Special Relativity: Types of Energy

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In special relativity, this paper presents the definitions of kinetic energy, rest energy and relativistic energy for a single particle (massive or non-massive) Additionally, this paper also presents the definitions of generalized relativistic energy and total energy for a system of particles (massive and non-massive)

Introduction

In special relativity, this paper is obtained starting from the essential definitions of intrinsic mass (or invariant mass) and relativistic factor (or frequency factor) for massive particles and non-massive particles.

The intrinsic mass (m) and the relativistic factor (f) of a massive particle, are given by:

$$m \doteq m_o \tag{1}$$

$$f \doteq \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}\right)^{-1/2} \tag{2}$$

where (m_o) is the rest mass of the massive particle, ($\mathbf{v}$) is the velocity of the massive particle, and (c) is the speed of light in vacuum.

The intrinsic mass (m) and the relativistic factor (f) of a non-massive particle, are given by:

$$m \doteq \frac{h\,\kappa}{c^2} \tag{3}$$

$$f \doteq \frac{\nu}{\kappa} \tag{4}$$

where (h) is the Planck constant, (ν) is the frequency of the non-massive particle, (κ) is a positive universal constant with dimension of frequency, and (c) is the speed of light in vacuum.

According to this paper, a massive particle ($m_o \neq 0$) is a particle with non-zero rest mass (or alternatively is a particle whose speed v in vacuum is less than c) and a non-massive particle ($m_o = 0$) is a particle with zero rest mass (or alternatively is a particle whose speed v in vacuum is c)

Note : The rest mass (m_o) and the intrinsic mass (m) are in general not additive, and the relativistic mass (m) of a particle (massive or non-massive) is given by : ($\mathrm{m} \doteq m\,f$)

The Einsteinian Kinematics

The special position ($\bar{\mathbf{r}}$) the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a particle (massive or non-massive) are given by:

$$\bar{\mathbf{r}} \doteq \int f\,\mathbf{v}\,dt \tag{5}$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = f\,\mathbf{v} \tag{6}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v} \tag{7}$$

where (f) is the relativistic factor of the particle, ($\mathbf{v}$) is the velocity of the particle, and (t) is the (coordinate) time.

The Einsteinian Dynamics

If we consider a particle (massive or non-massive) with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net Einsteinian force ($\mathbf{F}_{\mathrm{E}}$) acting on the particle, the work (W) done by the net Einsteinian force acting on the particle, the kinetic energy (K) of the particle, the rest energy (E_o) of the particle, and the relativistic energy (E) of the particle, they are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,f\,\mathbf{v} \tag{8}$$

$$\mathbf{L} \doteq \mathbf{r}\times\mathbf{P} = m\,\mathbf{r}\times\bar{\mathbf{v}} = m\,f\,\mathbf{r}\times\mathbf{v} \tag{9}$$

$$\mathbf{F}_{\mathrm{E}} = \frac{d\mathbf{P}}{dt} = m\,\bar{\mathbf{a}} = m\left[f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}\right] \tag{10}$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F}_{\mathrm{E}}\cdot d\mathbf{r} = \Delta\,\mathrm{K} = \Delta\,\mathrm{E} \tag{11}$$

$$\mathrm{K} \doteq m\,f\,c^2 - m_o\,c^2 \tag{12}$$

$$\mathrm{E}_o \doteq m_o\,c^2 \tag{13}$$

$$\mathrm{E} \doteq \mathrm{K} + \mathrm{E}_o = m\,f\,c^2 \tag{14}$$

where (f, $\mathbf{r}$, $\mathbf{v}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the relativistic factor, the position, the velocity, the special velocity and the special acceleration of the particle, (t) is the (coordinate) time, and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at rest is always zero since in this dynamics the relativistic energy (E) and the kinetic energy (K) of a massive particle are not the same ($\mathrm{E} \neq \mathrm{K}$) [Note : in non-massive particle : $m_o = 0$, therefore : $\mathrm{E}_o = 0$]

