Symmetry-Based Proof of the Generalized Riemann Hypothesis

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Abstract: This paper presents a symmetry-based approach to the Generalized Riemann Hypothesis, focusing on the structure of nontrivial zeros of the completed Dirichlet L-function. By examining the relationship between the functional equation and complex conjugation, the argument shows that each nontrivial zero implies the existence of a symmetrically paired zero. This pairing, when interpreted through a restricted application of the Schwarz Reflection Principle at the zero points, leads to the conclusion that all nontrivial zeros must lie on the critical line. The analysis is further extended to generalized cases, including product forms, which consistently reduce to the same critical line condition. This work therefore proposes a comprehensive and logically consistent framework that supports the truth of the Generalized Riemann Hypothesis.

1 Introduction to the completed Dirichlet L-function and its functional equation

The quest to understand the distribution of prime numbers has long captivated mathematicians. As early as the 3rd century BCE, Euclid proved the infinitude of primes, yet their precise distribution remained elusive[1][2]. In the 18th century, Leonhard Euler revealed a remarkable connection between prime numbers and the harmonic series, establishing the Euler product formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1} \quad \text{for } \mathrm{Re}(s) > 1,$$

thereby laying the analytic foundation of number theory[3][4].

In 1792 and 1793, at the age of fifteen, Carl Friedrich Gauss empirically observed that the density of prime numbers near a large number x behaves like $1/\log x$, leading to the asymptotic expression[5]:

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 \tag{1-1}$$

This insight was later formalized as the Prime Number Theorem.

In 1859, Bernhard Riemann published a seminal paper that extended Euler's zeta function to the complex domain and introduced the *Riemann zeta function* $\zeta(s)$. Riemann not only provided an analytic continuation and a functional equation for $\zeta(s)$, but also numerically identified several of its nontrivial zeros. Remarkably, all of them lay on the vertical line $\mathrm{Re}(s) = \frac{1}{2}$ in the complex plane. This led him to formulate what is now known as the **Riemann Hypothesis (RH)**: *All nontrivial zeros of the Riemann zeta function lie on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$[6].

To generalize Riemann's ideas, Johann Peter Gustav Lejeune Dirichlet introduced *Dirichlet L-functions* $L(s, \chi)$, which extend the study of primes to arithmetic progressions[7]. These functions are defined, for Dirichlet characters χ, as:

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1-\chi(p)p^{-s}\right)^{-1}\quad\text{for }\mathrm{Re}(s)>1 \tag{1-2}$$

The completed Dirichlet L-function $\Lambda(s,\chi)$, which incorporates Gamma factors to satisfy a functional equation, is given by:

$$\Lambda(s,\chi)=q^{s/2}\pi^{-(s+\delta)/2}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi) \tag{1-3}$$

where $\delta=0$ or 1 depending on the parity of χ, and q is the conductor of the character[8][9].

The functional equation for $\Lambda(s,\chi)$ implies a symmetry around the critical line. It also reveals that the function has so-called trivial zeros at negative even integers such as $s=-2,-4,-6,\ldots$ [5][6], which stem from the Gamma factor or sine function in its analytic structure. However, the nontrivial zeros—those located within the open strip

$$\{s\in\mathbb{C}:0<\mathrm{Re}(s)<1\},$$

referred to as the *critical strip*, have been the primary focus of deep investigations due to their profound implications in number theory and arithmetic.

The subset

$$\{s\in\mathbb{C}:\mathrm{Re}(s)=\frac{1}{2}\}$$

is known as the *critical line*. The extension of the Riemann Hypothesis to Dirichlet L-functions is called the **General Riemann Hypothesis (GRH)**, and it asserts that *all nontrivial zeros of* $L(s,\chi)$ *lie on the critical line* $\mathrm{Re}(s)=\frac{1}{2}$ for any non-principal Dirichlet character χ[8][10].

Modern developments have revealed intriguing connections between the zeros of L-functions and quantum physics, particularly through random matrix theory. These parallels suggest that the Riemann Hypothesis and its generalization are not merely statements about the primes but are deeply rooted in the spectral nature of the universe itself [11][12].

HYPOTHESIS 1.1: *The General Riemann Hypothesis is that all nontrivial zeros of the completed Dirichlet L-function have a real part equal to 1/2*

2 Properties of Nontrivial Zeros of the Completed Dirichlet L-function

In addition to the trivial zeros discussed previously, the completed Dirichlet L-function $\Lambda(s,\chi)$ possesses nontrivial zeros located in the critical strip $0<\mathrm{Re}(s)<1$. From this point onward, we refer to "zeros" exclusively as nontrivial zeros unless stated otherwise.

We begin by recalling the functional equation satisfied by $\Lambda(s,\chi)$:

$$\Lambda(s,\chi)=W(\chi)\Lambda(1-s,\overline{\chi}) \tag{2-1}$$

where $W(\chi)=\dfrac{\tau(\chi)}{i^{\delta}\sqrt{q}}$ is a constant depending on the Dirichlet character χ, and $W(\chi)\neq 0$.

