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ON VALIDITY IN NON STANDARD ANALYSIS OF RIEMANN- DINI THEOREM

AUTHOR: CIRO CESARANO

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INTRODUCTION

In this article we will use some fundamental concept of non standard analysis founded by mathematical logician Abraham Robinson in the sixties of twenty century subsequently it was simplified by Jerome Keysler, these concepts have already been proposed in a previous article of mine [1] we demonstrate that with the use of non-standard analysis and the definition of non-standard rearrangement introduced in [1] we overcome some paradoxes that often arise with numerical series **.**

COMMENT

This article is an extension and an application of arguments exposed and defined in another article of mine [1] however for completeness some definitions are repeated also in this article

DEFINITION AND PRELIMINARIES

We give here some concept and definition that we will use for the continuations of this article.Foundamentally Keisler approach is based on the following two principles [2][3].

THE EXTENSION PRINCIPLE

a) The real numbers form a subset of the hyperreal numbers, and the order relation $x< y$ for the real numbers is a subset of the order relation for the hyperreal numbers
b)There is a hyperreal number that is greater than zero but less than every positive real number
c) For every real function f of one or more variables we are given a corresponding hyperreal function f^* of the same number of variables , f^* is called the natural extension of f (in this article f^* is called the extension of f at non standard model of analysis) .
Furthermore with each relation X on R there is corresponding relation X* on R^* called the natural extension of X [7]

THE TRANSFER PRINCIPLE

Every real statement that hold for one or more particular real function holds for the hyperreal natural extensions of these functions, the transfer principle is equivalent to Leibniz' principle ,which is the property that for each real bounded sentence $\phi \in L$,is true if and only if ϕ^* is true . L is the language of the first order predicate [3]we still give the following definitions always of Keisler [2]:

DEFINITIONS

A hyperreal number b is said to be:
positive infinitesimal if b is positive but less than every positive real number,
negative infinitesimal if b is negative but greater than every negative real number.

A hyperreal number b is said to be:
finite if b is between two real numbers,
positive infinite if b is greater than every real number,
negative infinite if b is less than every real number.

ABSTRACT

By a simple extension and application of rearrangement definition of a simply convergent series, at non standard model of analysis called "non standard rearrangement" already introduced by [1] we overcome some paradoxes that often arise with numerical series to this end we give three significant examples of "standard" and "non standard rearrangement" of the harmonic series with alternate signs.

Instead notable result is that with the definition of " non standard rearrangement " introduced in [1] the commutative property of addition continues to hold even for simply convergent series (such as harmonic series with alternate) contrary to what is stated by Riemann-Dini theorem or
Riemann rearrangement theorem, Furthermore, by analyzing a famous result of Ramanujan and comparing it with results of non-standard analysis, we raise doubts about the coherence of the standard theory on divergent series and their regularization.

TEXT

Stated harmonic series with alternate signs :

$$S = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = Ln2 \qquad (1)$$

extending it at "non standard model" we have

$$S^* = \sum_{1}^{\omega} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = Ln2 \qquad (1)^*$$

with ω = infinite number positive , $*$ $=$ asterisk indicates the extension of ζ(s) at non standard model of analysis (see definitions and preliminaries) . We call "non standard rearranged" series of (1)* a series with both the conditions following ,the first condition is: "the rearranged series of (1)* must consist in a permutation of all standard terms (i.e with form $\frac{1}{N}$) contained in (1)* " the second condition is : "the rearranged series must consist in a permutation of all non standard terms (i.e with form $\frac{1}{N^*}$) contained in (1)*and no another terms "(N is the Natural numbers set , N^* is the infinite hypernatural numbers set i.e the extension of N at non standard model of analysis).Since real numbers are a subset of hyperreals (see extension principle) the non standard rearrangement could be defined by a single condition ,but to Better highlight the difference between two rearrangements two were written.We give an example of standard rearrangement (using only (1) condition) and correspondent “non standard rearrangement” (using (1) and (2) conditions)

$$S = \sum_{1}^{\infty}\frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty}\frac{1}{2n-1} - \sum_{1}^{\infty}\frac{1}{2n} \qquad (1).$$

