The Explicit abc Conjecture of Alan Barker Implies $c < R^2$ True

Abdelmajid Ben Hadj Salem[1,2*]

[1*]Residence Bousten 8, Bloc B, Av. Mosquée Raoudha, Soukra, 1181 Soukra-Raoudha, Tunisia.

Abstract

In this paper, we assume that the explicit $\boldsymbol{abc}$ conjecture of Alan Baker (2004) is true, we give the proof that $\boldsymbol{c < rad^2(abc)}$ is true, it is one of the keys to resolve the mystery of the $\boldsymbol{abc}$ conjecture. Some numerical examples are given.

Keywords: The $\boldsymbol{abc}$ conjecture, the explicit $\boldsymbol{abc}$ conjecture of Alan Baker, the conjecture $\boldsymbol{c < R^2}$, the exponential function.

MSC Classification: 11AXX , 11M26.

*To the memory of my **Father** who taught me arithmetic*
*To my wife **Wahida**, my daughter **Sinda**, my son **Mohamed Mazen** and my granddaughter **Rayhane** born May 12, 2025*
*To Prof. **A. Nitaj** for his work on the abc conjecture*

1 Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1].

It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1. ***(abc Conjecture)***: *For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2004, Alan Baker [1], [4] proposed the explicit version of the *abc* conjecture namely:

Conjecture 2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{5}$$

with $R = rad(abc)$ and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

In the following, we assume that the conjecture of Alan Barker is true, I will give an elementary proof of the conjecture $c < rad^2(abc)$ that constitutes one key to resolve the open *abc* conjecture. We give also some numerical examples.

2 The Proof of the $c < R^2$ Conjecture

Proof. : Let one triplet (a, b, c) of positive integers relatively prime with $c = a+b$ and :

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!}$$

Let $A = \dfrac{(LogR)^\omega}{\omega!}$, $rad(a) = \prod_{i=1,I} a_i, rad(b) = \prod_{j=1,J} b_j$, and $c = \prod_{l=1,L} c_l$, then $\omega = I + J + L$. we obtain:

$$\omega \ll (LogR = \sum_{i=1,I} Loga_i + \sum_{j=1,J} Logb_j + \sum_{l=1,L} Logc_l)$$

We can write R as:

$$R = e^{LogR} = 1 + LogR + \frac{(LogR)^2}{2!} + \cdots + A + \sum_{k=\omega+1}^{+\infty} \frac{(LogR)^k}{k!} \tag{6}$$

As $\frac{(LogR)^n}{n!} < \frac{(LogR)^{n+1}}{(n+1)!}$ for $n < LogR$ and $\omega \ll LogR \Rightarrow \omega+1 < LogR, \omega+2 < LogR$, it follows :

$$\frac{(LogR)^\omega}{\omega!} < \frac{(LogR)^{\omega+1}}{(\omega+1)!} < \frac{(LogR)^{\omega+2}}{(\omega+2)!} \Longrightarrow 2A < R :$$

$$2A < 1+LogR+\frac{(LogR)^2}{2!}+\cdots+\frac{(LogR)^\omega}{\omega!}+\frac{(LogR)^{\omega+1}}{(\omega+1)!}+\frac{(LogR)^{\omega+2}}{(\omega+2)!}+\sum_{k=\omega+3}^{+\infty}\frac{(LogR)^k}{k!}$$

I can take $A < \frac{5}{6}R$, then:

$$c < \frac{6}{5}R.\frac{(LogR)^\omega}{\omega!} = \frac{6}{5}R.A \leq \frac{6}{5}R.\frac{5}{6}R \Longrightarrow c < R^2 \tag{7}$$

The proof of the conjecture $c < R^2$ is finished.

Q.E.D

$\square$

We give below some numerical examples.

3 Examples

3.1 Example 1. of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6\,436\,343 \tag{8}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19\,683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6\,436\,343 \Rightarrow rad(c) = 23$. Then $R = rad(abc) = 2\times3\times109\times23 = 15\,042 \Longrightarrow R^2 = 226\,261\,764$.
$\omega = 4 \Longrightarrow A = \frac{(LogR)^4}{4!} = 356.64$, $R^2 > \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 6\,437\,590.238 > (c = 6\,436\,343)$. $\frac{A}{R} \approx 0.06 \ll \frac{5}{6} = 0.83$.

3.2 Example 2. of Nitaj

See [5]:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$R = rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$

$$\Longrightarrow R^2 = 831\,072\,936\,124\,776\,471\,158\,132\,100 > (c = 613\,474\,845\,886\,230\,468\,750)$$
$$\omega = 10 \Longrightarrow A = \frac{(LogR)^{10}}{10!} = 225\,312\,992.556 \Longrightarrow$$
$$R^2 > \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015,590 > (c = 613\,474\,845\,886\,230\,468\,750),$$
$$\frac{A}{R} = 7.815e-6 \ll \frac{5}{6} = 0.83$$

3.3 Example 3.

The example is of Ralf Bonse, see [2] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$R = rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$R = 1.5683959920004546031461002610848e+33 \Longrightarrow$$
$$R^2 = 2.4598659877230900595045886864952e+66$$
$$\omega = 10 \Longrightarrow A = \frac{(LogR)^{10}}{10!} = 1\,875\,772\,681\,108.203 \Longrightarrow$$
$$R^2 > \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 3.5303452259448631166310839830891e+45 >$$
$$c = 3.4136998783296235160378273576498e+44, \ \frac{A}{R} = 1.196e-21 \ll \frac{5}{6} = 0.83$$

4 Conclusion

Assuming that the explicit *abc* conjecture is true, we have given an elementary proof that the $c < R^2$ conjecture holds. We can announce the important theorem:
Theorem 3. *Assuming the explicit abc conjecture of Alan Baker is true, then the* $c < R^2$ *conjecture is true.*

References

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