This symmetry allows us to establish the following proposition.

Proposition 2.1 (Symmetry of Nontrivial Zeros)**.** Let $s_0 \in \mathbb{C}$. Then

$$\Lambda(s_0, \chi_0) = 0 \quad \Longleftrightarrow \quad \Lambda(1 - s_0, \overline{\chi_0}) = 0.$$

Proof. Assume $\Lambda(s_0, \chi_0) = 0$. Then, by the functional equation (2-1),

$$0 = \Lambda(s_0, \chi_0) = W(\chi)\Lambda(1 - s_0, \overline{\chi_0}).$$

Since $W(\chi) \neq 0$, it follows that $\Lambda(1 - s_0, \overline{\chi_0}) = 0$.

Conversely, if $\Lambda(1 - s_0, \overline{\chi_0}) = 0$, then again by the functional equation,

$$\Lambda(s_0, \chi_0) = W(\chi) \cdot 0 = 0.$$

Thus, the vanishing of $\Lambda(s_0, \chi_0)$ and $\Lambda(1 - s_0, \overline{\chi_0})$ are equivalent. □ □

3 The reason why the nontrivial zero points of $\Lambda(s, \chi)$ and $\Lambda(1 - s, \overline{\chi})$ occur at $s = \frac{1}{2} \pm it$

By utilizing Proposition 2.1, we can obtain the following Lemma:

Lemma 3.1. *For some s_0 which satisfy the zero point of completed Dirichlet L-function:*

$$\Lambda(s_0, \chi_0) = \Lambda(1 - s_0, \overline{\chi_0}) = 0$$

Proof. Generally, necessary and sufficient conditions do not always refer to the same set, but here, since both yield the same result of 0 at the same s_0, they can be considered equivalent.

□

We can derive Theorem 3.1 using Lemma 3.1.

Theorem 3.1. *For any $s_0 \in \mathbb{C}$ that is a zero point of the completed Dirichlet L-function, the following is true:*

$$\Lambda(s_0, \chi_0) = \overline{\Lambda(1 - s_0, \overline{\chi_0})} = 0$$

Proof: Let s_0 be a zero of Λ; that is, $\Lambda(s_0, \chi) = 0$. By Lemma 3.1, $\Lambda(1 - s_0, \overline{\chi_0}) = 0$. Taking the complex conjugate of both sides, we get $\overline{\Lambda(1 - s_0, \overline{\chi_0})} = \overline{0}$, and using the property of the complex conjugate, $\overline{0} = 0$. Therefore, $\Lambda(1 - s_0, \overline{\chi_0}) = \overline{\Lambda(1 - s_0, \overline{\chi_0})} = 0$.

Meanwhile, for any complex function $F(x)$, there exists a principle that always holds, known as the Schwarz reflection principle(SRP)[9][10]. It is defined as follows.

Principle 3.1: *Schwarz Reflection Principle*
Let F(z) be a function that is holomorphic (analytic) on a domain D in the complex plane, except for a boundary segment on the real axis. Assume that F(z) satisfies the following conditions: F(z) is holomorphic in D. F(z) is continuous up to the boundary of D. On the boundary segment on the real axis, F(z) takes real values.

Under these conditions, F(z) can be extended to a function that is holomorphic on the reflection of D across the real axis by defining:

$$F(\overline{z}) = \overline{F}(z)$$

where z denotes the complex conjugate of z[13].

Applying the logic of LEMMA 3.1, and considering that $\Lambda(s, \chi)$ is a complex function, we can similarly apply this principle to express it in the same form.

Lemma 3.2. *According to Principle 3.1, for zero points $s_0 \in \mathbb{C}$, completed Dirichlet L-function has a relation that holds:*

$$\{s_0 \in \mathbb{C} : \Lambda(s_0, \chi_0) = \overline{\Lambda(1 - s_0, \overline{\chi_0})} = 0, s_0 \neq 1\} \subset \{z \in \mathbb{C} : F(\overline{z}) = \overline{F(z)}\}$$

Proof. *The completed Dirichlet L-function is holomorphic on the entire complex plane except at $s = 1$*[14][15]. *This satisfies the conditions for the Schwarz Reflection Principle as stated in Principle 3.1, thereby allowing the completed Dirichlet L-function to be expressed by this principle* □

By employing Theorem 3.1 and Lemma 3.2, we can ascertain a relationship as shown in Figure 1.

