A Stronger Generalization of the Riemann Functional Equation

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Abstract

A major breakthrough introduced in this paper is a generalization of the Riemann functional equation that has a broader validity domain than the existing one from the literature. The insight that led to this new relation came from a new formula for the zeta function created herein that implies the Riemann functional equation. Further developments that stem from new formulae introduced previously are also discussed.

Summary

1 Introduction

This article presents further developments that stem from new formulae introduced previously for generalized harmonic progressions and their limits.

Among these developments is a generalization of a transformation of the $H_k(n)$ formula that had not been discovered before, and one more integral representation for the Riemann zeta function, $\zeta(k)$, that only holds at the positive integers greater than one. A demonstration of how this new zeta function formula can be extended from the positive integers to the right half-plane, $\Re(k) > 1$, is given.

More importantly, a closer inspection of this extended formula of the Riemann zeta function led to an unexpected discovery, it was found to be just a reformulation of the Riemann functional equation. This in turn provided insight into the possibility of replicating the method with a new formula for the Hurwitz zeta function, $\zeta(k, b)$, which was first introduced in paper [3] and then extended in paper [5]. As a result, an even broader generalization of the Riemann functional equation than the one available in the literature was discovered.

For comparison, the generalized Riemann functional equation from the literature states that,

$$\zeta(k,b) = \boldsymbol{i}\,(2\pi)^{k-1}\,\Gamma(1-k)\left(e^{-k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right), \tag{1}$$

if k is real and $0 < b \leq 1$. The new functional equation created herein allows k to be any real number and b to be any positive real number, except for occasional singularities (though it should hold at the limit when improper).

This paper is not long, but to make the demonstration easier to follow, it was organized in small sections and a content summary was also added.

2 Generalized harmonic progressions

In reference [1], new formulae were created for the generalized harmonic numbers, $H_k(n)$, and varied depending on whether k was odd or even. In reference [2], the same approach was used to produce formulae for a more general sum referred to as generalized harmonic progression, $HP_k(n)$.

For any integers a, b and $k \geq 1$,

$$\sum_{j=1}^{n}\frac{1}{(aj+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(an+b)^k} + \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2}\int_0^1\sum_{j=0}^{k}\frac{B_j\,(1-u)^{k-j}}{j!(k-j)!}\left(e^{2\pi\boldsymbol{i}(an+b)u} - e^{2\pi\boldsymbol{i}b\,u}\right)\cot\pi a u\,du, \tag{2}$$

where the right-hand side can be well-defined even when there are singularities on the left.

2.1 Generalized harmonic numbers

For the generalized harmonic numbers, as seen in [2], a single formula that does not depend on parity is,

$$\sum_{j=1}^{n}\frac{1}{j^k}=\frac{1}{2n^k}+\frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2}\int_0^1\sum_{j=0}^{k}\frac{B_j\,(1-u)^{k-j}}{j!(k-j)!}\left(e^{2\pi\boldsymbol{i}\,nu}-1\right)\cot\pi u\,du, \tag{3}$$

which is simply a particular case of formula (2).

Paper [1] mentioned how for the odd cases the formulae could be transformed based on a new integral representation for the zeta function at the odd integers (which stems from the formula itself). For example, for the case $k=3$,

$$\sum_{j=1}^{n}\frac{1}{j^3}=\frac{1}{2n^3}+\zeta(3)-\frac{\pi^3}{12}\int_0^1\left(u-u^3\right)\cos\pi n(1-u)\tan\frac{\pi u}{2}\,du,$$

although a generalization of that result was not produced.

But with formula (3) that is now straightforward. A simple investigation reveals that in that case the generalization is given by,

$$\sum_{j=1}^{n}\frac{1}{j^k}=\frac{1}{2n^k}+\zeta(k)+\frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2}\int_0^1\sum_{j=0}^{k-1}\frac{B_j\,(1-u)^{k-j}}{j!(k-j)!}e^{2\pi\boldsymbol{i}\,nu}\cot\pi u\,du, \tag{4}$$

that is, the harmonic numbers of order k are a function of $\zeta(k)$ (and vice-versa).

