Resolution Matrix Semantics for Deontic Logic

Andrey M. Kuznetsov

andreyk@telus.net

Submitted to Journal of Philosophical Logic 24.05.2025

Abstract

This paper introduces Resolution Matrix Semantics (RMS) [Kuznetsov, 2025] as a novel framework for formalizing deontic logic, inspired by Y. Ivlev's quasi-matrix approach [Ivlev, 1997]. Unlike traditional Kripkean semantics, RMS employs a truth-value-based approach with normative values—mandatory (m), indifferent (i), and forbidden (b)—to evaluate acts, and true (t) or false (f) for formulas, ensuring a clear distinction between normative and propositional domains. This truth value semantics is augmented with "undetermined" truth values (e.g. m/i, "either m, or i"), and RMS uses an interpretation and sub-interpretation mechanism to resolve these, enabling precise normative evaluations. We develop SDLm, a deontic system that mirrors Standard Deontic Logic (SDL) in its axiomatic structure but avoids well-known paradoxes, such as the Chisholm and Gentle Murderer Paradoxes. We prove SDLm's soundness and completeness and provide tableau rules for automated theorem verification. By prioritizing normative status over existential considerations, SDLm offers a robust, paradox-free framework for reasoning about obligations, permissions, and prohibitions, with potential applications in ethics, law, and artificial intelligence.

Introduction

In this paper, we introduce Resolution Matrix Semantics as a novel method to formalize the systems of deontic logic. The concept of substantive semantics, pioneered by Y. Ivlev ([Ivlev, 1985; Ivlev, 1988; Ivlev, 1991]) serves as a foundational inspiration for this approach. Ivlev suggested defining modal operators based on informal reasoning tailored to their area of applicability—such as epistemology, ethics, or physics—rather than formal relational structures, introducing the notion of an interpretation quasi-function that assigns truth values in a context-dependent manner. His modal systems, while lacking an obvious correspondence to Kripkean systems, offered an intriguing informal, substantive perspective that prioritized practical interpretation over abstract world-relations. In this paper, we adapt these ideas to build a deontic system that resembles the Standard Deontic Logic system (SDL) but at the same time avoids well-known deontic paradoxes, like the Chisholm or Gentle Murderer Paradox.

Standard Deontic Logic (SDL)

Deontic logic formalizes normative reasoning about obligation (O), permission (P), and prohibition, with applications in ethics, law, and artificial intelligence. Standard Deontic Logic (SDL), grounded in the Kripkean modal system KD ([Kripke, 1963], [Chellas, 1980]), utilizes a serial accessibility relation to ensure that every world has at least one deontically ideal world where obligations are fulfilled. In SDL, the formula Op indicates that proposition p holds in all accessible ideal worlds, governed by axioms including propositional tautologies, the distribution axiom $O(p \to q) \to (Op \to Oq)$, and the seriality axiom $Op \to \neg O \neg p$, alongside inference rules such as modus ponens and necessitation. SDL effectively handles straightforward normative reasoning—for instance, if keeping a promise (Op) implies honesty ($p \to q$), then honesty is obligatory (Oq). However, SDL struggles with paradoxes like the Chisholm Paradox, where conditional obligations lead to contradictions, and the Gentle Murderer Paradox, where implications (e.g., humane action implying killing) yield counterintuitive obligations due to the limitations of material implication.

Unlike SDL, which relies on the Kripkean KD system, our construction of a deontic logic within RMS deliberately avoids adopting KDm, the RMS-based alethic modal logic. Alethic modal logics like KDm focus on the existence or non-existence of situations, assigning truth values t and f to indicate whether a situation holds, with further distinctions (tn, tc, fc, fn) denoting necessary or contingent existence ([Kuznetsov, 2025]). In contrast, deontic logic centers on a proposition's normative status within a deontic context, where some propositions may be normatively indifferent. While alethic logics evaluate every proposition as true or false, necessarily or contingently, with the modal operator $\Box$ reflecting this status, deontic logic requires the operator O to capture the normative status of a proposition - whether it's obligatory, permitted, forbidden, or neutral. Thus, SDLm redefines the deontic modal operator to accommodate these normative distinctions, moving beyond KDm's alethic framework.

Resolution Matrix Semantics for Deontic Logic (SDLm)

Drawing as an upgraded version of Yuriy Ivlev's three-valued deontic system [Ivlev, Y. V. (1997)], SDLm reorients the evaluation of propositions within RMS to focus exclusively on their normative status. Unlike alethic modal logics, which assess whether propositions exist necessarily or contingently, deontic logic evaluates propositions as obligatory, permitted (neither obligatory nor prohibited, i.e., normatively indifferent), or prohibited within a given rule system. These normative categories are paramount when analyzing deontic statements, sidelining existential considerations. Truth values t and f emerge only at the level of modal formulas, where normative evaluations are expressed. For example, the formula "it is obligatory that p" (Op) takes the value t or f, reflecting p's normative standing, thereby prioritizing obligation, permission, or prohibition over alethic concerns of existence.

System SDLm

Language

The language of SDLm comprises:

- Propositional variables: p, q, etc., representing basic propositions (e.g., actions or states).
- Logical connectives for acts: ¬ (negation), ∧ (conjunction), ∨ (disjunction), → (implication).
- Logical connectives for formulas: ¬ (negation), ∧ (conjunction), ∨ (disjunction), → (implication).
- Deontic operator: O, representing obligation and P, representing permission
- Brackets: (and), for grouping expressions and ensuring clarity.

Note: This paper distinguishes between logical connectives for acts and formulas. Although they appear similar, their distinct truth table definitions must be considered.

Definition of Acts and Formulas

Acts (representing actions or states) are defined recursively:

- Basic propositions p, q are acts.
- If p and q are acts, then ¬p, p ∧ q, p ∨ q, p → q are acts.

Actions (acts) can have only normative (deontic) values: they are either mandatory (m) according to a normative code, normatively indifferent (i) within that code, or forbidden (b) by the code. To avoid confusion with the deontic operator O, we use 'm' instead of 'o' to denote obligatory acts in this paper.

Let s be an act in the system under consideration, and let I denote an interpretation function that assigns truth values to acts. The possible truth values are drawn from the set {m, m/i, i, i/b, b}, where m/i and i/b are the "indetermined" truth values; value m/i represents "either m or i" and i/b represents "either i or b", there is no certainty, which one, but always one of this pair, and never both. These concepts allow us to model situations where precision is elusive yet deontic distinctions remain relevant. We encounter a scenario where the precise value is indeterminate yet constrained within a specific domain. This does not violate the concept of a function, as it consistently selects a single value from its range (for example, truth value m/i - "either m or i") for each input, albeit one that "wanders" between m and i. Such situations arise across various domains.