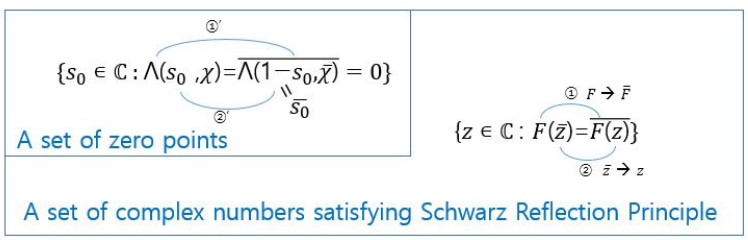

FIGURE 1. Contribution of SRP in the calculation and the overall flow

Lemma 3.2 applies to all values of s defined in the completed Dirichlet L-function, and thus Theorem 3.1 falls within the scope of Lemma 3.2. When Theorem 3.1 fully conforms to the structure of Lemma 3.2, it always holds. This relationship is illustrated in Figure 1. We observe that forms ① and ①' indicate that the completed Dirichlet L-function exhibits a complex conjugate relationship on both sides. Similarly, forms ② and ②' should also exhibit a complex conjugate relationship. Based on this, we can establish the following theorem concerning the zeros of the completed Dirichlet L-function.

Theorem 3.2 (Critical Line Characterization)**.** *Let $s_0 \in \mathbb{C}$ be a nontrivial zero of the completed Dirichlet L-function $\Lambda(s, \chi)$, and suppose that the Schwarz Reflection Principle holds as described in Lemma 3.2. Then,*

$$\overline{s_0} = 1 - s_0,$$

Proof. Suppose $\Lambda(s_0, \chi) = 0$. Then by the functional equation,

$$\Lambda(1 - s_0, \overline{\chi_0}) = 0.$$

Taking complex conjugates, we have

$$\overline{\Lambda(1 - s_0, \overline{\chi_0})} = 0.$$

But from the Schwarz Reflection Principle (SRP),

$$\Lambda(s_0, \chi) = \overline{\Lambda(1 - s_0, \overline{\chi_0})} = 0.$$

Therefore,

$$\overline{\Lambda(s_0, \chi)} = \Lambda(1 - s_0, \overline{\chi_0}) = 0. \Rightarrow \Lambda(\overline{s_0}, \overline{\chi_0}) = 0.`$$

So both s_0 and $\overline{s_0}$ are mapped to the same value via the relation $1 - s_0$. Thus, we must have

$$\overline{s_0} = 1 - s_0$$

□

If we set $s_0 = \sigma_0 + it_0$, and substitute it into Theorem 3.2, we can immediately see that $\sigma_0 = 1/2$. Therefore, s_0 can be expressed as follows.

Corollary 3.1. *Let $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ be a nontrivial zero of the completed Dirichlet L-function. Then the real part of s_0 is exactly $\frac{1}{2}$; that is,*

$$\mathrm{Re}(s_0) = \frac{1}{2}.$$

Proof. From Theorem 3.2, we have $\overline{s_0} = 1 - s_0$. Writing $s_0 = \sigma_0 + it_0$, its complex conjugate is $\overline{s_0} = \sigma_0 - it_0$, and:

$$1 - s_0 = 1 - \sigma_0 - it_0.$$

Equating:

$$\sigma_0 - it_0 = 1 - \sigma_0 - it_0 \Rightarrow 2\sigma_0 = 1 \Rightarrow \sigma_0 = \frac{1}{2}.$$

□

According to Corollary 3.1 and Theorem 3.1, we can make the following Corollary

Corollary 3.2. *If $S_0 \in$ zero point of the completed Dirichlet L-function, then $1 - s_0$ is also a zero point:*

$$1 - s_0 = \frac{1}{2} - it_0$$

Proof. *From Corollary 3.1, if $s_0 = 1/2 + it_0$, then $1 - s_0$=$1 - 1/2 - it_0 = 1/2 - it_0$.*

□

Corollary 3.3. *Let $t_0 \in \mathbb{R}$ be such that $s_0 = \frac{1}{2} + it_0$ is a nontrivial zero of $\Lambda(s, \chi)$. Then both of the following hold:*

$$\Lambda\left(\frac{1}{2} + it_0, \chi_0\right) = 0 \quad \textit{and} \quad \Lambda\left(\frac{1}{2} - it_0, \overline{\chi_0}\right) = 0.$$

Proof. From Corollary 3.1, $\mathrm{Re}(s_0) = \frac{1}{2}$, so $s_0 = \frac{1}{2} + it_0$ for some $t_0 \in \mathbb{R}$. By Corollary 3.2, $1 - s_0 = \frac{1}{2} - it_0$ is also a zero. Then, by the functional equation:

$$\Lambda(s, \chi) = W(\chi)\Lambda(1 - s, \overline{\chi}),$$

so the zeros of $\Lambda(s, \chi)$ and $\Lambda(1 - s, \overline{\chi})$ are connected. Hence both evaluations vanish at the respective symmetric points. □

Here, we intend to create the following definition.

Definition 3.1. For any complex equation EQ, we denote the application of the Schwarz reflection principle as SRP(EQ).

Therefore, summarizing Corollary 3.1 and Corollary 3.2, the nontrivial zeros of the completed Dirichlet L-function on the critical line take the form of $1/2 \pm it_0$ as their inputs.