2.2 Zeta at the positive integers

From the limits of the formulae created in [1] new integral representations for the Riemann zeta function at the positive odd integers (greater than one) were derived, such as,

$$\zeta(2k+1)=\frac{(-1)^k\pi^{2k+1}}{2}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}\tan\frac{\pi u}{2}\,du, \tag{5}$$

and,

$$\zeta(2k+1)=-\frac{(-1)^k(2\pi)^{2k+1}}{2}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)u^{2k+1-2j}}{(2j)!(2k+1-2j)!}\cot\pi u\,du \tag{6}$$

The limits of the real and imaginary parts of each individual component of the integral in (3) were deduced by educated guesses and their validity was confirmed by the integral representations derived thereafter, such as (5) and (6) (in the case of the formulae that were not created using the exponential function). That was first done in [3] and the statements were later refined in [4].

If k is real, the limit for the imaginary part can be stated as,

$$\lim_{n\to\infty}\int_0^1 (1-u)^k \sin 2\pi nu \cot \pi u \, du = \begin{cases} 1, & \text{if } k=0 \\ 1/2, & \text{if } k>0, \end{cases} \tag{7}$$

and for the real part as,

$$\int_0^1 (1-u)^k \left(\cos 2\pi nu - 1\right) \cot \pi u \, du \sim -\frac{H(n)}{\pi} + \int_0^1 \left(u^k - u\right) \cot \pi u \, du, \tag{8}$$

which only holds for positive real k. The integral to the left is zero if $k=0$ and n is a positive integer, while the integral to the right explodes out to infinity.

Paper [2] did not show how the values of the zeta function at the positive integers (greater than one) can be derived from (3), but from (4) that is now possible. First note that for positive integer n,

$$\int_0^1 \left(e^{2\pi \boldsymbol{i}\, nu} - 1\right) \cot \pi u \, du = \boldsymbol{i}, \tag{9}$$

and therefore (4) can be rewritten as,

$$\sum_{j=1}^{n} \frac{1}{j^k} = \frac{1}{2n^k} - \frac{B_k\,(2\pi \boldsymbol{i})^k}{2\,k!} + \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^k}{2} \int_0^1 \left(\sum_{j=0}^{k} \frac{B_j\,(1-u)^{k-j}}{j!(k-j)!} - \frac{B_k}{k!} \right) \left(e^{2\pi \boldsymbol{i}\, nu} - 1\right) \cot \pi u \, du,$$

Through the aforementioned limits, (7) and (8), if $k \neq 1$ is a non-negative integer then,

$$\begin{aligned} \zeta(k) &= -\frac{B_k\,(2\pi \boldsymbol{i})^k}{2\,k!} + \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^k}{2} \int_0^1 \sum_{j=0}^{k-1} \frac{B_j\,u^{k-j}}{j!(k-j)!} \cot \pi u \, du \\ &= \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^k}{2} \int_0^1 \left(\frac{B_k}{k!} e^{2\pi \boldsymbol{i}\, nu} - \sum_{j=0}^{k} \frac{B_j\,(1-u)^{k-j}}{j!(k-j)!} \right) \cot \pi u \, du, \end{aligned} \tag{10}$$

where the rightmost formula is obtained from equations (3) and (4) and n is any positive integer. Oddly, these formulae hold even for $k=0$.

2.3 Extending $H_k(n)$ and $\zeta(k)$

How can one obtain the extension of these formulae from positive integer to complex k?

Looking at formula (2), the discrete sum within the integral can be extended to complex k as follows,

$$\sum_{j=0}^{k+1} \frac{B_j\,u^{k+1-j}}{j!(k+1-j)!} = \sum_{q=-1}^{k} \frac{B_{q+1} u^{k-q}}{(q+1)!(k-q)!} = \frac{u^{k+1}}{(k+1)!} - \frac{u^k}{k!} - \sum_{q=0}^{k} \frac{\zeta(-q)\,u^{k-q}}{q!(k-q)!},$$

since for integer q,

$$\frac{B_{q+1}}{q+1} = \begin{cases} -1/2, & \text{if } q = 0 \\ -\zeta(-q), & \text{if } q > 0 \end{cases},$$