Suppose that under interpretation I, the act s is assigned the indeterminate truth value m/i, i.e., $|s|_I = m/i$. Then, there exist sub-interpretation functions I_1 and I_2, generated from I, such that:

- In sub-interpretation I_1, $|s|_{I_1} = m$,
- In sub-interpretation I_2, $|s|_{I_2} = i$.

A sub-interpretation I' of I is defined as a function that resolves each indeterminate truth value (m/i) assigned by I into exactly one of its corresponding determinate values (m or i for m/i; i or b for i/b), consistently across all sub-acts, while preserving the truth value assignments for all determinate values (m, i, b) as given by I. **The act s is called universally mandatory in the interpretation I if and only if s is universally mandatory in every sub-interpretation I' generated from I**, where universal mandate in a sub-interpretation I' means that $|s|_{I'} \in \{m\}$. Thus, s is universally mandatory under I if and only if, for all sub-interpretations I' of I, the truth value $|A|_{I'}$ is m.

When handling indeterminate truth values u/v within an act, we must maintain consistency by assigning the same value from u/v to all sub-acts sharing identical truth values.

Formulas (representing normative evaluations) are defined recursively:

- If p is an act, then Op is a formula.
- If A and B are formulas, then $\neg A$, $A \land B$, $A \lor B$, $A \rightarrow B$ are formulas.

Similarly to the definition of interpretation and sub-interpetation function for the acts, the formula A can take the truth value t or f, when the act has indetermined truth value. For example, if act s takes values m/i, then formula Os takes either t or f, depending on whether the act takes value m or i. **A formula is valid (takes only values t) in the interpretation I if and only if A is valid in every sub-interpretation I' generated from I.**

RMS Semantics for SDLm

SDLm's semantics for acts uses RMS with truth values {m (mandatory), i (indifferent), b (forbidden)}, ordered as b < i < m; we use b not f for "forbidden" to avoid confusion with the truth value f. Negation (¬) and implication (→) serve as primitive connectives, with conjunction (∧) and disjunction (∨) defined accordingly. Some binary connectives yield indeterminate truth values (e.g., m/i, i/b), consistent with RMS's approach to modal logic [1].

Negation (¬)

The truth value of ¬p reverses the normative status:

p	¬p
m	b
i	i
b	m

This reflects intuitive deontic reasoning: if p, "follow strict safety rules in an explosive area", is mandatory (m), then ¬p, "do not follow safety rules" is forbidden (b). If p, "eat an apple", is normatively indifferent (i), then ¬p, "do not eat an apple" remains indifferent (i).

Conjunction (∧)

p ∧ q	m	i	b
m	m	i	b
i	i	i/b	b
b	b	b	b

Conjunction takes the minimum of p and q's truth values.

For example, if p, "be at work at 8 a.m." is mandatory (m) and q, "get a cappuccino" is indifferent (i), then p ∧ q , "start work at 8 a.m. and get a cappuccino" is indifferent (i).

If p, "steal money from a bank" is forbidden (b) and q ,"sing a song" is indifferent (i), then p ∧ q "steal money while singing" is forbidden (b).

When p = i and q = i, p ∧ q = i/b, as context matters: p, "eat an apple" and q, "sing a song" yield I, "eat an apple while singing", but p, "smoke" and q, "stay in a highly explosive area" yield b, "smoke in an explosive area".

Disjunction (∨)

p v q	m	i	b
m	m	m	m
i	m	i/m	i
b	m	i	b

Disjunction takes the maximum of p and q's truth values. When p = i and q = i, p ∨ q = i/m. For example, p, "finish my work today" and q, "eat an apple" yield i, "finish work today or eat an apple", but p, "finish my work today" and q, "finish my work tomorrow" yield m "finish work today or tomorrow" if the work must be completed by tomorrow.

Implication (→)

p → q	m	i	b
m	m	i	b
i	m	i/m	i
b	m	m	m

When p = i and q = i, p → q = i/m.

For example, p, "finish my work today" and q, "eat an apple" yield i "if I finish my work today, I'll eat an apple", but p, "stay in a room with flammable substances" and q, "do not smoke" yield m, "if in a flammable room, you must not smoke".

Universally Mandatory Acts

In SDLm, as in Standard Deontic Logic (SDL), we treat acts that are structurally analogous to tautologies as universally mandatory acts. This approach stems from the fact that all reasoning within any normative code relies, directly or indirectly, on the principles of classical logic. These principles are inherently embedded in the framework of any legal or normative system. If we were to classify tautologies as normatively indifferent acts, this would be incorrect, as normative indifference implies that a normative code is neutral regarding whether an act is performed. For example, consider the inference: "If a person is obliged to complete their paper and write a review for a colleague's paper by Friday, then they are clearly obliged to review their colleague's work." Classical logical reasoning permeates all forms of normative reasoning, where maintaining logical consistency is a fundamental requirement.

If tautologies were deemed normatively indifferent, it would suggest that classical logical inferences could sometimes be disregarded or even replaced with their opposites, since both a statement and its negation would be permissible in a normatively indifferent context. This, however, is untenable. Consequently, we regard tautologies in classical logic as an integral part of the normative context, neither indifferent nor prohibited, but mandatory. Thus, in SDLm, as in SDL, all tautologies are considered universally mandatory acts.

In SDLm, universally mandatory acts, such as $p \vee \neg p$, consistently take the truth value m (mandatory), and this should be reflected in the table definitions for logical connectives provided earlier. For the act $p \vee \neg p$, when p is assigned i (indifferent, with $\neg p$ also i), the disjunction $p \vee \neg p$ should take only the value m from the possible values m / i, ensuring it is always mandatory. Thus, $p \vee \neg p$ invariably takes the value m for any combination of truth values assigned to p and $\neg p$.

Conversely, for the act $p \wedge \neg p$, when both p and $\neg p$ are indifferent (i), the conjunction $p \wedge \neg p$ should always take the value b (prohibited), not the indeterminate i/b. For all other truth values of p, $p \wedge \neg p$ similarly evaluates to b. This act, being the opposite of a universally mandatory act, is universally forbidden, as contradictions must not occur in normative reasoning.

For the act $p \rightarrow p$, when p is indifferent (i), the implication $p \rightarrow p$ should take only the value m from the m/ i. For all other truth values of p, $p \rightarrow p$ consistently maintains the value m, reinforcing its status as a universally mandatory act.

