Approaching Legendre's conjecture within a limited boundary

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Abstract (203 words)

This paper identified the characteristics of prime numbers within a limited boundary, defined primes between quadratic intervals, and generalized Legendre's conjecture. Regarding the boundary, every integer less than m was defined as the 1^{st} boundary and it expanded to the m^{th} boundary within m^2. Thus, each boundary contained m elements. Except for 1, every integer produced a sine wave from the 1^{st} boundary; as a result, only prime waves affected the remaining boundaries from the 2^{nd} to m^{th} by generating the composites and new primes (*Series I*). Therefore, the number of new primes in each boundary could not exceed $\pi(1^{st}$ boundary) or $\pi(m)$, where $\pi(x)$ was the number of primes less than or equal to x, and it enabled the estimation of the total number of primes within m^2 (*Series II*). Based on *Series I* and *II*, the quadratic intervals between $\pi(m^2)$ and $\pi((m + 1)^2)$ were identical to the sum of the last two boundaries, expressed as $2 \cdot \beta_{m+1} \cdot \pi(m)$, where β_{m+1} was the ratio of $\pi((m + 1)^2)$ to $\pi(m + 1) \cdot (m + 1)$ (*Series III*). This led to the conclusion that Legendre's conjecture satisfied while

$$0.8986 \cdot \pi(P) < 2 \cdot \beta_P \cdot \pi(P) < \pi(P) \text{ (prime } P > 113\text{), or}$$

$$2 \cdot \beta_{m+1} \cdot \pi(m) \leq 2 \cdot \pi(m) \text{ (integer } m \geq 2\text{).}$$

Keywords: Boundary, Composites, Legendre's conjecture, New primes, Primes

1. Introduction

Including Goldbach and the twin prime conjectures, Legendre's conjecture, which stated that there was a prime in the quadratic interval, was listed as one of the unsolved problems in the field of prime research [2]. Regardless, these problems were related to the characteristics of primes. Therefore, by analyzing the problems through the prime characteristics, it might help to generalize the structure of Legendre's conjecture and extend this understanding to identify similarities with other unsolved problems.

To address Legendre's conjecture, the characteristics of primes were initially identified [1] through the cause-and-effect relationship among primes, composites, and new primes within a limited boundary in *Series I*. Based on these characteristics, the median number of primes per boundary was estimated, which enabled to estimation of the total number of primes either m^2 or $(m + 1)^2$ in *Series II*. Combining the *Series I* and *II*, the number of primes between m^2 and $(m + 1)^2$ was estimated in *Series III*, thereby supporting a proof of Legendre's conjecture.

2. Materials and Methods

The sine waves and plotted data were visualized using an online graphing calculator (Desmos, www.desmos.com). The visualized images were exported, modified in Illustrator (CS6, Adobe, CA, USA), and used for Figures 1 and 2. The number of primes and n^{th} primes were identified from an online resource (The N^{th} Prime Page, www.t5k.org).

2.1. Series I: Characteristics of primes within a limited boundary

In this paper, a boundary was defined as follows: when an arbitrary positive integer n was chosen, the set of consecutive integers less than n was considered the 1^{st} boundary. Except for 1, each integer produced a sine wave in the form of '$Y_n = sin\,(\frac{180}{n} \cdot x)$' (Figure 1A), as shown below

$$Y_2 = sin\,(\frac{180}{2} \cdot x),\ Y_3 = sin\,(\frac{180}{3} \cdot x),\ Y_4 = sin\,(\frac{180}{4} \cdot x),\ Y_5 = sin\,(\frac{180}{5} \cdot x), \ldots,\ y_n = sin\left(\frac{180}{n} \cdot x\right).$$

On the *x-axis*, composite waves such as Y_4 overlapped with prime waves like Y_2. Thus, the product of '$Y_2 \cdot Y_3 \cdot Y_4 \cdot Y_5 \cdot \ldots \cdot Y_n$' and '$Y_2 \cdot Y_3 \cdot Y_5 \cdot \ldots \cdot Y_P$' yielded the same result (Figure 1B), and this relationship was expressed as

$$\prod_{2 \le n} sin\,(\frac{180}{n} \cdot x) = \prod_{P \le n} sin\left(\frac{180}{P} \cdot x\right)$$

, where P represented the primes less than or equal to n in the 1^{st} boundary. The product of prime waves in the 1^{st} boundary directly generated or connected the composites between the 2^{nd} and n^{th} boundaries within n^2. Thus, the primes in the 1^{st} boundary and the composites in the remaining boundaries were connected by a cause-and-effect relationship.