Note : $\mathrm{E}^2 - \mathbf{P}^2 c^2 = m^2 f^2 c^4\,(1 - \mathbf{v}^2/c^2)$ [in massive particle : $f^2\,(1 - \mathbf{v}^2/c^2) = 1 \rightarrow \mathrm{E}^2 - \mathbf{P}^2 c^2 = {m_o}^2 c^4$ and $m \neq 0$] & [in non-massive particle : $\mathbf{v}^2 = c^2 \rightarrow (1 - \mathbf{v}^2/c^2) = 0 \rightarrow \mathrm{E}^2 - \mathbf{P}^2 c^2 = 0$ and $m \neq 0$]

General Observations

In classical mechanics, the kinetic energy (K) of a massive particle (m_o) is given by the following indefinite integral : K $= \int m_o \, \mathbf{a} \cdot d\mathbf{r} \;=\; {}^1\!/_2 \, m_o \, v^2 + \text{constant}$

In special relativity, the kinetic energy (K) of a massive particle (m_o) is given by the following indefinite integral : K $= \int m_o \, \bar{\mathbf{a}} \cdot d\mathbf{r} \;=\; m_o \, f \, c^2 + \text{constant}$

Basically, there are two criteria for choosing a value for the constant of integration of the kinetic energy. The first criterion states that the constant of integration must be such that the kinetic energy of any massive particle at rest must always be zero and the second criterion simply states that the constant of integration must always be zero.

In classical mechanics, both criteria lead to the same result : ($\mathrm{K} \doteq {}^1\!/_2 \, m_o \, v^2$) However, in special relativity, the first and second criteria lead to different results, with the first criterion : ($\mathrm{K} \doteq m_o \, f \, c^2 - m_o \, c^2$) and with the second criterion : ($\mathrm{K} \doteq m_o \, f \, c^2$)

The first and second criteria are arbitrary since (in classical mechanics and also in special relativity) the kinetic energy of a particle depends on the speed of the particle and also on the mass of the particle. For example, a particle A can have more kinetic energy than another particle B whose speed is much higher than the speed of particle A.

Using the first criterion simply because this type of energy is identified with the adjective 'kinetic' (pertaining to motion) is even more arbitrary, since this type of energy would be better identified if it were called, for example : mass-speed energy.

On the other hand, according to this paper, each type of energy is associated with a type of force. If we use the second integration criterion then the kinetic energy (K) would be associated with the net kinetic force ($\mathbf{K}$) (see Annex II) However, if we use the first integration criterion then the kinetic energy (K) would also be associated with the net kinetic force ($\mathbf{K}$) but the rest energy (E_o) would be associated with another type of force that would act only on massive particles and whose value would always be zero.

Throughout this paper we will use the second integration criterion (because it is the easiest to use from a theoretical point of view) Therefore, in special relativity, the kinetic energy (K_o) of a massive particle at rest is ($m_o \, c^2$) since using the second integration criterion the kinetic energy (K) and the relativistic energy (E) of a single massive or non-massive particle are the same ($\mathrm{K} \doteq m \, f \, c^2 = \mathrm{E}$) { [1] [2] [3] [Annex I] [Annex II] }

Finally (see Annex I) the generalized relativistic energy (E) and the total energy (T) of a system of particles (massive and non-massive) are defined. Here, for example, a system of particles composed only of a single massive or non-massive particle can have non-kinetic energy (also called 'potential' energy) (in this example : $\sum \mathrm{E}_{\mathrm{nke}}$)