In classical logic, every tautology, and thus every universally mandatory act, can be expressed in conjunctive normal form, with elementary disjunctions such as $p \vee \neg p$ as its conjuncts. Consequently, every universally mandatory act consistently takes the truth value m (mandatory).

Formulas and Modal Operator

Formulas in SDLm are modalized acts, evaluated as true (t) or false (f), akin to classical binary logic. Acts are propositions relevant to a rule system, not evaluated for existential truth but for normative status (m, i, b). In alethic modal logics, the necessity operator $\Box$ captures distinctions among tn, tc, fc, and fn, but in deontic logic, the operator O operates on normative values, avoiding alethic contexts that often lead to paradoxes in SDL.

Definition of Negation:

p	¬p
m	b
i	i
b	m

Definition of Implication:

p → q	m	i	b
m	m	i	b
i	m	i/m*	i
b	m	m	m

*) if p=q, then m

Definition of deontic operators O and P:

p	Op	Pp
m	t	t
i	f	t
b	f	f

The permission operator P is defined via Op ≡ ¬P(¬p), meaning p is obligatory if and only if ¬p is not permitted.

Axioms and Inference Rules

Axioms:

- **Propositional Tautologies**: All classical propositional tautologies.
- **Distribution Axiom**: O(p → q) → (Op → Oq).
- **Seriality Axiom**: Op → Pp.

Inference Rules:

- **Modus Ponens**: From A and A → B, infer B.

- **Necessitation Rule**: If p is a universally mandatory act, then Op is a valid formula.

Soundness Theorem

SDLm is sound: every theorem A in SDLm is valid (takes t in all interpretations).

- **Tautologies**: Valid, as formulas take t or f, and classical tautologies hold.
- **Distribution Axiom**: $O(p \to q) \to (Op \to Oq)$. Assume it takes f, so $O(p \to q) = t$ and $Op \to Oq = f$, implying $Op = t$ and $Oq = f$. Thus, $p = m$ and $q = i$ or b, so $p \to q = i$ or b, and $O(p \to q) = f$, a contradiction.
- **Seriality Axiom**: $Op \to Pp$, or equivalently, $Op \to \neg O(\neg p)$. Assume it takes f, so $Op = t$ and $\neg O(\neg p) = f$. Then $p = m$, $\neg p = b$, $O(\neg p) = f$, and $\neg O(\neg p) = t$, a contradiction.
- **Modus Ponens**: Preserves validity, as in classical logic.
- **Necessitation Rule**: If p is universally mandatory act but $Op = f$, then $p = i$ or b. Since p is a universally mandatory act, it takes only o. Therefore, we get a contradiction.

Completeness Theorem

SDLm is complete: every valid formula A is provable ($SDLm \vdash A$).

Lemma 1: A consistent set Γ extends to a maximal consistent set T such that:

- For all formulas A, either $A \in T$ or $\neg A \in T$.
- If $T \vdash B$ and $\Gamma \subseteq T$, then $B \in T$.
- $B \lor C \in T$ if and only if $B \in T$ or $C \in T$.
- $B \land C \in T$ if and only if $B \in T$ and $C \in T$.

Proof: Use a Henkin-style construction, enumerating formulas and adding consistent ones to ensure maximality, consistent with SDLm's axioms.

Lemma 2: There exists an interpretation $|\cdot|\_T$ such that:

- $|p|\_T = m$ if and only if $Op \in T$.
- $|p|\_T = i$ if and only if $\neg Op \in T$ and $\neg O\neg p \in T$.
- $|p|\_T = b$ if and only if $O\neg p \in T$.
- $|A|\_T = t$ if and only if $A \in T$; A is a formula.

We can see that the act p can take only one value from {m, i, b} and no other values can be taken. In some cases, p can take indeterminate truth values, but this value doesn't violate the definition of interpretation $|\cdot|\_T$ – in case of indetermined truth values, we still have only one value u or v from indetermined value u/v, and no other values can be taken by this act.

Let's show that $|\cdot|\_T$ aligns with described SDLm's semantics for logical connectives for the acts and for the formulas, and for the deontic operator O.

Proof:

Verify RMS negation:

- **If p = m then ¬p = b**

Assume p = m, then Op ∈ T, therefore, O(¬¬p) ∈ T (using universally mandatory act p→¬¬p, necessitation rule gives O(p→¬¬p) ∈ T, and with Op ∈ T, Distribution axiom and MP rule we get O(¬¬p) ∈ T). Therefore, O(¬(¬p)) ∈ T, so ¬p = b. Similarly, for p = b.

- **If p = i then ¬p = i**

Now assume p = i, therefore Pp ∈ T and P¬p ∈ T. So, P(¬(¬p)) ∈ T and P¬¬(¬p) ∈ T, and ¬p = i.

Verify RMS implication:

- **If p = b, then p→q = m**

Assume p = b, so O¬p ∈ T. ¬ p →(p→ q) – universally mandatory act, so O(¬ p →(p→ q)) ∈ T, by D axiom and with O¬p ∈ T, we get O(p→ q)) ∈ T.

- **If q = m, then p→q = m**

Assume q = m, so Oq ∈ T. q →(p→ q) – universally mandatory act, so O(q →(p→ q)) ∈ T, by D axiom and with Oq ∈ T, we get O(p→ q)) ∈ T.

Theorem 1. Op ∧ Oq →O(p ∧ q)

1. Op assumption

2. Oq assumption
3. $p \to (q \to p \wedge q)$ universally mandatory act
4. $O(p \to (q \to p \wedge q))$ 3, Nec
5. $Op \to O(q \to p \wedge q))$ 4, D, MP
6. $O(q \to p \wedge q))$ 1,5, MP
7. $Oq \to O(p \wedge q)$ 6, D, MP
8. $O(p \wedge q)$ 2,7, MP

- **If p = m, q = b, then p→q = b**

$Op \in T$ and $O\neg q \in T$. By theorem 1, $O(p \wedge \neg q) \in T$, $O\neg (p \to q) \in T$, $p \to q = b$.