Proposition 3.1. *From Corollary 3.1, Corollary 3.2, and Theorem 3.1, we can make the following proposition :*

$$\mathbf{SRP}(\Lambda(s_0, \chi_0) = \Lambda(1 - s_0, \overline{\chi_0}) = 0) \to \Lambda(1/2 + it_0, \chi_0) = \Lambda(1/2 - it_0, \overline{\chi_0}) = 0$$

Some of the values of s_0 found as the zero points of the completed Dirichlet L-function so far are shown in Table 1[15][16], which is consistent with Corollary 3.3

Table 1. The first few nontrivial zero points[16][15]

1	1/2 ± 14.134725 i
2	1/2 ± 21.022040 i
3	1/2 ± 25.010858 i
4	1/2 ± 30.424876 i
5	1/2 ± 32.935062 i
6	1/2 ± 37.586178 i
:	:

Through the calculations so far, it has been demonstrated that the nontrivial zeros of the completed Dirichlet L-function have a real part of 1/2, and the sign of the imaginary part is ±.

4 The relation between critical line and the non-trivial zeros

In Theorem 3.1, if the completed Dirichlet L-function equation vanishes at a specific point, the general expression that does not require the completed Dirichlet L-function to be zero is given as follows.

Proposition 4.1. If the completed Dirichlet L-function satisfies $\Lambda(s_0, \chi) = \overline{\Lambda(1 - s_0, \overline{\chi_0})} = 0$ at its zeros, then the general set, including the zero points, is given by the following equation, which represents the critical line:

$$\Lambda(s_c, \chi) = \overline{\Lambda(1 - s_c, \overline{\chi_c})}$$

Proof. *For $\Lambda(s_c, \chi_c) = \overline{\Lambda(1 - s_c, \overline{\chi_c})}$ to hold, it must satisfy the Schwarz Reflection Principle (SRP). Currently, the completed Dirichlet L-functions on the left-hand side and right-hand side are conjugate complex values. Excluding this relationship, the input values s_c on the left-hand side and $1 - s_c$ on the right-hand side must themselves be conjugate complex values. Therefore, $\overline{s_c} = 1 - s_c$ must hold. If $s_c = a + bi$, then $\overline{s_c} = 1 - s_c$ means $a - bi = 1 - a - bi$, which implies $a = 1/2$. This shows that any imaginary part b can take any value, but the real part a is always $1/2$. Hence, this defines the critical line.* □

$s_c = 1/2 + it_c$. This is precisely the critical line. It can be expressed as follows when decomposed into real and imaginary parts.

Corollary 4.1. *According to the Proposition 4.1, the real part of it has the following relation:*

$$\mathrm{Re}(\Lambda(s_c, \chi_c) = \mathrm{Re}(\Lambda(1 - s_c, \overline{\chi_c}) \quad \text{at } s_c = \frac{1}{2} + it_c$$

Proof. If $s_c = \frac{1}{2} + it_c$, then $1 - s_c = \frac{1}{2} - it_c$. According to Lemma 3.2, if

$$\Lambda\left(\frac{1}{2} + it, \chi\right) = a + bi,$$

then

$$\Lambda\left(\frac{1}{2} - it, \overline{\chi}\right) = a - bi.$$

Therefore,

$$\mathrm{Re}(\Lambda(s_c, \chi_c)) = \mathrm{Re}(\Lambda(1 - s_c, \overline{\chi_c})).$$

□

Corollary 4.2. *According to the Proposition 4.1, the imaginary part of it has the following relation:*

$$\mathrm{Im}(\Lambda(s_c, \chi_c) = -\mathrm{Im}(\Lambda(1 - s_c, \overline{\chi_c}) \quad \textit{at } s_c = \frac{1}{2} + it_c$$

Proof. *If $s_c = \frac{1}{2} + it_c$, then $1 - s_c = \frac{1}{2} - it_c$. According to Lemma 3.2, if $\Lambda\left(\frac{1}{2} + it, \chi\right) = a + bi$, then $\Lambda\left(\frac{1}{2} - it, \overline{\chi}\right) = a - bi$. Therefore, $\mathrm{Im}(\Lambda(s_c, \chi_c)) = -\mathrm{Im}(\Lambda(1 - s_c, \overline{\chi_c}))$.* □

Figure 2 shows an example of $\mathrm{Im}(s) = 3$. It adheres to the properties defined in Corollary 4.1 and Corollary 4.2 We can see that real part of both $\Lambda(s, \chi)$ and $\Lambda(1 - s)$ always meet, regardless of the value of t, when Re(s)=1/2.