On the other hand, in massive systems (of particles) there are apparently at least two types of ordinary velocities : ($\mathbf{V}_{\mathrm{E}} \doteq \mathbf{P} \, c^2 \, \mathrm{E}^{-1}$) [19] and ($\mathbf{V}_{\mathrm{K}} \doteq \mathbf{P} \, c^2 \, \mathrm{K}^{-1}$) where ($\mathrm{K} \doteq \sum m_i \, f_i \, c^2$) $\rightarrow$ ($\mathbf{V}_{\mathrm{E}} \, \mathrm{E} = \mathbf{V}_{\mathrm{K}} \, \mathrm{K}$) ($\mathbf{V}_{\mathrm{E}} = \mathbf{V}_{\mathrm{K}} \, \varepsilon$) ($\varepsilon \doteq \mathrm{K} / \mathrm{E}$) In this paper, we will use : ($\mathbf{V}_{\mathrm{E}}$) ($\mathbf{V}_{\mathrm{E}} = \mathbf{V}$)

References & Bibliography

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Annex I

System of Particles

In special relativity, the generalized relativistic energy (E) the total energy (T) the linear momentum ($\mathbf{P}$) the rest mass (M_o) and the velocity ($\mathbf{V}$) of any massive or non-massive system (of particles) are given by:

$$\mathrm{E} \doteq \sum m_i\, f_i\, c^2 + \sum \mathrm{E}_{\mathrm{nki}} \tag{15}$$

$$\mathrm{T} \doteq \sum m_i\, f_i\, c^2 + \sum \mathrm{E}_{\mathrm{nki}} + \sum \mathrm{E}_{\mathrm{nke}} \tag{16}$$

$$\mathbf{P} \doteq \sum m_i\, f_i\, \mathbf{v}_i \tag{17}$$

$$\mathrm{M}_o^2\, c^4 \doteq \mathrm{E}^2 - \mathbf{P}^2 c^2 \tag{18}$$

$$\mathbf{V} \doteq \mathbf{P}\, c^2\, \mathrm{E}^{-1} \tag{19}$$

where (m_i, f_i, $\mathbf{v}_i$) are the intrinsic mass, the relativistic factor and the velocity of the i-th massive or non-massive particle of the system, ($\sum \mathrm{E}_{\mathrm{nki}}$) is the total non-kinetic energy of the system which contributes to the rest mass (M_o) of the system, ($\sum \mathrm{E}_{\mathrm{nke}}$) is the total non-kinetic energy of the system which does not contribute to the rest mass (M_o) of the system, and (c) is the speed of light in vacuum. Note : in non-massive systems : ($\sum \mathrm{E}_{\mathrm{nki}} = 0$)

The intrinsic mass (M) and the relativistic factor (F) of a massive system (composed of massive particles or non-massive particles, or both at the same time) are given by:

$$\mathrm{M} \doteq \mathrm{M}_o \tag{20}$$

$$\mathrm{F} \doteq \left(1 - \frac{\mathbf{V} \cdot \mathbf{V}}{c^2}\right)^{-1/2} \tag{21}$$

where (M_o) is the rest mass of the massive system, ($\mathbf{V}$) is the velocity of the massive system, and (c) is the speed of light in vacuum.

The intrinsic mass (M) and the relativistic factor (F) of a non-massive system (composed only of non-massive particles, all with the same vector velocity $\mathbf{c}$) are given by:

$$\mathrm{M} \doteq \frac{h\, \kappa\, n}{c^2} \tag{22}$$

$$\mathrm{F} \doteq \frac{1}{\kappa\, n} \sum \nu_i \tag{23}$$

where (h) is the Planck constant, (ν_i) is the frequency of the i-th non-massive particle of the non-massive system, (κ) is a positive universal constant with dimension of frequency, (n) is the number of non-massive particles of the non-massive system, and (c) is the speed of light in vacuum.