If 'Y_1' or '$sin\,(\frac{180}{1} \cdot x)$' was divided by the product of prime waves, the primes in the 1^{st} boundary and the composites between the 2^{nd} and n^{th} boundaries could not be defined, while the specific odd numbers were passively remained on the *x-axis* and they were all new primes (Figure 1C). Therefore, the primes in the 1^{st} boundary and the new primes between the 2^{nd} and n^{th} boundaries were also indirectly connected by the cause-and-effect relationship. Overall, the cause-and-effect relationship between primes in the 1^{st} boundary and the composites and new primes between the 2^{nd} and n^{th} boundaries within n^2 was expressed as

$$\frac{sin(180 \cdot x)}{\prod_{P \le n} sin\,(\frac{180}{P} \cdot x)} = 0.$$

2.2. Series II: Estimation of the total number of primes within a limited boundary

The number of primes in the 1^{st} boundary limited the number of new primes between the 2^{nd} and n^{th} boundaries. Thus, a cause-and-effect relationship between $\pi(1^{st}$ boundary) and $\pi(n^{th}$ boundary) was expressed as

$$\pi(1^{st}\text{ boundary}) = \frac{1}{\gamma}\cdot\pi(n^{th}\text{ boundary})$$

, where $\pi(x)$ denoted the number of primes less than or equal to x and $0 < \gamma \leq 1$. As the boundary increased from the 1^{st} to n^{th} within n^2, more primes involved, from $\pi(\sqrt{n})$ to $\pi(\sqrt{n^2})$, which produced more composites and affected a decreasing pattern in the number of new primes per boundary. Additionally, the waves of primes in the 1^{st} boundary, except for 2, had an asymmetric structure with respect to composites in the 2^{nd} boundary. Considering the rhythmic wave of primes in *Series I*, the asymmetric structure between 1^{st} and 2^{nd} could expand into adjacent boundaries, such as the 2^{nd} and 3^{rd}, 3^{rd} and 4^{th}, and so on, until it reached the $(n-1)^{th}$ and n^{th}. Due to the asymmetric structure, the primes in the 1^{st} boundary could not produce or connect entire composites in the 2^{nd} boundary; as a result, at least one new prime should exit in the 2^{nd} boundary and it symmetrically but partially matched to the primes in the 1^{st} boundary. Similarly, the passively remaining new primes were also symmetrically but partially paired between adjacent boundaries. Overall, the number of passively remaining new primes decreased and showed oscillatory behavior. In other words, $\pi(1^{st}$ boundary) was always the maximum while $\pi(n^{th}$ boundary) approached close to the minimum. By averaging the maximum $\pi(1^{st}$ boundary) and the near-minimum $\pi(n^{th}$ boundary), the median number of primes per boundary was estimated as

$$\frac{\pi(1^{st}\ boundary) + \pi(n^{th}\ boundary)}{2}$$

, and it enabled the estimation of the total number of primes within n^2 by multiplying the total number of boundaries as

$$\left(\frac{\pi(1^{st}\ boundary) + \pi\left(n^{th}\ boundary\right)}{2}\right) \cdot n_B = \pi(n^2)$$

, where n_B was the total number of boundaries. As $\pi(n^{th}$ boundary) could be replaced with $\gamma \cdot \pi(1^{st}$ boundary), the total number of primes within n^2 was expressed as follows

$$\left(\frac{\pi(1^{st}\ boundary) + \gamma \cdot \pi(1^{st}\ boundary)}{2}\right) \cdot n_B = \pi(n^2)$$

$$\left(\frac{1+\gamma}{2}\right) \cdot \pi(1^{st}\ boundary) \cdot n_B = \pi(n^2)$$

$$\frac{1+\gamma}{2} = \frac{\pi(n^2)}{\pi(1^{st}\ boundary) \cdot n_B}.$$

γ was defined between 0 and 1, so $\frac{1+\gamma}{2}$ could be defined as follows

$$0 < \gamma \leq 1$$

$$1 < \gamma + 1 \leq 2$$

$$0.5 < \frac{1+\gamma}{2} \leq 1.$$

As $\frac{1+\gamma}{2}$ was identical to the prime ratio, thus,

$$0.5 < \frac{\pi(n^2)}{\pi(1^{st}\ boundary) \cdot n_B} \leq 1.$$

Also, the 1^{st} boundary contained n elements and the total number of boundaries, n_B, was n within n^2. Therefore, the above prime ratio was simplified as