Theorem 2. $O(p \wedge q) \to Op \wedge Oq$

1. $O((p \wedge q)$ assumption
2. $(p \wedge q) \to p$ universally mandatory act
3. $O((p \wedge q) \to p)$ 2, Nec
4. $O(p \wedge q) \to Op$ 1, 3, D, MP
5. $(p \wedge q) \to q$ universally mandatory act
6. $O((p \wedge q) \to q)$ 5, Nec
7. $O(p \wedge q) \to Oq$ 1, 6, D, MP
8. $O(p \wedge q) \to Op \wedge Oq$ 4,7, tautology

- **If p = q = i, then p→q = i/m**

We have p = q = i, so $\neg Op \in T$ and $\neg O\neg p \in T$ and $\neg Oq \in T$ and $\neg O\neg q \in T$. Assume $p \to q = b$, then $O\neg(p \to q) \in T$, then $O(p \wedge \neg q) \in T$, then by theorem 2, $Op \in T$ and $O\neg q \in T$, a contradiction.

- **If p = m and q = i, then p→q = i**

We have p = m, so $Op \in T$ and q = i, so $\neg Oq \in T$ and $\neg O\neg q \in T$. Assume $p \to q = m$, then $O(p \to q) \in T$, by D axiom, using $Op \in T$ and MP, we get $Oq \in T$, a contradiction. Now assume that $p \to q = b$, then $O\neg (p \to q) \in T$, so $O(p \wedge \neg q) \in T$, and by theorem 2, $Op \in T$ and $O\neg q \in T$, a contradiction.

- **If p = i and q = b , then p→q = i**

We have p = i, so ¬Op ∈ T and ¬O¬p ∈ T, and q = b, so O¬q ∈ T. Assume p→q = o, then O(p→q) ∈ T, then O(¬q→¬p) ∈ T (using tautology (universally mandatory act), Nec, D and MP). Then, by D and MP, we get O(¬q)→O(¬p) ∈ T, then, using O¬q ∈ T and MP, O(¬p) ∈ T, a contradiction.

Verify RMS deontic operator O:

- **If p = m then Op = t**

Assume p = m, then Op ∈ T, which makes Op = t.

- **If p = i then Op = f**

Assume p = i, then ¬Op ∈ T and ¬O¬p ∈ T, therefore, Op = f.

- **If p = b then Op = f**

Assume p = b, then O¬p ∈ T, O(¬p)→P(¬p), so P(¬p) ∈ T, then P(¬p) → ¬Op ∈ T, so ¬Op ∈ T, therefore, Op = f.

All SDLm axioms are valid in the interpretation |·|_T, and validity is preserved.

Now assume a valid formula E is not provable in SDLm, then ¬E is consistent, extends to a maximal set T (Lemma 1), and ¬E is valid in |·|_T (Lemma 2), contradicting E's validity. Thus, SDLm is complete.

We have demonstrated that the SDLm system is sound and complete under the proposed RMS semantics. It is equivalent to the KDm system in that it employs the same axioms and inference rules. However, SDLm distinguishes between two distinct categories—acts and formulas—which enables it to avoid well-known deontic paradoxes, as will be illustrated in subsequent sections.

Tableau Rules for the system SDLm

Before addressing deontic paradoxes, we demonstrate the construction of tableau rules for the SDLm system, adapting the approach used for classical bivalent logic. In the paper [Kuznetsov, 2025], a tableau method for Kripkean modal logic systems was proposed based on RMS semantics. Here, we extend this method to SDLm, tailoring it to the system's unique structure.

At the formula level, where modalized statements such as Op and Pp are evaluated, SDLm employs classical logic with truth values t (true) and f (false). Consequently, all classical tableau rules apply to formulas. However, SDLm requires additional rules to handle the transition from formulas to acts, reflecting the normative evaluation of acts as mandatory (m), indifferent (i), or forbidden (b). For instance, if the formula Op ("it is obligatory that p") holds, the act p is assigned the deontic value m (mandatory).

Based on the definition table for the SDLm deontic operator O, we formulate a specific tableau rule to capture the transition from formulas to acts:

TO

T(Op)
Mp

This rule indicates that if "it is obligatory that p" (T(Op)) holds, the act p takes the deontic value m (mandatory).

Similarly, for the negation of an obligation, if "it is false that p is obligatory" (F(Op)), the act p takes one of two possible deontic values: either i (indifferent) or b (forbidden).

FO

F(Op)	
Ip	Bp

In these rules, Mp, Ip, and Bp denote that the act p takes the deontic value m, i, or b, respectively. Note that M, I, and B are not operators but represent deontic truth values for acts, analogous to T (true) and F (false) for formulas.

We also define specific SDLm tableau rules for acts, based on their logical connectives: negation (¬) and implication (→).

M¬

M(¬p)
Bp

I¬

I(¬p)
Ip

B¬

B(¬p)
Mp

M→

M(p → q)		
Bp	Mq	Ip, Iq

I→

I(p → q)		
Mp, Iq	Ip, Iq	Ip, Bq

B→

B(p → q)
Mp, Bq

To complement these, we incorporate classical tableau rules for SDLm formulas, using negation and implication as primitive connectives:

T¬

T(¬A)
FA

F¬

F(¬A)
TA

T →

T(A → B)	
FA	TB

F→

F(A → B)
TA, FB

Since SDLm uses only negation and implication, these rules suffice for formula decomposition.

To determine whether a formula A is a theorem of SDLm, we begin with the assumption F(A) and apply all applicable tableau rules to decompose the formula. If decomposition results in both T(A) and F(A) on each branch of the tableau tree, the formula A is a theorem. If the decomposition involves transitions to acts, we continue applying the act-specific rules until one of two outcomes occurs: either each branch contains contradictory deontic values for the same act (e.g., Mp and Bp), indicating that A is a theorem, or at least one branch remains open (no contradictory deontic values for any act), indicating that A is not a theorem.

Consider a classical logic tautology, which is also a theorem in SDLm: (Op → Oq) → (¬Oq → ¬Op).

1. F (Op→ Oq) → (¬Oq→¬ Op)) assumption
2. T(Op→ Oq), F(¬Oq→¬ Op) from1, **F→**
3. T(Op→ Oq), T(¬Oq), F(¬ Op) from 2**, F→**
4. T(Op→ Oq), F(Oq), T(Op) from 3, **T¬, F¬**

5.1. F(Op), F(Oq), T(Op) from 4, **T→**

5.2. T(Oq), F(Oq), T(Op) from 4, **T→**

Both branches (5.1 and 5.2) are closed, as each contains both T(A) and F(A) for the same formula. Thus, (Op → Oq) → (¬Oq → ¬Op) is a theorem of SDLm.