By exchanging the terms of series (1) so as to have a sequence of two positive terms followed by a negative term we have:

$$(1+\tfrac{1}{3}) - \frac{1}{2} + (\frac{1}{5}+\frac{1}{7}) - \frac{1}{4} + (\frac{1}{9}+\frac{1}{11}) - \frac{1}{6} + \dots \qquad (2)$$

The (2) is an example of standard rearrangement of the (1).We demonstrate that (2) convergent to $\frac{3}{2}S = \frac{3}{2}Ln2$,in fact dividing (1) by 2 we have

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots \qquad (3)$$

adding term by term the (2) and the (3)

$$S + \frac{1}{2}S = \frac{3}{2}S = \frac{3}{2}Ln2 = 1 + 0 - \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{4} - \frac{1}{4} + \frac{1}{5} + 0 - \frac{1}{6} + \frac{1}{6} + \frac{1}{7} + 0 - \frac{1}{8} - \frac{1}{8} + \dots \qquad (4). \quad \text{or}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = S$$

That is $\frac{3}{2}S = S$ (4)

We can arrive at the same result by applying Ohm's rearrangement theorem with it is shown that in (1) taking p positive and q negative terms we have the sum : S = ln(2)+(1/2)ln(p/q),in previous example was $p=2$ and $q = 1$,instead with our definition of "non standard rearrangement" it is always $p^* = q^*$ both infinite numbers [2][3] as seen in [1]). The result (4) is in according with Riemann theorem of rearrangement,instead in non standard analysis using the two definitions of the non standard rearrangement of series (1*) we have:

$$S^* = \sum_{1}^{\omega}\frac{(-1)^{n+1}}{n} = \sum_{1}^{\frac{\omega}{2}}\frac{1}{2n-1} - \sum_{1}^{\frac{\omega}{2}}\frac{1}{2n} \qquad (1^*)$$

Dividing by 2 the (1 *)we have:

$$\frac{1}{2}S^{*}=\sum_{1}^{\frac{\omega}{2}}\frac{1}{4n-2}\ -\ \sum_{1}^{\frac{\omega}{2}}\frac{1}{4n} \qquad (2^*)$$

if we associate as in (2) for each pair of positive numbers p a single negative number q we should write (2*) as:

$$\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n-1}\ -\ \sum_{1}^{\frac{\omega}{4}}\frac{1}{2n}$$

it would be the standard rearrangement extended to the non-standard model regardless of the definition of non-standard rearrangement ,in fact since p^* =ω/2 while q*=ω/4 therefore $p^* \neq$ q* and the (2*) could not be a non-standard rearrangement of (2) since it should always be $p^* = q^*$ .Adding S^* with $\frac{1}{2}S^*$ we have:

$$S^{*}+\frac{1}{2}S^{*}=\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n-1}\ -\ \sum_{1}^{\frac{\omega}{2}}\frac{1}{2n}+\sum_{1}^{\frac{\omega}{2}}\frac{1}{4n-2}\ -\ \sum_{1}^{\frac{\omega}{2}}\frac{1}{4n} \qquad (3^*)$$

as it is $\sum_{1}^{\frac{\omega}{2}}\frac{1}{4n}=\frac{1}{2}\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n}$ and

$\sum_{1}^{\frac{\omega}{2}}\frac{1}{4n-2}=\frac{1}{2}\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n-1}$ we have:

$$S^*+\frac{1}{2}S^{*}=(1+\frac{1}{2})\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n-1}\ -\ \sum_{1}^{\frac{\omega}{2}}\frac{1}{2n}=(1+\frac{1}{2})\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n}=(1+\frac{1}{2})[\sum_{1}^{\frac{\omega}{2}}(\frac{1}{2n-1}-\frac{1}{2n})]$$

(4*)