$$0.5 < \frac{\pi(n^2)}{\pi(n) \cdot n} \leq 1.$$

2.3. Series III: Approaching Legendre's conjecture using the characteristics of primes

Let $\beta = \frac{1+\gamma}{2}$, then, β_n was expressed as follows

$$\beta_n = \frac{\pi(n^2)}{\pi(n) \cdot n}$$

$$\pi(n^2) = \beta_n \cdot \pi(n) \cdot n$$

, where $\pi(n^2)$ was the total number of primes, $\pi(n)$ was the number of primes in the 1st boundary, and n was the total number of boundaries within n^2. Using the characteristics of primes from *Series I*, $\pi(n^2)$ was reorganized based on the number of primes in the 1st boundary with m elements by increasing the total number of boundaries from n to $(n + 1)$, where $n = (m + 1)$ and $m \geq 2$ (Figure 1D). Thus, the total number of primes within n^2 was expressed by either $\pi(n)$ or $\pi(m)$, as follows

$$\pi(n^2) = \beta_n \cdot \pi(n) \cdot n$$

$$= \beta_n \cdot \pi(n - 1) \cdot (n + 1)$$

$$= \beta_{m+1} \cdot \pi(m) \cdot (m + 2)$$

Within n^2, the total number of primes less than m^2 could also be expressed using $\pi(m)$ and β_{m+1} as

$$\pi(m^2) = \beta_{m+1} \cdot \pi(m) \cdot m$$

, and it led to Legendre's conjecture, which stated the number of primes between $\pi(m^2)$ and $\pi(n^2)$, as follows

$$\pi(n^2) - \pi(m^2) = \beta_{m+1} \cdot \pi(m) \cdot (m + 2) - \beta_{m+1} \cdot \pi(m) \cdot m$$

$$= 2 \cdot \beta_{m+1} \cdot \pi(m).$$

Using the defined β_{m+1}, which ranged between 0.5 and 1 in *Series II*, Legendre's conjecture, expressed as $2 \cdot \beta_{m+1} \cdot \pi(m)$, was written as follows

$$0.5 < \beta_{m+1} \leq 1$$

$$1 < 2 \cdot \beta_{m+1} \leq 2$$

$$\pi(m) < 2 \cdot \beta_{m+1} \cdot \pi(m) \leq 2 \cdot \pi(m).$$

3. Results

Within a limited n^2 boundary, the integers $(m^2 - 1)$, m^2, $(n^2 - 1)$, and n^2 should not be primes, where $n = (m + 1)$, as they were composed of at least two different prime factors or formed quadratic expressions (red areas in Figure 1D). As a result, Legendre's conjecture was identical to the sum of the last two boundaries, regardless of whether the 1st boundary consisted of n or m elements within n^2 (Figure 1D). Therefore, the expression of Legendre's conjecture, $2 \cdot \beta_{m+1} \cdot \pi(m)$, in *Series III* was identical to the sum of the last two boundaries either $\pi((n-1)^{\text{th}}) + \pi(n^{\text{th}})$ (1st Legendre's conjecture) or $\pi((m+1)^{\text{th}}) + \pi((m+2)^{\text{th}})$ (2nd Legendre's conjecture).

Considering the characteristics of primes, π(1st boundary) showed the maximum value, while the number of new primes per boundary showed a decreasing pattern with oscillation. As a result, the last two boundaries approached near-minimum values. Logically, therefore, π(1st boundary) and the last two boundaries, $\pi((m+1)^{\text{th}})$ and $\pi((m+2)^{\text{th}})$, could be expressed as

$$\pi(1^{\text{st}} \text{ boundary}) = \pi(m) = \pi(m)_{\max}, \text{ and}$$

$$\pi((m+1)^{\text{th}}) \approx \pi((m+2)^{\text{th}}) \rightarrow \pi(m)_{\min}.$$

; it led to reorganize Legendre's conjecture defined in *Series III* as follows

$$\pi(m) < 2 \cdot \beta_{m+1} \cdot \pi(m) \leq 2 \cdot \pi(m)$$

$$\pi(m) < \pi(m+1)^{\text{th}} + \pi(m+2)^{\text{th}} \leq 2 \cdot \pi(m)$$

$$\pi(m)_{\max} < 2 \cdot \pi(m)_{\min} \leq 2 \cdot \pi(m)_{\max}.$$

The approximated value of $2 \cdot \pi(m)_{\min}$ was always less than or equal to $2 \cdot \pi(m)_{\max}$; thus, it followed that

$$2 \cdot \pi(m)_{\min} \leq 2 \cdot \pi(m)_{\max}, \text{ or } 2 \cdot \beta_{m+1} \cdot \pi(m) \leq 2 \cdot \pi(m) \quad \text{[True]}$$

