dimensional Antisymmetric Tensors

Recall that a three-dimensional axial vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ can be written as

$$C_\alpha = \varepsilon_{\alpha\beta\gamma} A_\beta B_\gamma = \frac{1}{2}\varepsilon_{\alpha\beta\gamma} C_{\beta\gamma}, \tag{54}$$

where $C_{\beta\gamma} = A_\beta B_\gamma - A_\gamma B_\beta$ is the antisymmetric tensor associated with $\mathbf{A}$ and $\mathbf{B}$.

9.2 Antisymmetric Second-Rank Tensors in Four Dimensions

A four-dimensional antisymmetric tensor A_{ik} satisfies

$$A_{ik} = -A_{ki}. \tag{55}$$

All diagonal elements vanish ($A_{ii} = 0$). In matrix notation,

$$(A_{ik}) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \tag{56}$$

The purely spatial part ($i, k = 1, 2, 3$) corresponds to an axial vector:

$$(A_{ik}) = \begin{pmatrix} 0 & -2C_3 & 2C_2 \\ 2C_3 & 0 & -2C_1 \\ -2C_2 & 2C_1 & 0 \end{pmatrix} \tag{57}$$

where $C_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma} C_{\beta\gamma}$.

9.3 Decomposition into Polar and Axial Vectors

The components $A_{0\alpha}$ ($\alpha = 1, 2, 3$) form a three-dimensional polar vector, while $A_{\alpha\beta}$ form an axial vector. Thus, the most general antisymmetric 4×4 tensor may be written as

$$(A_{ik}) = \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{pmatrix} \tag{58}$$

where $p = (p_x, p_y, p_z)$ is a polar vector and $a = (a_x, a_y, a_z)$ is an axial vector.

In covariant notation:

$$A_{ik} = (-p, a), \qquad A^{ik} = (p, a). \tag{59}$$

This structure underlies the tensorial formulation of the electromagnetic field strength and its dual, and connects the algebraic properties of three-dimensional vectors to their relativistic generalizations.

10 Conclusion

This work has provided a unified, rigorous, and physically motivated treatment of tensors and pseudotensors, with special attention to their explicit transformation properties under both proper and improper coordinate transformations. By deriving general transformation laws—including the role of the Jacobian determinant's sign—and providing explicit constructions for objects such as the Levi-Civita symbol and higher-rank antisymmetric tensors, we have elucidated the precise mathematical and physical distinctions between ordinary tensors, pseudotensors, and tensor densities.

The central results establish not only the technical foundation required for correct tensorial calculations across physics but also illuminate the deep interplay between algebraic structure, orientation, and symmetry principles. This has far-reaching implications for a variety of fields: from the formulation of parity-violating interactions and topological field theories to the modern classification of matter phases and the quantization of gauge systems [21, 19, 14, 18]. The framework provided here is directly applicable to current research challenges, including the study of anomalies, dualities, and geometric phases in condensed matter and high-energy physics [20, 22].

Furthermore, by clarifying the connection between geometric structures (such as orientation, volume forms, and Hodge duality) and the physical observables they underpin, we pave the way for future advances in the mathematical formulation of field theories, the geometric quantization program, and the exploration of new symmetry-protected topological phenomena. The ability to distinguish and correctly manipulate tensors and pseudotensors is no longer merely a technical subtlety but a prerequisite for meaningful participation in the ongoing revolutions of modern physics.

We expect that this systematic exposition will prove invaluable not only for theoretical investigations but also for computational applications, such as in numerical relativity, topological data analysis, and modern quantum simulations where orientation-dependent quantities play an essential role. By bridging the mathematical formalism with physical intuition and experimental relevance, this work contributes to the enduring project of unifying geometry and physics at the most fundamental level.

References

[1] J. A. Schouten, *Tensor Analysis for Physicists*, Oxford University Press, 1951.

[2] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley, 2004.

[3] T. Frankel, *The Geometry of Physics: An Introduction*, Cambridge University Press, 2011.

[4] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman, 1973.

[5] R. M. Wald, *General Relativity*, University of Chicago Press, 1984.

[6] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, 1995.

[7] F. A. Bais, J. D. Drouffe, *Gauge Theory and Variational Principles*, Princeton University Press, 2011.

[8] B. Schutz, *Geometrical Methods of Mathematical Physics*, Cambridge University Press, 1980.

[9] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, 1999.

[10] M. Fecko, *Differential Geometry and Lie Groups for Physicists*, Cambridge University Press, 2006.

[11] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, Butterworth-Heinemann, 1975.

[12] W. Greiner, *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics*, Springer, 2003.

[13] J. D. Jackson, *Classical Electrodynamics*, 3rd Edition, Wiley, 1998.

[14] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Oxford University Press, 2000.

[15] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman, 1973.

[16] M. Nakahara, *Geometry, Topology and Physics*, 2nd Edition, Taylor & Francis, 2003.

[17] W. Thirring, *Classical Mathematical Physics: Dynamical Systems and Field Theories*, Springer, 2013.

[18] S. Weinberg, *The Quantum Theory of Fields, Vol. II: Modern Applications*, Cambridge University Press, 1996.

[19] X.-L. Qi, S.-C. Zhang, "Topological insulators and superconductors," *Rev. Mod. Phys.* **83**, 1057 (2011).

[20] M. Z. Hasan, C. L. Kane, "Colloquium: Topological Insulators," *Rev. Mod. Phys.* **82**, 3045 (2010).

[21] X.-G. Wen, "Colloquium: Zoo of quantum-topological phases of matter," *Rev. Mod. Phys.* **89**, 041004 (2017).

[22] M. Fukugita, T. Yanagida, *Physics of Neutrinos and Applications to Astrophysics*, Springer, 2013.

[23] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Oxford University Press, 2000.