Therefore, using an identity for $\zeta(-k, u+1)$ from [5],

$$\sum_{j=0}^{k+1} \frac{B_j\, u^{k+1-j}}{j!(k+1-j)!} = -\frac{u^k}{k!} - \frac{1}{k!}\left(-\frac{u^{k+1}}{k+1} + \sum_{q=0}^{k} \binom{k}{q} \zeta(-q)\, u^{k-q}\right)$$

$$= -\frac{1}{k!}\left(u^k + \zeta(-k, u+1)\right) = -\frac{\zeta(-k,u)}{k!},$$

and if the rightmost function is differentiated once with respect to u then,

$$\sum_{j=0}^{k} \frac{B_j\, u^{k-j}}{j!(k-j)!} = -\frac{\zeta(-k+1,u)}{(k-1)!} \tag{11}$$

This extended formula in turn implies,

$$\sum_{j=1}^{n} \frac{1}{j^k} = \frac{1}{2n^k} - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)} \int_0^1 \zeta(-k+1, 1-u)\left(e^{2\pi\boldsymbol{i}\,nu} - 1\right) \cot \pi u\, du$$

$$= \frac{1}{2n^k} + \zeta(k) - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)} \int_0^1 \left(\zeta(-k+1, 1-u) - \zeta(-k+1)\right) e^{2\pi\boldsymbol{i}\,nu} \cot \pi u\, du, \tag{12}$$

where the first formula holds for $\Re(k) > 0$, when equation (3) is used, and the second formula holds for $\Re(k) > 1$, when equation (4) is used.

3 The extended zeta formula

Through the extended function from the previous section, (11), the Riemann zeta function formula in (10) can be extended from the non-negative integers to $\Re(k) > 1$,

$$\zeta(k) = \frac{(2\pi\boldsymbol{i})^k\, \zeta(-k+1)}{2\,\Gamma(k)} - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)} \int_0^1 \left(\zeta(-k+1, u) - \zeta(-k+1)\right) \cot \pi u\, du \tag{13}$$

From this formula the Riemann functional equation can be deduced, as described next. Since for real k the imaginary part on the right-hand side must be zero, it can be used to deduce a closed-form for the real-valued integral,

$$\int_0^1 \left(\zeta(-k+1, u) - \zeta(-k+1)\right) \cot \pi u\, du = \tan \frac{k\,\pi}{2}\, \zeta(-k+1),$$

and the above formula when replaced in the real part of the initial equation,

$$\zeta(k) = \frac{(2\pi)^k}{2\,\Gamma(k)}\, \zeta(-k+1) \left(\cos\frac{k\,\pi}{2} - \cos\frac{(k+1)\pi}{2} \tan\frac{k\,\pi}{2}\right) = \frac{(2\pi)^k}{2\,\Gamma(k)}\, \zeta(-k+1) \sec\frac{k\,\pi}{2},$$

leads to the Riemann functional equation. This explains why the analytic continuation of the zeta function appears in the new formula.

3.1 The extended Hurwitz zeta formula

The prior result was the insight that led to the following discovery, whose derivation requires both parameters, k and b, to be real.

Throughout this section, let b be a positive non-integer number, unless otherwise noted. The extended Hurwitz zeta function formula from reference [5], which holds in the right half-plane $\Re(k) > 1$, is,

$$\zeta(k,b) = \sum_{j=0}^{\infty} \frac{1}{(j+b)^k} = \frac{1}{2\,b^k} + \frac{(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)} \int_0^1 \left(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, u) - \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right) \cot \pi u \, du \tag{14}$$

If the shift is $-b$ though, the picture gets more complicated and the above formula does not hold (the real part is right, but not the imaginary part).