, while $2 \cdot \pi(m)_{\min}$ could not be directly compared with $\pi(m)_{\max}$, it remained debatable whether or not

$$\pi(m)_{\max} < 2 \cdot \pi(m)_{\min}, \text{ or } \pi(m) < 2 \cdot \beta_{m+1} \cdot \pi(m) \qquad \text{[Debatable].}$$

3.1. A result on the minimal number in Legendre's conjecture

If m was 2, then β_3 and $\pi(2)$ were 0.6667 and 1, respectively. Thus, Legendre's conjecture, was satisfied as

$$\pi(2) < 2 \cdot \beta_3 \cdot \pi(2) \leq 2 \cdot \pi(2)$$

$$1 < 1.3334 \leq 2 \quad \text{[True]}.$$

3.2. A result on an arbitrarily large number in Legendre's conjecture

If m was an arbitrarily large number 5477224, then $\beta_{5477225}$ and $\pi(5477224)$ were 0.4814 and 379333, respectively. Thus, Legendre's conjecture was ranged as

$$\pi(5477224) < 2 \cdot \beta_{5477225} \cdot \pi(5477224) \leq 2 \cdot \pi(5477224)$$

$$379333 < 365221.8124 \leq 758666$$

, and it partially satisfied Legendre's conjecture as shown below

$$365221.8124 \leq 758666 \quad \text{[True]}$$

$$379333 < 365221.8124 \quad \text{[Not true]}.$$

4. Discussions

Using the characteristics of primes in *Series I*, the total number of primes was estimated using β_{m+1} in *Series II*, and it led to conclude that Legendre's conjecture was identical to the sum of the last two boundaries, expressed as $2 \cdot \beta_{m+1} \cdot \pi(m)$, where $m \geq 2$, and defined in *Series III* as

$$2 \cdot \beta_{m+1} \cdot \pi(m) \leq 2 \cdot \pi(m).$$

In this section, it would be discussed whether the defined Legendre's conjecture in *Series III* could be narrowed with the actual value of β_{m+1}.

In *Series III*, β_{m+1} was defined as

$$\beta_{m+1} = \frac{\pi((m+1)^2)}{\pi(m+1)\cdot(m+1)}$$

, where $\pi((m+1)^2)$ was the total number of primes, $\pi(m + 1)$ was the number of primes in the 1st boundary, and $(m + 1)$ was the total number of boundaries within $(m + 1)^2$. Let any positive integer $(m + 1)$ was placed between two consecutive primes as

$$P_1 < (m + 1) < P_2$$

, where P_1 and P_2 were primes, then, the number of primes was expressed as

$$\pi(P_1) = \pi(m + 1) < \pi(P_2).$$

As $(m + 1)$ increased, $\pi((m + 1)^2)$ constantly increased, while $\pi(m + 1)$ was equal to $\pi(P_1)$. Consequently, the relationship between β_{P1} and β_{m+1} was expressed as

$$\beta_{P1} < \beta_{m+1}$$

, and it implied that

$$2\cdot\beta_{P1}\cdot\pi(P_1) < 2\cdot\beta_{m+1}\cdot\pi(m).$$

However, it did not imply the relationship between β_{P1} and β_{P2} due to the variable number of integers between two consecutive primes. Overall, it was possible to conclude that $2\cdot\beta_{P1}\cdot\pi(P_1)$ could represent the minimum value of $2\cdot\beta_{m+1}\cdot\pi(m)$ in Legendre's conjecture.

Using the 168 consecutive primes between 2 and 1009, actual $2\cdot\beta_P\cdot\pi(P)$ was calculated and plotted; it showed that β_2 initiated at 1 (maximum), decreased to 0.4493 at β_{113} (minimum), and then increased with oscillatory behavior (Figure 2). Using the prime counting function, ideal β_P was calculated as follows

$$\beta_P = \frac{\pi(P^2)}{\pi(P)\cdot P} = \frac{\frac{P^2}{ln(P^2)}}{\frac{P}{ln(P)}\cdot P} = \frac{P^2\cdot ln(P)}{2\cdot P^2\cdot ln(P)} = 0.5$$

, where $\pi(P) = \frac{P}{ln(P)}$ in the prime counting function, and this supported the hypothesis that the minimum value of β_{113}, 0.4493, would converge to 0.5 as P increased. As infinite many primes increased [3] after 113, the hypothesis was acceptable, allowing β_P to range

$$0.4493 < \beta_P < 0.5$$

, where $P > 113$. Therefore, Legendre's conjecture, expressed as $2 \cdot \beta_P \cdot \pi(P)$, could be narrowed with primes as

$$0.8986 \cdot \pi(P) < 2 \cdot \beta_P \cdot \pi(P) < \pi(P)$$

, where prime $P > 113$.