Next, we verify the Seriality Axiom, Op → Pp, equivalently expressed as Op → ¬O¬p:

1. F(Op → ¬ O¬p) assumption
2. T(Op), F (¬ O¬p) from 1, **F→**
3. T(Op), T (O¬p) from 2, **F¬**
4. Mp, M¬p from 3, **TO**
5. Mp, Bp from 4, **M¬**

Step 5 yields a contradiction, as the act p is both mandatory (Mp) and forbidden (Bp). Thus, the branch is closed, and Op → Pp is a theorem of SDLm.

We now examine the Distribution Axiom, O(p → q) → (Op → Oq):

1. F(O(p → q) → (Op → Oq)) assumption
2. T(O(p → q)), F(Op → Oq) from 1, **F→**
3. T(O(p → q)), T(Op), F(Oq) from 2, **F→**

4.1. M(p → q)), Mp, Iq, from 3, **FO**

4.2 . M(p → q)), Mp, Bq, from 3, **FO**

4.1.1 Bp, Mp, Iq from 4.1, **M→**

4.1.2 Mq, Mp, Iq from 4.1, **M→**

4.1.3 Ip, Iq, Mp, Iq from 4.1, **M→**

4.2.1 Bp, Mp, Bq from 4.2, **M→**

4.2.2 Mq, Mp, Bq from 4.2, **M→**

4.2.3 Ip, Iq, Mp, Bq from 4.2, **M→**

Each branch contains at least one act with multiple deontic values (e.g., Mp and Bp, or Mq and Iq), indicating a contradiction. Thus, the Distribution Axiom is a theorem of SDLm.

Finally, we consider a formula that is not a theorem of SDLm, such as O(p → q) → Oq:

1. F(O(p → q) →Oq)
2. T(O(p → q)), F(Oq)

3.1 M(p → q), Iq
3.2 M(p → q), Bq

3.1.1 Bp, Iq from 3.1, **M→**
3.1.2 Mq, Iq from 3.1, **M→**
3.1.3 Ip, Iq, Iq from 3.1, **M→**

3.2.1 Bp, Bq from 3.2, **M→**
3.2.2 Mq, Bq from 3.2, **M→**

3.2.3 Ip, Iq, Bq from 3.2, **M→**

Branches such as 3.1.1 and 3.2.1 remain open, as they contain no contradictory deontic values for any act. Thus, $O(p \rightarrow q) \rightarrow Oq$ is not a theorem of SDLm.

The tableau method adapted for SDLm provides an efficient and systematic approach for automatically determining whether a given formula is a theorem of the system, leveraging the clear distinction between acts and formulas to ensure robust and consistent reasoning.

Chisholm Paradox

The Chisholm Paradox involves a set of intuitively consistent deontic statements that lead to a contradiction in SDL. Consider the following scenario, adapted from Chisholm's original formulation:

1. It is obligatory that you keep your promise (Op).
2. If you keep your promise, it is obligatory that you act honestly ($p \rightarrow Oq$).
3. If you do not keep your promise, it is obligatory that you do not act honestly ($\neg p \rightarrow O\neg q$).
4. You do not keep your promise ($\neg p$).

These statements appear coherent in natural normative reasoning: you must keep your promise, and keeping it entails an obligation to be honest, while breaking it entails an obligation to avoid honesty (perhaps to mitigate harm). However, in SDL, these statements are logically inconsistent.

Why the Chisholm Paradox Arises in SDL

In SDL, obligations are modeled using a Kripkean framework where Op holds if p is true in all deontically ideal worlds accessible from the current world, and the seriality axiom ($Op \rightarrow \neg O\neg p$) ensures such ideal worlds exist. Let's analyze the paradox step-by-step in SDL:

- **Statement 1: Op**. This asserts that p (keeping the promise) is true in all ideal worlds.
- **Statement 4: ¬p**. This indicates that in the current world, the promise is not kept, which is possible in SDL since Op does not entail p (obligations can be violated).
- **Statement 2: p → Oq**. In SDL, this material implication is equivalent to $\neg p \vee Oq$. Since $\neg p$ is true (from statement 4), the implication holds, but it also allows cases where p holds and Oq is true (q is obligatory in all ideal worlds).

- **Statement 3: ¬p → O¬q**. Similarly, this is equivalent to ¬¬p ∨ O¬q, or p ∨ O¬q. Since ¬p is true, this implication requires O¬q (¬q is true in all ideal worlds).

The contradiction emerges when combining these statements:

- From Op (statement 1), all ideal worlds satisfy p.
- From ¬p → O¬q (statement 3) and ¬p (statement 4), we derive O¬q, meaning all ideal worlds satisfy ¬q.
- Since p → q is equivalent to ¬p ∨ q, and p is true in all ideal worlds (from Op), q must also be true in all ideal worlds to avoid contradicting p → Oq (statement 2).
- Thus, SDL derives both Oq (q in all ideal worlds) and O¬q (¬q in all ideal worlds), violating the seriality axiom (Op → ¬O¬p), as Oq ∧ O¬q is impossible.

The paradox arises in SDL because material implication (p → Oq) inadequately captures conditional obligations, treating them as disjunctions (¬p ∨ Oq) that fail to distinguish between factual and normative dependencies. This issue with contrary-to-duty imperatives, where obligations conditional on violations lead to contradictions, has been extensively discussed in the literature [Åqvist, 1967; Carmo and Jones (2010)].

Additionally, SDL's Kripkean semantics assumes a uniform accessibility relation, forcing all propositions into a binary framework (true or false in ideal worlds), which cannot accommodate the nuanced interplay of conditional obligations and factual violations in the Chisholm set.

Chisholm Paradox in SDLm: No Longer a Paradox

In the SDLm system, the Chisholm Paradox does not arise, as statements 2 and 3 from the original formulation are not well-formed formulas. In SDLm, acts are evaluated through a normative lens, assigned values of mandatory (m), normatively indifferent (i), or forbidden (b), rather than as true or false. Formulas, such as Op, are statements about the deontic status of acts and are evaluated as true (t) or false (f). Consequently, statements like "if you keep your promise, it is obligatory that you act honestly" (p → Oq) are invalid in SDLm, as they mix acts (p, an act) with formulas (Oq, a normative statement). In SDLm, an act like "keeping your promise" cannot directly imply a normative statement but implies another act within the deontic framework. Acts exist in the normative world, while statements about acts belong to the propositional world of normative formulas, which are either true or false.

To align the Chisholm Paradox with SDLm's semantics, we reformulate the statements to operate solely at the level of formulas, focusing on normative obligations to avoid mixing categories:

1. It is obligatory that you keep your promise (Op).
2. It is obligatory that, if you keep your promise, then you act honestly ($O(p \rightarrow q)$).
3. It is obligatory that, if you do not keep your promise, then you do not act honestly ($O(\neg p \rightarrow \neg q)$).
4. You do not keep your promise ($\neg p$).