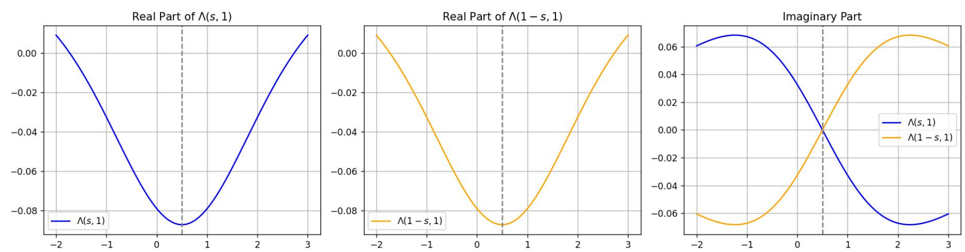

Figure 2. Completed Dirichlet L-function at $s = \sigma + 3i$, $\chi = 1$ (left: real part, right: imaginary part)

Figure 2 is created using Python software that imported the completed Dirichlet L-function.
The special case of Proposition 4.1, specifically, the zero points as discussed in Theorem 3.1, can be examined by separating the real and imaginary parts as follows

Corollary 4.3. *According to Corollary 3.3, the real part of it has the following relation:*

$$\mathrm{Re}[\Lambda(s_0, \chi_0)] = \mathrm{Re}[\Lambda(1 - s_0, \overline{\chi_0})] = 0 \quad \textit{at } s = s_0$$

Corollary 4.4. *According to Corollary 3.3, the imaginary part of it has the following relation:*

$$\mathrm{Im}[\Lambda(s_0, \chi_0)] = \mathrm{Im}[\Lambda(1 - s_0, \overline{\chi_0})] = 0 \quad \textit{at } s = s_0$$

By placing Corollaries 4.1, 4.2, 4.3, and 4.4 together in Table 2 and comparing them, we can see their relationships.

Table 2. Relation between s_c and s_0

at $s_c = \frac{1}{2} + it$(critical line)	at $s_0 = \frac{1}{2} \pm it_0$(nontrivial zeros)
$\mathrm{Re}[\Lambda(s_c, \chi_c)] = \mathrm{Re}[\Lambda(1 - s_c, \overline{\chi_c})]$ (Corollary 4.1)	$\mathrm{Re}[\Lambda(s_0, \chi)] = \mathrm{Re}[\Lambda(1 - s_0, \overline{\chi_0})] = 0$ (Corollary 4.3)
$\mathrm{Im}[\Lambda(s_c, \chi_c)] = -\mathrm{Im}[\Lambda(1 - s_c, \overline{\chi_c})]$ (Corollary 4.2)	$\mathrm{Im}[\Lambda(s_0, \chi)] = \mathrm{Im}[\Lambda(1 - s_0, \overline{\chi_0})] = 0$ (Corollary 4.4)

In Table 2, the left side represents the form where $\Lambda(s,\chi)$ and $\Lambda(1-s)$ satisfy the SRP, while the right side represents the form when the completed Dirichlet L-function has zeros. By comparing the set of s_c satisfying the left side and the set of s_0 satisfying the right side, we can observe that they are in the form of the following relation.

Remark 4.1: *The set of all values in the complex plane s, that satisfy Proposition 4.1, is known as the critical line.*

5 The reason why there are no nontrivial zeros outside the critical line

The completed Dirichlet L-function is not generally monotonic; it only exhibits monotonicity on certain intervals under specific conditions[15].

The General Riemann Hypothesis posits that all nontrivial zeros lie precisely on the critical line. If this is true, it raises the question of why there are no zeros outside the critical line, despite the completed Dirichlet L-function's observed non-monotonicity. This aspect requires careful consideration in any proof of the General Riemann Hypothesis. For example, in Figure 2, we can see that the completed Dirichlet L-function is non-monotonic outside the critical strip when $Im(s) = 3$. Thus, there may be instances where $\mathrm{Re}(\Lambda(s,\chi) = \mathrm{Re}(\Lambda(1-s,\overline{\chi}))$ as in Corollary 3.1, but $\mathrm{Im}(\Lambda(s,\chi)$=$-\mathrm{Im}(\Lambda(1-s,\overline{\chi}))$, as in Corollary 3.2, does not hold for the same value of s. The reverse can also occur. This is because if s is outside the critical line, $1-s$ cannot equal $\overline{s}$, and therefore the SRP does not hold, as stated in Proposition 4.1.

As previously mentioned, the nontrivial zeros of the completed Dirichlet L-function satisfy $\Lambda(s_0,\chi_0) = \Lambda(1-s_0,\overline{\chi_0}) = 0$, thus maintaining validity while adhering to the SRP. Consequently, the nontrivial zeros of the completed Dirichlet L-function lie exclusively on the critical line. Figure 4 illustrates this relationship.