Since in the square bracket we have the harmonic series with alternating signs calculated up to ω/2 it converges to Ln2 for the Cauchy convergence test therefore have $S^*+\frac{1}{2}S^{*}=\frac{3}{2}S^{*}$, by the Cauchy convergence test we see that the harmonic series with alternating signs always converges to the same quantity (standard)if calculated for any infinite number[1][2]. Now we generalize and formalize the two conditions which together constitute the new concept of "non standard rearrangement"of a series (1*) for the first condition that characterize the classical standard rearrangement of a series we have:give the series $\sum_{k} a_k$ with real or complex terms and one bijective function $\pi : N \rightarrow N$ it's called rearranged series of $\sum_{k} a_k$ according π the series $\sum_{k} a_{\pi(k)}$ .The second condition is the following:given the series $\sum_{k} a_k$ * with non standard and complex terms (infinitesimal complex and infinitesimal real numbers) and one bijective non standard function (see definition and preliminaries)$\pi^* : N^* \rightarrow N^*$ (N^* are also called infinite hypernatural numbers [6]).It is called rearranged series according to π^* the series:$\sum_{k} a_{\pi^*(k)}$ .The biunivocity of π and π^* ensure in particular that the number of terms of the rearranged series have the same number of terms (positives and negatives) as the originals series respectively in N and in N^*.It is shown in standard analysis (Ohm's rearrangement theorem) that in (1) taking p positive and q negative terms we have the sum : S = ln(2)+(1/2)ln(p/q). instead with our definition of "non standard rearrangement" it is always $p^* = q^*$ both infinite numbers . The validity of the commutative property in (1)* should not be surprising as this property is obviously valid for real numbers,according to the transfer principle it is also valid for hyperreal numbers in fact we have:

$(\forall x\ \forall y) \in R\ (x+y=y+x)$

it is true in R,extending it at non standard model we have : $(\forall x^*\ \forall y^*) \in R^*$ $(x^*+y^*=y^*+x^*)$ (R*= set of hyperreal numbers)(Following common usage we omit the asterisk for the sum between hyperreal numbers).Why is Riemann -Dini theorem valid with " standard rearrangement" despite the commutativity of addition?The answer is simple in fact formalizing this theorem (by the first order predicate logic) we have: let $\sum_{k} a_k$ a simply convergent series

$(\forall x \in R \cup \{-\infty, +\infty\}\ \exists \pi : N \leftrightarrow N)$:

$$\lim\ N \rightarrow \infty \sum_{k=1}^{N} a_{\pi(k)} = x$$

Extending at non standard model we have:

$(\forall x^* \in R^* \ \exists \pi^* : N^* \leftrightarrow N^*)$:

$$\sum_{k=1}^{\omega} a_{\pi^*(k)} = x^*$$

It is precisely the limit with $N \rightarrow \infty$ $\sum_{k=1}^{N} a_{\pi(k)}$ or in non standard model $\sum_{k=1}^{\omega} a_{\pi^*(k)}$ that introduces (or omits) an infinite quantity of infinitesimals whose sum is different from zero (and from being infinitesimal) not existing in the original series (1)* these quantities are excluded only with the" non standard rearrangement" as seen before.We give another example of standard rearrangement in which apparently as many positive terms as negative terms are counted ,still obtaining a different result from (1) [or ln2] which however with non standard rearrangement the result of this rearrangement respects the commutative property of addition for simply convergent series.Be given

$$S = (1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots.) = (1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots) + (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots.) - 2(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots.) = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) - (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots..) = 0$$

Or

$$S = \sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n} - 2\sum_{n=1}^{\infty} \frac{1}{2n} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n} = 0$$

Therefore for this standard rearrangement S =0.

Applying the non standard rearrangement we have. $S^* = \sum_{1}^{\frac{\omega}{2}} \frac{1}{2n-1} - \sum_{1}^{\frac{\omega}{2}} \frac{1}{2n} = \sum_{1}^{\frac{\omega}{2}} \frac{1}{2n-1} + \sum_{1}^{\frac{\omega}{2}} \frac{1}{2n} - 2\sum_{1}^{\frac{\omega}{2}} \frac{1}{2n} = \sum_{1}^{\omega} \frac{1}{n} - 2\sum_{1}^{\frac{\omega}{2}} \frac{1}{2n}$ (In this step according to the standard rearrangement this sum is zero since $S = \sum_{n=1}^{\infty} \frac{1}{n} - 2\sum_{n=1}^{\infty} \frac{1}{2n} = 0$)