5. Conclusions

Within a limited n^2 boundary, the number of primes in the 1st boundary limited the number of new primes between the 2nd and nth boundaries. Therefore, the number of new primes in the nth (last) boundary should not exceed the number of primes in the 1st boundary. Considering the asymmetrically paired primes in the 1st boundary and the composites in the 2nd boundary, except the prime 2, at least one new prime should exist, which was symmetrically but partially paired with the primes in the 1st boundary. This relationship between the 1st and 2nd boundaries could expand to other adjacent boundaries, including the last two $(n-1)$th and nth boundaries, thereby ensuring that at least one new prime existed per boundary, which qualitatively satisfied Legendre's conjecture.

The prime characteristics within a limited boundary also implied that the number of primes in Legenedre's conjecture was defined by the sum of the last two boundaries, expressed as

$$2 \cdot \beta_{m+1} \cdot \pi(m) \leq 2 \cdot \pi(m)$$

, where integer $m \geq 2$. If the consecutive primes were applied instead of m, Legendre's conjecture was narrowed and quantitatively satisfied while

$$0.8986 \cdot \pi(P) < 2 \cdot \beta_P \cdot \pi(P) < \pi(P)$$

, where prime $P > 113$.

5. Reference

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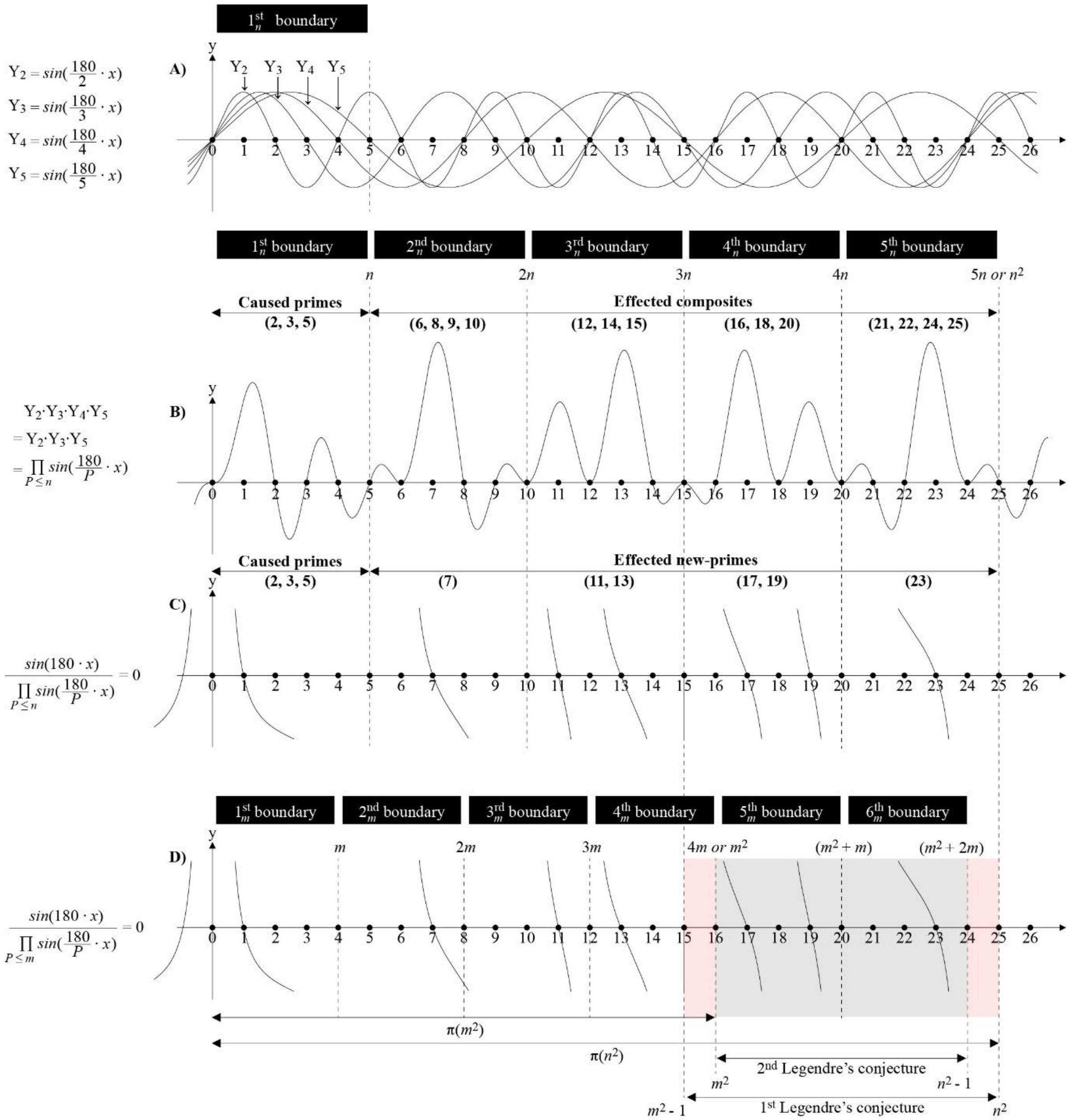
Figure 1. Approaching to Legendre's conjecture using the characteristics of primes. A) Any positive integer less than n ($n \geq 3$) was defined as the 1^{st} boundary, and each integer produced sine waves, except for 1. B) The product of prime waves was directly connected to the composites between the 2^{nd} and n^{th} boundaries within n^2. C) The wave of '$sin(180 \cdot x)$' was divided by the product of prime waves, and the connected primes and composites could not be defined, while specific odd numbers passively remained on the *x-axis*; these were all new primes between the 2^{nd} and n^{th} boundaries. D) Considering the prime-free red areas, Legendre's conjecture was shown to be identical to the sum of the last two boundaries, regardless of whether n (1^{st} Legendre's conjecture) or m (2^{nd} Legendre's conjecture) elements were used in the 1^{st} boundary.