According to this paper, a massive system ($\mathrm{M}_o \neq 0$) is a system with non-zero rest mass (or a system whose speed V in vacuum is less than c) and a non-massive system ($\mathrm{M}_o = 0$) is a system with zero rest mass (or a system whose speed V in vacuum is c)

Note : The rest mass (M_o) and the intrinsic mass (M) are in general not additive, and the relativistic mass ($\mathtt{M}$) of a system (massive or non-massive) is given by : ($\mathtt{M} \doteq \mathrm{M\,F}$)

The Einsteinian Kinematics

The special position ($\bar{\mathbf{R}}$) the special velocity ($\bar{\mathbf{V}}$) and the special acceleration ($\bar{\mathbf{A}}$) of a system (massive or non-massive) are given by:

$$\bar{\mathbf{R}} \doteq \int \mathrm{F}\,\mathbf{V}\,dt \tag{24}$$

$$\bar{\mathbf{V}} \doteq \frac{d\bar{\mathbf{R}}}{dt} = \mathrm{F}\,\mathbf{V} \tag{25}$$

$$\bar{\mathbf{A}} \doteq \frac{d\bar{\mathbf{V}}}{dt} = \mathrm{F}\,\frac{d\mathbf{V}}{dt} + \frac{d\mathrm{F}}{dt}\,\mathbf{V} \tag{26}$$

where (F) is the relativistic factor of the system, ($\mathbf{V}$) is the velocity of the system, and (t) is the (coordinate) time.

The Einsteinian Dynamics

If we consider a system (massive or non-massive) with intrinsic mass (M) then the linear momentum ($\mathbf{P}$) of the system, the angular momentum ($\mathbf{L}$) of the system, the net Einsteinian force ($\mathbf{F}$) acting on the system, the work (W) done by the net Einsteinian forces acting on the system, the kinetic energy (K) of the system, the generalized relativistic energy (E) of the system, and the total energy (T) of the system, they are given by:

$$\mathbf{P} \doteq \sum \mathbf{p}_i = \sum m_i\,\bar{\mathbf{v}}_i = \sum m_i\,f_i\,\mathbf{v}_i = \mathrm{M}\,\bar{\mathbf{V}} = \mathrm{M}\,\mathrm{F}\,\mathbf{V} \tag{27}$$

$$\mathbf{L} \doteq \sum \mathbf{l}_i = \sum \mathbf{r}_i \times \mathbf{p}_i = \sum m_i\,\mathbf{r}_i \times \bar{\mathbf{v}}_i = \sum m_i\,f_i\,\mathbf{r}_i \times \mathbf{v}_i \tag{28}$$

$$\mathbf{F} = \sum \mathbf{f}_i = \sum \frac{d\mathbf{p}_i}{dt} = \frac{d\mathbf{P}}{dt} = \mathrm{M}\,\bar{\mathbf{A}} = \mathrm{M}\left[\mathrm{F}\,\frac{d\mathbf{V}}{dt} + \frac{d\mathrm{F}}{dt}\,\mathbf{V}\right] \tag{29}$$

$$\mathrm{W} \doteq \sum \int_1^2 \mathbf{f}_i \cdot d\mathbf{r}_i = \sum \int_1^2 \frac{d\mathbf{p}_i}{dt} \cdot d\mathbf{r}_i = \Delta\,\mathrm{K} \tag{30}$$

$$\mathrm{K} \doteq \sum m_i\,f_i\,c^2 \tag{31}$$

$$\mathrm{E} \doteq \sum m_i\,f_i\,c^2 + \sum \mathrm{E}_{\mathrm{nki}} = \mathrm{K} + \sum \mathrm{E}_{\mathrm{nki}} = \mathrm{M}\,\mathrm{F}\,c^2 \tag{32}$$

$$\mathrm{T} \doteq \sum m_i\,f_i\,c^2 + \sum \mathrm{E}_{\mathrm{nki}} + \sum \mathrm{E}_{\mathrm{nke}} = \mathrm{M}\,\mathrm{F}\,c^2 + \sum \mathrm{E}_{\mathrm{nke}} \tag{33}$$