Since $2\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n}=\sum_{1}^{\frac{\omega}{2}}\frac{1}{n}$ we obtain $S^{*}=\sum_{1}^{\omega}\frac{1}{n}-\sum_{1}^{\frac{\omega}{2}}\frac{1}{n}=\sum_{\frac{\omega}{2}+1}^{\omega}\frac{1}{n}$

Using (E.M.F) we have

$$\sum_{\frac{\omega}{2}+1}^{\omega}\frac{1}{n}\sim\left\|\frac{n^{-1}}{1}\right\|_{\frac{\omega}{2}}^{\omega}=In2$$

Therefore $S^{*}=In2$ in accordance with commutative property ,in this case p*=q*=ω/2 , therefore it is a non standard rearrangement of (1*) .Let us now look at a well-known result of Ramanujan in the light of non-standard rearrangement. As is known, Ramanujan arrived at the following result

$$\sum_{1}^{\infty}n=1+2+3+4+......=-\frac{1}{12} \qquad (5)$$

This result is clearly absurd, let's see how it should be interpreted according to the non-standard rearrangement introduced in [1] it obviously in non-standard analysis becomes

$$\sum_{1}^{\omega}n=1+2+3+4+......=-\frac{1}{12} \qquad (5^{*})$$

Applying the formula (lll), obtained with a non standard rearrangement of (1*) see[1] we have with s=−1 and R =ω − 1,

$$\zeta^{*}(-1)=\sum_{1}^{\omega-1}n-\frac{1}{2}\omega^{2}+\frac{1}{2}\omega-\omega-\frac{1}{12}\omega^{0} \quad (5^{*})$$ (The number $-\frac{1}{12}$ is the only real number in second member of (5*)).

In (5*) $\omega^{0}=e^{0ln\omega}=e^{0}=1$.Therefore we have (adding and subtracting the positive infinite number ω to the second member)

$\zeta^{*}(-1)=\sum_{1}^{\omega}n-\frac{1}{2}\omega^{2}-\frac{1}{2}\omega-\frac{1}{12}$ That is (since $\zeta^{*}(-1)=-\frac{1}{12}$)

$$\sum_{1}^{\omega}n=\frac{1}{2}\omega^{2}+\frac{1}{2}\omega=\frac{1}{2}\omega(\omega+1) \qquad (6^{*})$$

Where $\zeta^*(-1)$ is the extension at non standard model of analysis of zeta function of Riemann obtained by non standard rearrangement calculated in point -1 as seen in [1] . Furthermore, the following simple formula is well known.

$$\sum_{1}^{m} n = \frac{1}{2} m(m+1) \quad (\forall m, \forall n) \in N$$

it is true for all natural numbers m and n , therefore by transfert principle (see preliminaries and definitions) is true too the following formula extended at non standard model of analysis $\sum_{1}^{m^*} n = \frac{1}{2} m^*(m^*+1) \quad (\forall m^*, \forall n^*) \in N^*$ (That is the (6*) with m*= ω) . Where N^* it is the ensemble of infinite hypernatural numbers .Let us now see how the great indian mathematician Ramanujan obtained his absurd result and compare it step by step with the corresponding results in the non-standard model. Let us give the following series called Grandi's

$$S = 1-1+1-1+1-1+1-1+..... \qquad (7)$$

it is an oscillating series which can assume 2 values depending on whether an even or odd number of addends is counted. Let M be the number of addends of (7) $\forall n \in N$ $M = 2n+1 \Rightarrow S=1$; $\forall n \in N$ $M=2n \Rightarrow S=0$ *which in the non − standard model becomes* $\forall n^* \in N^*$ $M^*=2n^*+1 \Rightarrow S^*=1$, $\forall n^* \in N^*$ $M^*=2n^* \Rightarrow S^*=0$.However, the value of $\frac{1}{2}$ is commonly attributed to S (the same value is also given with the Cesaro and Abel summations)in non standard analysis separating the positive and negative terms of(7)we have:

$$S^* = \sum_{1}^{\frac{\omega}{2}}(1) + \sum_{1}^{\frac{\omega}{2}}(-1) = \frac{1}{2}\omega - \frac{1}{2}\omega = 0 \qquad (7^*)$$