In this reformulation, statements 1 and 2, combined with SDLm's Distribution Axiom ($O(p \rightarrow q) \rightarrow (Op \rightarrow Oq)$) and modus ponens, yield Oq (it is obligatory to act honestly). However, $O\neg q$ (it is obligatory to not act honestly) cannot be derived from statement 3, as this would require $O\neg p$ (it is obligatory to not keep your promise) in statement 4. Instead, statement 4 provides only $\neg p$, an act, not a normative statement ($O\neg p$). If we were to assume both Op and $O\neg p$, a contradiction would already exist, as SDLm's semantics prevent contradictory obligations ($Oq \wedge O\neg q$) via the normative truth value assignments.

Thus, the Chisholm Paradox is avoided in SDLm due to its foundational principle: acts and normative statements about acts (formulas) are distinct categories. Mixing these categories, as SDL does through material implication, leads to the Chisholm Paradox and other issues, such as the Gentle Murderer Paradox, which we address in the next section.

The Gentle Murderer Paradox in SDL

The Gentle Murderer Paradox is a classic challenge in deontic logic that reveals the shortcomings of Standard Deontic Logic (SDL) when addressing conditional obligations tied to forbidden acts. However, due to the sensitive nature of discussing murder, we opt not to use this example. Instead, we present a structurally similar but less severe scenario involving a maid disturbing her boss, which we term the Gentle Disturbance Paradox. This paradox, like its predecessor, arises in SDL due to the limitations of material implication and Kripkean relational semantics, which struggle to reconcile normative prohibitions with conditional obligations. This chapter introduces the Gentle Disturbance Paradox, describes its formulation, and explains why it emerges as a paradox in SDL, paving the way for a later analysis of how SDLm, based on Resolution Matrix Semantics (RMS), resolves this issue through its truth-value-based approach.

The Gentle Disturbance Paradox

The Gentle Disturbance Paradox involves a scenario where a forbidden act (disturbing the boss) appears to generate an obligation to perform it in a specific manner (gently). Consider the following statements, which seem intuitively consistent in normative reasoning:

1. It is obligatory that the maid does not disturb her boss for a couple of hours after lunch ($O\neg d$).
2. If the maid disturbs her boss, it is obligatory that she does so gently ($d \rightarrow Og$).
3. The maid disturbs her boss (d).

These statements reflect a reasonable normative framework: the maid is prohibited from disturbing her boss during a specific period, but if she does, she should do so gently to minimize disruption. However, in SDL, this set of statements leads to a counterintuitive result, suggesting an obligation to disturb the boss gently, which conflicts with the prohibition against disturbing.

Why the Gentle Disturbance Paradox Arises in SDL

In SDL, obligations are modeled using a Kripkean framework where a formula Op holds if proposition p is true in all deontically ideal worlds accessible from the current world, with a serial accessibility relation ensuring such ideal worlds exist. Let's analyze the paradox step-by-step in SDL:

- **Statement 1: $O\neg d$**. This asserts that $\neg d$ (not disturbing the boss) is true in all ideal worlds, reflecting the prohibition against disturbance.
- **Statement 3: d**. This indicates that in the current world, the maid disturbs the boss, which is consistent with SDL, as $O\neg d$ does not entail $\neg d$ (obligations can be violated).
- **Statement 2: $d \rightarrow Og$**. In SDL, this material implication is equivalent to $\neg d \vee Og$. Since d is true (from statement 3), $\neg d$ is false, so the implication requires Og to be true, meaning g (disturbing gently) must hold in all ideal worlds.

The paradox emerges when combining these statements:

- From $O\neg d$ (statement 1), all ideal worlds satisfy $\neg d$, meaning no disturbance occurs in any ideal world.
- From $d \rightarrow Og$ (statement 2) and d (statement 3), we derive Og, implying that g (disturbing gently) is true in all ideal worlds.
- However, g typically entails d (disturbing gently implies disturbing), as gently disturbing the boss is a specific form of disturbance. Thus, Og suggests d is true in all ideal worlds (Od), contradicting $O\neg d$, which requires $\neg d$ in all ideal worlds.

The counterintuitive outcome is that SDL derives Og—an obligation to disturb the boss gently—despite O¬d prohibiting disturbance. This conflicts with the seriality axiom ($Op \rightarrow \neg O\neg p$), as O¬d and Og (implying d) cannot coexist without contradicting the prohibition. The paradox arises because SDL's material implication ($d \rightarrow Og$) fails to distinguish between factual conditions (d) and normative obligations (Og), treating the conditional as a disjunction ($\neg d \vee Og$) that forces Og when d holds. Furthermore, SDL's Kripkean semantics relies on a binary evaluation (true or false in ideal worlds), which cannot adequately handle the normative tension between a forbidden act and a conditional obligation tied to its violation.

Gentle Disturbance Paradox in SDLm: No Paradox Arises

In SDLm, the Gentle Disturbance Paradox is resolved by ensuring that statements adhere to the system's strict distinction between acts (evaluated normatively as mandatory m, indifferent i, or forbidden b) and formulas (normative statements evaluated as true t or false f). The original formulation of the paradox in SDL includes the statement $d \rightarrow Og$, which is not well-formed in SDLm because it mixes an act (d, disturbing the boss) with a formula (Og, the obligation to disturb gently). To address this, we reformulate the paradox to preserve the normative intent of statement 2 ("if the maid disturbs the boss, she must do so gently") while ensuring all statements are valid in SDLm's syntax, operating at the level of formulas or acts consistently.

Reformulated Statements in SDLm

To align with SDLm's semantics, we rephrase the Gentle Disturbance Paradox as follows:

1. It is obligatory that the maid does not disturb her boss for a couple of hours after lunch (O¬d).
2. It is obligatory that, if the maid disturbs the boss, she does so gently ($O(d \rightarrow g)$).
3. The maid disturbs the boss (d). – we still have to keep this statement.

These statements reflect the normative framework: the maid is prohibited from disturbing her boss (O¬d), but if she does disturb him, there is a normative obligation to do so gently ($O(d \rightarrow g)$). Here, $d \rightarrow g$ is an act (an implication between acts d and g, where g is the act of disturbing gently), and $O(d \rightarrow g)$ is a formula asserting its obligation, consistent with SDLm's syntax. Statement 3, d, is an act indicating a violation of the obligation, which is permissible in SDLm as obligations do not entail their fulfillment.