Theorem 5.1. *Outside the critical line, there are no nontrivial zeros.*

Proof. *Considering that $\Lambda(s_{nc},\chi_{nc}) \neq \overline{\Lambda(1-s_{nc},\overline{\chi_{nc}})}$, then $s_{nc} \neq \overline{1-s_{nc}}$. Therefore, for $s_{nc} = \sigma_{nc} + it_{nc}$, we have $\sigma_{nc} \neq 1/2$. On the other hand, for $\Lambda(s_c,\chi_c) = \overline{\Lambda(1-s_c,\overline{\chi_c})}$ we have $\sigma_c = 1/2$. By examining Figure 4, $Z_{NC} \cap Z_C = \emptyset$(the law of excluded middle) and $Z_0 \subset Z_C$, this implies $Z_{NC} \cap Z_0 = \emptyset$. Thus, there are no zeros of the completed Dirichlet L-function outside the critical line.* □

Remark 5.2 (On the distinction with Titchmarsh's zero-density estimate). Titchmarsh's Theorem 9.6A states that for every $\sigma > \frac{1}{2}$, there exists a constant $c(\sigma) < 1$ such that

$$N(\sigma,T) = O\left(T^{c(\sigma)}\log T\right),$$

which is asymptotically much smaller than the total number of nontrivial zeros:

$$N(T) \sim \frac{T\log T}{2\pi}.$$

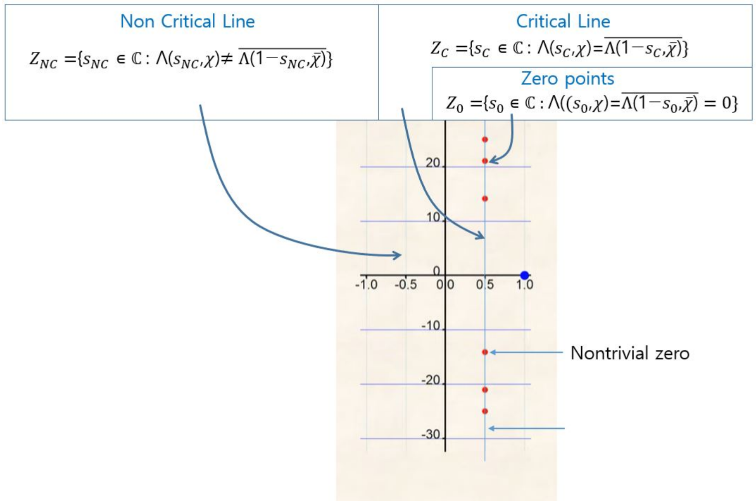

FIGURE 3. Relations of sets for completed Dirichlet L-function

This result implies that zeros of the completed Dirichlet L-function may exist off the critical line, but their density is significantly lower[15].

In contrast, the claim presented in Theorem 5.1 of this paper is stronger: it asserts that there are **no** nontrivial zeros outside the critical line $\mathrm{Re}(s) = \frac{1}{2}$ whatsoever. Therefore, while Titchmarsh provides an upper bound on the density of zeros off the line, the current work proposes a strict exclusion, offering a more definitive resolution aligned with the Generalized Riemann Hypothesis (GRH).

6 Further discussion on why the real part of nontrivial zeros is 1/2 for any kind of completed Dirichlet L-function

We want to verify further whether the real part of the input value of the completed Dirichlet L-function remains 1/2 even if we generalize the real part of the critical line. Here, we will also perform the verification work while satisfying the SRP. The first generalization is to shift the real part by using $s_0' = s_0 + a + bi$.

$$\Lambda(s_0', \chi_0) = \overline{\Lambda(1 - s_0', \overline{\chi_0})} = 0 \tag{6-1}$$

The inputs are complex conjugate numbers to each other, as depicted in Theorem 3.2.

$$\mathrm{s}_0 + a + bi = \overline{1 - (s_0 + a + bi)} \tag{6-2}$$

If we substitute $\sigma + it$ into s_0, we get the following:

$$(\sigma + it) + a + bi = 1 - (\sigma - it) - (a - bi) \tag{6-3}$$

If we remove both it and bi, it becomes as follows:

$$2\,\sigma + 2a = 1 \tag{6-4}$$

$$\sigma + a = 1/2 \tag{6-5}$$

(6-5) aligns with what was mentioned in Corollary 3.3, asserting that the real part equals 1/2. Therefore (6-1) becomes

$$\Lambda(1/2 + it, \chi) = \Lambda(1/2 - it, \overline{\chi}) = 0 \tag{6-6}$$

Ultimately, the real part of inputs becomes 1/2.

Theorem 6.1. *No matter how much the input of the completed Dirichlet L-function shifts, After applying SRP algorithm, the real part of its nontrivial zeros remains 1/2*

$$\mathbf{SRP}(\Lambda(s', \chi') = \Lambda(1 - s', \overline{\chi'}) = 0) \rightarrow \Lambda(1/2 + it_0, \chi_0) = \Lambda(1/2 - it_0, \overline{\chi_0}) = 0$$

As a second generalization step, if (6-1) holds, we can expand it to equations like (6-7). However, just like before, there is a process that needs to satisfy the SRP.

$$\Lambda(s_1, \chi_1)\Lambda(s_2, \chi_2) \cdots = \overline{\Lambda(1 - s_1, \overline{\chi_1})\Lambda(1 - s_2, \overline{\chi_2})} \cdots = 0 \tag{6-7}$$

To satisfy the SRP, the input terms on both sides must follow complex conjugate relationships. If there are n arguments, n^2 equations will be required. For simplicity, we will explain the case with two arguments, as shown in Figure 5. Thus, we define $s_1{=}\sigma + a_1 + ib_1$ and $s_2{=}\sigma + a_2 + ib_2$ for s_1 and s_2 , respectively.