In the case of M* being even number as in the previous case M*=ω then $S^*=0$,if instead M* is odd Number as in the following case with M*=ω+1 then $S^*=1$

$$S^* = \sum_{1}^{\frac{\omega}{2}+1} 1 + \sum_{1}^{\frac{\omega}{2}} -1 = \frac{1}{2}\omega + 1 - \frac{1}{2}\omega = 1$$

Instead using the well-known relation[1]

$$\eta(s) = (1 - \frac{2}{2^s})\, \zeta(s) \qquad (8)$$

with $s=0$ we obtain:

$\eta(0)=(-1)\zeta(0)$ (9) (since $\zeta(0) = -\frac{1}{2}$ [1]) or

1-1+1-1+1-1+1-1+.....= $S = \frac{1}{2}$

However (8) in the first member converges only with a >0 see[1] therefore the second member or $\zeta(s)$ can be calculated by means of (8) only with a>0 and s≠1 [1] so (9) was obtained by calculating the (8) beyond the convergence domain (since it was calculated with a =0). The next step is the following:

1−2+3−4+5−6+.......=U (10)

0+1−2+3−4+5−.......=U (11)

Adding (10) and (11) we have:

1−1+1−1+1−1+1−1+..... =2U (12)

Since we have seen that the Grandi series, i.e. the first member of (12), is usually given the value $\frac{1}{2}$ we obtain U= 1/4, this value is in agreement with the regularization obtained with the Abel summation method. Using (8) again abusing it by calculating with a=−1 (i.e. beyond the convergence domain of $\eta(s)$) we obtain :

$\eta(-1)=(-3)\zeta(-1)$ or

1−2+3−4+5−6+.......= $\frac{1}{4}$

value which is in agreement with the result found before (and in agreement with the Abel summation method).Extending (10) and (11) at non standard analysis we have:

$1-2+3-4+5-6+....-\ \omega = U^*$ (10*)

$0 + 1-2+3-4+5-.... +(\omega-1)-\ \omega = U^*$ (11*) therefore

$(10^*)+(11^*)=2U^*=1-1+1-1+1-1+....-\ \omega =-\ \omega$ therefore $U^*=-\frac{1}{2}\ \omega$

It was considered that the(10*)and (11*)were calculated up to ω where it is considered an even (infinite)hypernatural number and therefore ,as seen before,if the number of addends is even then the Grandi series is 0 (unlike ($\omega\pm 1$) are to be considered odd hypernatural numbers).We can also calculate with non standard analysis U* by separating the positive terms from the negative ones with the following non-standard series

$$U^* = \sum_{1}^{\frac{\omega}{2}}(2n-1) - \sum_{1}^{\frac{\omega}{2}}(2n) = \sum_{1}^{\frac{\omega}{2}}(-1) = \ -\frac{1}{2}\,\omega \qquad (10^*)$$

As a further next step consider the following 2 series

$1+2+3+4+5+6+\ldots\ldots = T \qquad (13)$

$1-2+3-4+5-6+\ldots\ldots = U \qquad (14)$ consider $(13)-(14)$ we have:

$T-U = 0+4+0+8+0+12+0+\ldots\ldots = 4(0+1+0+2+0+3+\ldots\ldots) = 4T \qquad (15)$

being as seen before $U=\frac{1}{4}$

the (15) becomes $3T=-U=-\frac{1}{4}$ Meaning what:

$$T=\sum_{1}^{\infty} n = -\frac{1}{12} \qquad (16)$$

This paradoxical result can also be obtained by calculating (8) at the point s= − 1 we can in fact write (8) as:

$\eta(-1)=(-3)\,\zeta(-1)$ with $\zeta(s)=\sum_{1}^{\infty} n^{-s}$ (see [1]) therefore:

$\zeta(-1)=\sum_{1}^{\infty} n$ being $\eta(-1)=1-2+3-4+5-6+\ldots. = \frac{1}{4}$ (as before seen)

obtain for this :