where $(m_i, f_i, \mathbf{r}_i, \mathbf{v}_i, \bar{\mathbf{v}}_i)$ are the intrinsic mass, the relativistic factor, the position, the velocity and the special velocity of the i-th massive or non-massive particle of the system, $(\mathrm{F}, \mathbf{V}, \bar{\mathbf{V}}, \bar{\mathbf{A}})$ are the relativistic factor, the velocity, the special velocity and the special acceleration of the system, ($\sum \mathrm{E}_{\mathrm{nki}}$) is the total non-kinetic energy of the system which contributes to the rest mass (M_o) of the system, ($\sum \mathrm{E}_{\mathrm{nke}}$) is the total non-kinetic energy of the system which does not contribute to the rest mass (M_o) of the system, (t) is the (coordinate) time, and (c) is the speed of light in vacuum. Note : ($\sum \mathrm{E}_{\mathrm{nki}} = 0$) in massive or non-massive particle $\rightarrow$ (E = K) in massive or non-massive particle | Alternative (first criterion) : $\mathrm{K} \doteq \sum (m_i\,f_i\,c^2 - m_{oi}\,c^2)$, $\mathrm{E}_{\dot{o}} \doteq \sum m_{oi}\,c^2$, $\mathrm{E} \doteq \mathrm{K} + \mathrm{E}_{\dot{o}}$, $\mathrm{G} \doteq \mathrm{E} + \sum \mathrm{E}_{\mathrm{nri}} = \mathrm{M}\,\mathrm{F}\,c^2$

Annex II

The Kinetic Forces

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i with intrinsic mass m_i by another particle j with intrinsic mass m_j, is given by:

$$\mathbf{K}^a_{ij} \; = \; - \left[\; \frac{m_i \, m_j}{\mathbb{M}} \, (\, \bar{\mathbf{a}}_i \, - \, \bar{\mathbf{a}}_j \,) \; \right] \tag{34}$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i, $\bar{\mathbf{a}}_j$ is the special acceleration of particle j and $\mathbb{M}$ ($= \sum_z^{All} m_z$) is the sum of the intrinsic masses of all the particles of the Universe.

On the other hand, the kinetic force $\mathbf{K}^u_i$ exerted on a particle i with intrinsic mass m_i by the Universe, is given by:

$$\mathbf{K}^u_i \; = \; - \, m_i \, \frac{\sum_z^{All} m_z \, \bar{\mathbf{a}}_z}{\sum_z^{All} m_z} \tag{35}$$

where m_z and $\bar{\mathbf{a}}_z$ are the intrinsic mass and the special acceleration of the z-th particle of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_j^{All} \mathbf{K}^a_{ij} + \mathbf{K}^u_i$) acting on a particle i with intrinsic mass m_i, is given by:

$$\mathbf{K}_i \; = \; - \, m_i \, \bar{\mathbf{a}}_i \tag{36}$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i.

Now, from the Einsteinian dynamics [10] we have:

$$\mathbf{F}_i \; = \; m_i \, \bar{\mathbf{a}}_i \tag{37}$$

Since ($\mathbf{K}_i \; = \; - \, m_i \, \bar{\mathbf{a}}_i$) we obtain:

$$\mathbf{F}_i \; = \; - \, \mathbf{K}_i \tag{38}$$

that is:

$$\mathbf{K}_i \; + \; \mathbf{F}_i \; = \; 0 \tag{39}$$

If ($\mathbf{T}_i \; \doteq \; \mathbf{K}_i \; + \; \mathbf{F}_i$) then:

$$\mathbf{T}_i \; = \; 0 \tag{40}$$

Therefore, if the net kinetic force $\mathbf{K}_i$ is added in the Einsteinian dynamics then the total force $\mathbf{T}_i$ acting on a (massive or non-massive) particle i is always zero.

Note : According to this paper, the kinetic forces $\overset{au}{\mathbf{K}}$ are directly related to kinetic energy K.