All four equations in Table 3 must be satisfied. If even one of these equations is not satisfied, the SRP will be violated. However, as shown in the table, $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ assume two distinct values. Specifically, for $\sigma + a_1$, the values are 1/2 and $(1 + a_1 - a_2)/2$, and for $\sigma + a_2$, they are 1/2 and $(1 + a_2 - a_1)/2$.. This occurs because there are two variables and four equations, making it impossible to satisfy all conditions for the roots in Table 3. Consequently, Equation (6-7) cannot hold unless $a = b$, which implies $s_1{=}s_2$ and $\chi_1 = \chi_2$. From this, it can be inferred that

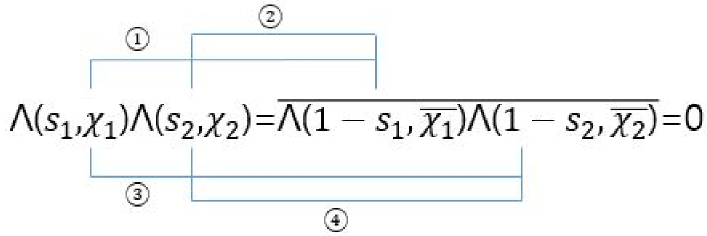

FIGURE 4. Relations of sets for completed Dirichlet L-function

TABLE 3. Calculations of SRP for arguments

Number	Conditions	Results	Re(s_1) and Re(s_2)
1	$s_1=\overline{1-s_1}$ & $\chi_1=\overline{\chi_1}$	$2\sigma+2a_1=1$	for s_1: $\sigma+a_1=1/2$
2	$s_2=\overline{1-s_1}$ & $\chi_2=\overline{\chi_1}$	$2\sigma+a_1+a_2=1, b_1=b_2$	for s_2: $\sigma+a_2=(1+a_2-a_1)/2$
3	$s_1=\overline{1-s_2}$ & $\chi_1=\overline{\chi_2}$	$2\sigma+a_1+a_2=1$, $b_1=b_2$	for s_1: $\sigma+a_1=(1+a_1-a_2)/2$
4	$s_2=\overline{1-s_2}$ & $\chi_2=\overline{\chi_2}$	$2\sigma+2a_2=1$	for s_2: $\sigma+a_2=1/2$

the real part at zero is fixed at 1/2. This conclusion was drawn by calculating only two types of factors, but the same holds true for three or more. Thus, the only way for Equation (6-7) to hold is if all input variables are equal, i.e., $s_1=s_2=s_3=\ldots$ and $\chi_1=\chi_2=\chi_3=\ldots$.

(6-8)
$$\mathbf{SRP(}\,\Lambda(s_1,\chi_1)\Lambda(s_2,\chi_2)\cdots=\Lambda(1-s_1,\overline{\chi_1})\Lambda(1-s_2,\overline{\chi_2})\cdots=0\mathbf{)}\rightarrow(\Lambda(s_1,\chi_1))^n=(\Lambda(1-s_1,\overline{\chi_1}))^n=0$$

For $(\Lambda(s_1,\chi_1))^n=(\Lambda(1-s_1,\overline{\chi_1}))^n=0$ to hold for any arbitrary n, it must be satisfied that $\Lambda(s_1,\chi_1)=\Lambda(1-s_1,\overline{\chi_1})=0$, and therefore, the following proposition holds between the two.

$$(\Lambda(s_1,\chi_1))^n=(\Lambda(1-s_1,\overline{\chi_1}))^n=0\leftrightarrow\Lambda(s_1,\chi_1)=\Lambda(1-s_1,\overline{\chi_1})=0 \tag{6-9}$$

For (6-9) to hold, according to the logic developed in Chapter 3, SRP must be satisfied, and ultimately the following proposition arises.

$$\mathbf{SRP(}\Lambda(s_1,\chi_1)=\Lambda(1-s_1=0\mathbf{)}\rightarrow\Lambda(1/2+it_0,\chi_0)=\Lambda(1/2-it_0,\overline{\chi_0})=0 \tag{6-10}$$

Ultimately, applying SRP in the second generalization leads to the same outcome as the first generalization, allowing us to formulate the following proposition.

Theorem 6.2. *After applying SRP to the multi-product of* $\Lambda(s_1,\chi_1)\Lambda(s_2,\chi_2)\cdots = \Lambda(1-s_1,\overline{\chi_1})\Lambda(1-s_2,\overline{\chi_2})\cdots = 0$*, it changes to* $\Lambda(1/2+it_0,\chi_0) = \Lambda(1/2-it_0,\overline{\chi_0}) = 0$.