Analysis in SDLm

Let's analyze these statements using SDLm's axioms, inference rules, and semantics:

- **Statement 1: O¬d**
  This is a formula, asserting that the act ¬d (not disturbing the boss) is mandatory. In SDLm, |¬m|_I = m, so |O¬m|_I = t.
- **Statement 2: O(d → g)**
  This is a formula, where d → g is an act representing the implication "if the maid disturbs the boss, she does so gently." Unlike the original d → Og, which mixed an act and a formula, O(d → g) is well-formed, as d → g is an act, and the deontic operator O applies to acts to produce formulas. This statement captures the normative rule that, in case of disturbance, it must be done gently, as might be codified in a verbal or written normative code.
- **Statement 3: d**
  This is an act, indicating that the maid disturbs the boss in the current context. In SDLm, this does not imply O(d), as obligations can be violated (i.e., O¬d does not entail ¬d).

Derivation in SDLm

We now show that SDLm avoids the paradox by deriving the appropriate obligations without contradiction, using SDLm's axioms (propositional tautologies, Distribution Axiom: O(p → q) → (Op → Oq), Seriality Axiom: Op → Pp), and inference rules (Modus Ponens, Necessitation Rule).

Step 1: Deriving O(m → g)

The normative intent of statement 2 is that it is obligatory for the maid to disturb gently if she disturbs the boss. We can derive O(m → g) formally:

- The act ¬m → (m → g) is universally mandatory, as it is a tautology in classical logic (¬m → (m → g) is equivalent to m ∨ (¬m ∨ g), which simplifies to true).
- By the Necessitation Rule (if an act is universally mandatory, then its obligation is valid), we obtain O(¬m → (m → g)).
- Using the Distribution Axiom: O(¬d → (d → g)) → (O¬d → O(d → g)), and Modus Ponens with O¬d (from statement 1), we derive O(d → g).

Thus, statement 2, O(d → g), is formally justified, representing the norm that if the maid disturbs the boss, she must do so gently.

Step 2: Checking for Og

In SDL, the paradox arises because d → Og and d imply Og, suggesting an obligation to disturb gently, which conflicts with O¬d. In SDLm, we investigate whether Og can be derived:

- From O(d → g) (statement 2), we apply the Distribution Axiom: O(d → g) → (Od → Og).
- To derive Og, we would need Od (if Om and Od → Og, then Og by Modus Ponens).
- However, statement 3 provides only d (an act), not Od (a formula). In SDLm, d does not imply Od, as acts are not equivalent to their obligations.
- Moreover, statement 1, O¬d, indicates that |d|_I = b (forbidden, since |¬d|_I = o and |d|_I = b by the negation table). Thus, Om is false (|Od|_I = f), as |d|_I ≠ o.

Therefore, Og cannot be derived, as Od is not present, avoiding the paradoxical obligation to disturb gently.

Step 3: Deriving O¬g

Interestingly, SDLm allows us to derive an obligation not to disturb the boss gently (O¬g), reinforcing the prohibition against disturbance:

- The act g → d (if the maid disturbs gently, then she disturbs) is universally mandatory, as it is a tautology (disturbing gently implies disturbing).
- By contraposition, ¬d → ¬g (if the maid does not disturb, she does not disturb gently) is also a universally mandatory act.
- By the Necessitation Rule, we obtain O(¬d → ¬g).
- Using the Distribution Axiom: O(¬d → ¬g) → (O¬d → O¬g), and Modus Ponens with O¬d (statement 1), we derive O¬g.

This result indicates that it is obligatory not to disturb the boss gently (O¬g), aligning with the prohibition against disturbance (O¬d), as gentle disturbance is a form of disturbance.

Step 4: Checking for Contradiction with Od

The critical issue in SDL is the derivation of Od (an obligation to disturb), which contradicts O¬d. In SDLm, we confirm that Od cannot be derived:

- From O(d → g), we need Od to derive Og, but Od is not implied by d or any other statement.

- The presence of O¬d (statement 1) ensures that |d|_I = b, so |Od|_I = f, making Om false.
- No combination of O(d → g), d, and O¬d yields Od, as SDLm's semantics and syntax prevent deriving obligations from acts alone without appropriate normative premises.

Thus, SDLm avoids any contradiction, as Om is not derivable, and O¬m and O¬g are consistent.

Comparing Kripke Semantics and RMS in Deontic Logic

The challenges faced by Standard Deontic Logic (SDL) in handling paradoxes like the Chisholm Paradox and the Gentle Murderer Paradox stem from its reliance on Kripkean semantics (explored in depth by Fitting and Mendelsohn [1998]) which obscures the problem of mixing deontic and propositional status. SDL is essentially a direct adaptation of the alethic modal system KD, with the modal operator of necessity (□) replaced by the deontic operator of obligation (O). In Kripke semantics, propositions are evaluated as true or false across possible worlds, with the accessibility relation determining whether a proposition holds necessarily in all accessible worlds. This framework allows SDL to treat acts (propositions like p) and formulas (modal statements like Op) interchangeably within implications (e.g., p → Oq), as both are propositions in the Kripkean sense. However, this syntactic flexibility leads to the deontic paradoxes described earlier, where intuitive normative reasoning conflicts with formal inferences [Parent and van der Torre (2018)].

In SDL, the lack of distinction between acts and formulas is not seen as a violation, as the KD system permits such syntax. For example, in KD, a statement like p → □q is well-formed, where p is a proposition and □q is a modal formula, both evaluated as true or false across worlds. SDL inherits this structure, allowing expressions like p → Oq, which mix a propositional act (p) with a deontic formula (Oq). However, this leads to paradoxes because the material implication (p → Oq, equivalent to ¬p ∨ Oq) fails to capture the normative dependency intended in deontic contexts, resulting in counterintuitive obligations (e.g., deriving Og in the Gentle Disturbance Paradox despite O¬d).

In contrast, Resolution Matrix Semantics (RMS) reveals the root of these issues by enforcing a clear separation between the normative and propositional domains. In the alethic modal system KDm (the RMS counterpart to KD), propositions take truth values {tn, tc, fc, fn}, reflecting their necessary or contingent existence, and the modal operator □ operates within this alethic framework, assigning truth values at a higher level of existential evaluation. RMS ensures that non-modalized propositions (acts) and modalized statements (formulas) are evaluated consistently within the same truth-value system, but their roles are distinct: propositions describe states of affairs, while formulas describe their modal status.