1^{st}_n boundary
$Y_2 = sin(\frac{180}{2} \cdot x)$
$Y_3 = sin(\frac{180}{3} \cdot x)$
$Y_4 = sin(\frac{180}{4} \cdot x)$
$Y_5 = sin(\frac{180}{5} \cdot x)$
A)
1^{st}_n boundary
2^{nd}_n boundary
3^{rd}_n boundary
4^{th}_n boundary
5^{th}_n boundary
n
$2n$
$3n$
$4n$
$5n$ or n^2
Caused primes
(2, 3, 5)
Effected composites
(6, 8, 9, 10)
(12, 14, 15)
(16, 18, 20)
(21, 22, 24, 25)
B)
$Y_2 \cdot Y_3 \cdot Y_4 \cdot Y_5$
$= Y_2 \cdot Y_3 \cdot Y_5$
$= \prod_{P \leq n} sin(\frac{180}{P} \cdot x)$
Caused primes
(2, 3, 5)
Effected new-primes
(7)
(11, 13)
(17, 19)
(23)
C)
$\frac{sin(180 \cdot x)}{\prod_{P \leq n} sin(\frac{180}{P} \cdot x)} = 0$
1^{st}_m boundary
2^{nd}_m boundary
3^{rd}_m boundary
4^{th}_m boundary
5^{th}_m boundary
6^{th}_m boundary
D)
m
$2m$
$3m$
$4m$ or m^2
$(m^2 + m)$
$(m^2 + 2m)$
$\frac{sin(180 \cdot x)}{\prod_{P \leq m} sin(\frac{180}{P} \cdot x)} = 0$
$\pi(m^2)$
$\pi(n^2)$
2nd Legendre's conjecture
m^2
$n^2 - 1$
$m^2 - 1$
1st Legendre's conjecture
n^2

Figure 2. The actual values of β_P were plotted, where P was 168 consecutive primes between 2 and 1009. The maximum value, β_2, was 1, and it gradually decreased to a minimum of 0.4493 at β_{113}. It was hypothesized that β_P converged to 0.5, based on the prime counting function between $\pi(P^2)$ and $\pi(P)\cdot P$. Since infinitely many primes are known to exist, the hypothesis was acceptable, and β_P was defined within the range $0.4493 < \beta_P < 0.5$, where prime $P > 113$.