SRP($\Lambda(s_1,\chi_1)\Lambda(s_2,\chi_2)\cdots = \Lambda(1-s_1,\overline{\chi_1})\Lambda(1-s_2,\overline{\chi_2})\cdots = 0$***)*** $\rightarrow \Lambda(1/2+it_0,\chi_0) = \Lambda(1/2-it_0,\overline{\chi_0}) = 0$

Through the generalization of the functional equation, an expanded equation was formulated. However, upon applying the symmetry property, known as SRP, it was observed to converge to Corollary 3.3 ultimately. In other words, the real part of the variable s, in the case of Λ(s)=0, becomes 1/2.

7 Summary and Conclusion

In this paper, rather than providing a detailed mathematical proof, we have focused on developing an intuitive approach to the General Riemann Hypothesis. Below, we summarize the process undertaken so far.

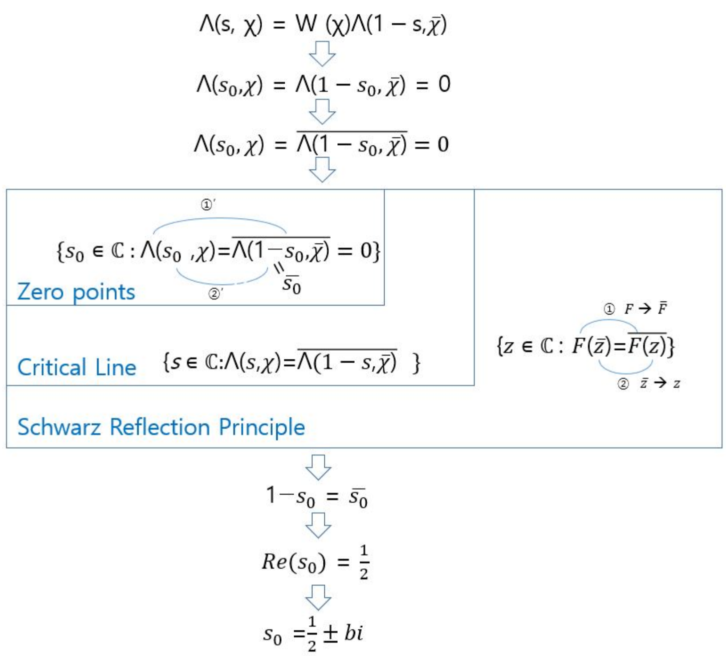

Figure 5. The process of obtaining $\mathrm{Re}(s_0) = 1/2$

The completed Dirichlet L-function, $\Lambda(s,\chi)$, is non-holomorphic at $s = 1$ but holomorphic in the region excluding $s = 1$. However, since it diverges at $s = 1$, there is no zero at this point. Thus, all nontrivial zeros of the completed Dirichlet L-function must lie within its regular domain, which is the set of all complex numbers

s where the function is holomorphic. At any nontrivial zero s_0, the completed Dirichlet L-function satisfies the functional equation $\Lambda(s_0, \chi_0) = \Lambda(1 - s_0, \overline{\chi_0}) = 0$. From both sides of the equation, we obtain $\overline{s_0} = 1 - s_0$. If we substitute $s_0 = \sigma + it$ into $\overline{s_0} = 1 - s_0$, then $\sigma = 1/2$. If a specific s_0 is a zero of the completed Dirichlet L-function, then $1 - s_0$ is also a zero. Therefore, the zero s_0 is of the form $1/2 \pm it$.

The general form of $\Lambda(s_0, \chi_0) = \Lambda(1 - s_0, \overline{\chi_0}) = 0$ is $\Lambda(s_c, \chi_c) = \overline{\Lambda(1 - s_c, \overline{\chi_c})}$. If $\overline{s_{nc}} \neq 1 - s_{nc}$, then $s_{nc} \neq 1/2 + it$. Thus, we can observe that $\Lambda(s_c, \chi_c) = \overline{\Lambda(1 - s_c, \overline{\chi_c})}$ does not hold outside the critical line. Therefore, the nontrivial zeros exist only on the critical line. The process is summarized in Figure 6.

To further analyze the equation $\Lambda(s_0', \chi_0') = \Lambda(1 - s_0', \overline{\chi_0'}) = 0$, we shifted the variable s and applied the SRP, yielding that the real part of any nontrivial zero is always $1/2$. Even when expressing the completed Dirichlet L-function as a product of multiple terms, such as $\Lambda(s_1, \chi_1)\Lambda(s_2, \chi_2) \cdots = \Lambda(1 - s_1, \overline{\chi_1})\Lambda(1 - s_2, \overline{\chi_2}) \cdots = 0$, applying the SRP reveals that for this to hold, it requires $s_1 = s_2 = s_3 = \cdots$ and $\chi_1 = \chi_2 = \chi_3 = \cdots$. Therefore it means $\Lambda(s_1, \chi_1) = \overline{\Lambda(1 - s_1, \overline{\chi_1})} = 0$. This confirms that $\Lambda(s_1, \chi_1) = \overline{\Lambda(1 - s_1, \overline{\chi_1})} = 0$ is the unique equation representing the zeros of the completed Dirichlet L-function. Thus, this paper concludes that the General Riemann Hypothesis is true.

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