$\sum_{1}^{\infty} n = -\frac{1}{12}$ but both in the first member where $\eta(s)$ converges only with $a>0$ [1] and in the second member where $\zeta(s)=\sum_{1}^{\infty} n^{-s}$ converges only with $a>1$ [1]there are therefore violations of the convergence interval(or radius of convergence) of these two functions since they are calculated at point $s=-1$.We can calculate (14) to the non standard model as follows:

$$T^* - U^* = 4\sum_{1}^{\frac{\omega}{2}} n \qquad (15^*)$$

($it\ was\ calculated\ up\ to$ $\frac{1}{2}\omega$ because the integers in(14) always alternate with zero) since:

$\sum_{1}^{\frac{\omega}{2}} n = \frac{1}{2}\left(\frac{1}{2}\omega\right)\left(\frac{1}{2}\omega+1\right) = \frac{1}{8}\omega^2 + \frac{1}{4}\omega$ (using $\sum_{1}^{m^*} n = \frac{1}{2}$ m*(m*+1) with m*= $\frac{1}{2}\omega$)

therefore :

$$T^* - U^* = 4\sum_{1}^{\frac{\omega}{2}} n = \frac{1}{2}\omega^2 + \omega \qquad (15^*)$$

Being as seen before in (10*)

$$U^* = \sum_{1}^{\frac{\omega}{2}}(2n-1) - \sum_{1}^{\frac{\omega}{2}}(2n) = \sum_{1}^{\frac{\omega}{2}}(-1) = -\frac{1}{2}\omega$$

the T* of (16*) becomes

$$T^* = \frac{1}{2}\omega^2 + \frac{1}{2}\omega \qquad (16^*) \quad \text{or}$$

$$\sum_{1}^{\omega} n = \frac{1}{2}\omega^2 + \frac{1}{2}\omega = \frac{1}{2}\omega(\omega+1)$$

in perfect agreement with (6*). As for Ramanujan's proof, step by step, we started with the Grandi series, which is given a value of $\frac{1}{2}$ (although for any number of addends we calculate whether they are even or odd, whether they are numerically finite or infinite, we can only obtain 0 or 1, never $\frac{1}{2}$, which in any case is the arithmetic mean between the only two possible values)and then we performed some very simple operations with divergent series (as seen above,it was only an addition of a divergent series with itself, the first time starting from 1 and the second time starting from 0 , and a difference between two divergent series), if such operations had been performed with convergent series, these operations would have been completely legitimate. We obtained an absurd result: the sum of all positive integers is $-\frac{1}{12}$! Should we perhaps assume that in standard analysis the theory of divergent series is inconsistent?While with non-standard analysis, by following the same steps, we obtained consistent results as if we had been dealing with convergent series. The same main summation methods, such as Cesaro's and Abel's, used for the regularization of divergent series, give results that are replicable, as seen above, by mathematical abuses, that is, from considering a relationship valid beyond its convergence interval. Furthermore, many important convergent series, such as the series $\eta(s)$ (with >0) of Dirichlet can be written as the difference between two divergent series (with $a \leq 1$)

$$\eta(s) = \sum_{n=1}^{\infty} (2n - 1)^{-s} - \sum_{n=1}^{\infty} (2n)^{-s}$$

therefore in the interval $0<a<1$ the Dirichlet eta function is both convergent and constituted by the difference between two divergent series.Therefore the possible inconsistency of the theory divergent series could also propagate to the theory of analytic functions which generally in their domain of definition can be represented by convergent series (or by the difference between two divergent series) .

FINAL CONSIDERATIONS

If inconsistencies were demonstrated in the classical or standard theory of divergent series, the definition of mathematical proof would also have to be looked at differently, since in an inconsistent system it is possible to prove both a mathematical proposition and its negation. It would be possible, that is, to prove a certain proposition, making it a theorem in all respects, and even after a long time, one could instead demonstrate the falsity of that theorem, thus deducing the undecidability of the original proposition. This, however, would most likely be due to the inconsistent mathematical system in which the proposition in question was stated or proved. However, perhaps the constant reworking by mathematicians and physicists, with their renormalizations, regularizations, and abstruse methods of summation, could hide such inconsistencies indefinitely, while making the underlying mathematics increasingly complex

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