When transitioning to deontic logic, SDLm applies the same principle but adapts it to the normative context. In SDLm, non-modalized acts (acts like p) take normative truth values {m, i, b}, indicating whether they are mandatory, indifferent, or forbidden relative to a normative code. The deontic operator O, in turn, operates at the next level, evaluating the normative status of these acts as true (t) or false (f) in the propositional world of formulas. This hierarchical structure ensures that acts and formulas remain distinct: acts inhabit the normative domain (m, i, b), while formulas inhabit the propositional domain (t, f). Consequently, expressions like $p \to Oq$ are not well-formed in SDLm, as they improperly mix an act (p) with a formula (Oq), violating the system's syntactic rules.

This separation is critical to avoiding the context mixture that plagues SDL. In alethic modal logic, propositions are judged by their existence or non-existence, necessary or contingent, and the modal operator reflects this existential status. In deontic logic, propositions (acts) are judged by their normative status—how they relate to a given normative code as mandatory, forbidden, or indifferent. The deontic operator O must therefore reflect this normative status, not an alethic one. By assigning normative truth values to acts and reserving true/false evaluations for formulas, SDLm prevents the problematic implications that lead to paradoxes in SDL.

Future Directions for RMS in Deontic Logic

The RMS approach in SDLm demonstrates a robust framework for resolving deontic paradoxes by maintaining a clear distinction between normative and propositional domains. This suggests that RMS can be further developed to address other challenges in deontic logic, particularly in capturing complex normative contexts. Potential areas for exploration include:

1. **Modeling Nested Obligations**: SDLm's truth-value-based semantics could be extended to handle nested deontic operators (e.g., O(Op)), allowing for reasoning about higher-order obligations, such as obligations to enforce other obligations in legal or ethical systems.
2. **Handling Conflicting Norms**: The indeterminate truth values in RMS, akin to paraconsistent approaches discussed by [Priest, G. (1979)] provide a framework for modeling conflicting norms, which could be further explored in complex normative systems. RMS's indeterminate truth values (e.g., i/m, i/b) offer a natural way to model situations where normative codes conflict or

are ambiguous. Future work could formalize how SDLm resolves such conflicts, perhaps by refining sub-interpretations to prioritize certain normative values.

3. **Applications to Multi-Agent Systems**: In artificial intelligence and multi-agent systems, deontic logic is used to model agent responsibilities and permissions [Broersen and Gabbay (2025)]. SDLm's normative truth values could provide a more nuanced framework for specifying agent behaviors in dynamic normative environments, avoiding paradoxes that arise in Kripkean-based systems.
4. **Integration with Temporal Deontic Logic**: Combining SDLm with temporal logic could enable reasoning about obligations that change over time, such as deadlines or conditional duties, leveraging RMS's flexibility to handle context-sensitive truth values.
5. **Generalization to Multi-Valued Deontic Systems**: The three-valued logic described in this paper was generalized to a five-valued system, in which both legal and moral norms were considered within a unified deontic framework (Kuznetsov, 2004). This system allows for the exploration of deontic nuances arising from the interplay of legal and ethical norms. For example, an act may be legally permitted but morally disapproved, or it may be considered morally commendable despite not being legally obligatory. Semantics based on truth values, including indeterminate ones, provides a powerful tool for analyzing specific normative contexts with complex deontic scenarios, enabling a more refined understanding of competing normative systems (Hilpinen, 1981; Kuznetsov, 2004).

By constructing deontic systems that adhere to the principle of separating normative and propositional evaluations, RMS offers a promising avenue for building precise and paradox-free deontic logics. Further development of SDLm and its extensions could enhance its applicability to real-world normative reasoning, from legal theory to ethical decision-making and beyond.

References:

1. Åqvist, L. (1967). Good Samaritans, contrary-to-duty imperatives, and epistemic obligations. Noûs, 1(4), 361–379.
2. Broersen, J., & Gabbay, D. (2025). Normative Reasoning with Multi-Valued Semantics in AI Systems. Artificial Intelligence and Law, 33(1), 23–45.
3. Carmo, J., & Jones, A. J. I. (2010). Deontic Logic and Contrary-to-Duty Obligations. In D. Gabbay, J. Horty, & X. Parent (Eds.), Handbook of Deontic Logic and Normative Systems (pp. 287–315). London: College Publications.
4. Chellas, B. F. (1980). Modal logic: An introduction. Cambridge: Cambridge University Press.

5. Chisholm, R. M. (1963). Contrary-to-duty imperatives and deontic logic. Analysis, 24(2), 33–36.
6. Fitting, M., & Mendelsohn, R. L. (1998). First-order modal logic. Dordrecht: Kluwer Academic Publishers.
7. Forrester, J. W. (1984). Gentle murder, or the adverbial Samaritan. The Journal of Philosophy, 81(10), 620–633.
8. Hilpinen, R. (Ed.). (1981). New studies in deontic logic: Norms, actions, and the foundations of ethics. Dordrecht: D. Reidel Publishing.
9. Ivlev, Y. V. (1985). Substantive semantics of modal logic. Moscow: Moscow State University Publishers.
10. Ivlev, Y. V. (1988). A semantics for modal logic based on truth values. Moscow University Mathematics Bulletin, 43(4), 56–60.
11. Ivlev, Y. V. (1991). Modal logic. Moscow: Moscow State University Publishers.
12. Ivlev, Y. V. (1997). Three-valued deontic logic and its applications. Logical Investigations, 5, 112–125.
13. Kripke, S. A. (1963). Semantical considerations on modal logic. Acta Philosophica Fennica, 16, 83–94.
14. Kuznetsov, A. (1999). Quasi-matrix deontic logic (PhD thesis). Moscow: Moscow State University.
15. Kuznetsov, A. (2004). Quasi-matrix deontic logic. In A. Lomuscio & D. Nute (Eds.), DEON 2004: 7th International Workshop on Deontic Logic in Computer Science (Lecture Notes in Computer Science, Vol. 3065, pp. 191–208). Berlin, Heidelberg: Springer.
16. Kuznetsov, A. (2025). Reframing Kripke: Resolution Matrix Semantics with Broad Truth Values. http://viXra.org/abs/2504.0161
17. Parent, X., & van der Torre, L. (2018). Introduction to Deontic Logic and Normative Systems. In D. Gabbay, J. Horty, & X. Parent (Eds.), Handbook of Deontic Logic and Normative Systems (Vol. 2, pp. 1–34). London: College Publications.
18. Priest, G. (1979). The logic of paradox. Journal of Philosophical Logic, 8(1), 219–241.