To understand why, it is necessary to analyze what happens when the shift is negative. By definition, a real power of a positive number is always a positive number[1], but if the base is negative, the result can be a non-real number. For example, if $-1 < -b < 0$, then all the terms of the Hurwitz zeta series are positive and hence well-behaved, except term $j = 0$. For any other b, only the terms of the series that have $j - b < 0$ contribute to the imaginary part of $\zeta(k,-b)$, that is,

$$\Im(\zeta(k,-b)) = \Im\left(\sum_{j=0}^{\lfloor b \rfloor} \frac{1}{(j-b)^k}\right) \tag{15}$$

Fortunately, it is not hard to fix the formula (14) when the shift is $-b$, as a simple observation of the patterns reveals that, surprisingly, the difference can be attributed to the imaginary part of the terms that become negative, as in the below,

$$\zeta(k,-b) = \sum_{j=0}^{\infty} \frac{1}{(j-b)^k} = \frac{1}{2(-b)^k} + \boldsymbol{i}\,\Im\left(\frac{1}{(-b)^k} + 2\sum_{j=1}^{\lfloor b \rfloor} \frac{1}{(j-b)^k}\right) + \frac{(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)}\,\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right) - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)} \int_0^1 \left(e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b}, -k+1, u) - \mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) \cot \pi u \, du \tag{16}$$

3.2 Reflection symmetry relations

When the two equations, (14) and (16), are added together or subtracted from each other, each part becomes either purely real or purely imaginary regardless of the value of b. That is essential for the method to work since the idea is to form a system of equations, so that the unknown part, the integral, can be replaced by an expression involving well-known functions, leading to a closed-form.

[1] When the exponentiation is multi-valued, for example, $4^{1/2} = \pm 2$, the positive root is taken by definition.

This is the difference between the two reflection formulae,

$$\zeta(k,b)-\zeta(k,-b)=\frac{1}{2\,b^k}-\frac{1}{2(-b)^k}-\boldsymbol{i}\,\Im\left(\frac{1}{(-b)^k}+2\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right)$$
$$+\frac{(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$-\frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)}\int_0^1\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-k+1,u)$$
$$-\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du,\quad (17)$$

and the formula for the addition is not shown since it is trivially similar. These formulae allow the real and imaginary parts to be separated.

4 Difference formulae

This section was created just to organize the calculations better and make them easier to follow. A system of equations is created with the purpose of replacing the integral with an equivalent closed-form.

4.1 The real part

The real part for the difference, formula (17), is

$$\zeta(k,b)-\Re(\zeta(k,-b))=\frac{1}{2\,b^k}-\Re\left(\frac{1}{2(-b)^k}\right)$$
$$+\frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\sin\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$-\frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\sin\frac{(k+1)\pi}{2}\int_0^1\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-k+1,u)$$
$$-\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du\quad (18)$$

4.2 The imaginary part

The imaginary part for the difference is,

$$
\begin{aligned}
-\boldsymbol{i}\,\Im(\zeta(k,-b)) = -\boldsymbol{i}\,\Im&\left(\frac{1}{2(-b)^k}+\sum_{j=0}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right)\\
&+\frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\
-\frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{(k+1)\pi}{2}\int_0^1&\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-k+1,u)\\
&-\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du
\end{aligned}
$$

Notice the cancellation of $\Im\zeta(k,-b)$ with its respective equivalent formula from (15). The equation can therefore be simplified as below,

$$
\begin{aligned}
\frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{(k+1)\pi}{2}\int_0^1&\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-k+1,u)\\
&-\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du=\\
-\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^k}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right)&+\frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)
\end{aligned}
\tag{19}
$$

4.3 The closed-form

When the closed-form of the integral from (19) is replaced in equation (18), the below is obtained,

$$
\begin{aligned}
\zeta(k,b)-\Re(\zeta(k,-b)) = \frac{1}{2\,b^k}-\Re&\left(\frac{1}{2(-b)^k}\right)\\
&+\frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\sin\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\
-\boldsymbol{i}\tan\frac{(k+1)\pi}{2}\left(-\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^k}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right)\right.&\left.+\frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\right)
\end{aligned}
$$

It can be simplified using the below identity,

$$
\boldsymbol{i}\sin\frac{k\,\pi}{2}-\boldsymbol{i}\tan\frac{(k+1)\pi}{2}\cos\frac{k\,\pi}{2}=\boldsymbol{i}\csc\frac{k\,\pi}{2},
\tag{20}
$$

which then gives the final relation for the difference,

$$\zeta(k,b) - \Re(\zeta(k,-b)) = \frac{1}{2\,b^k} - \Re\left(\frac{1}{2(-b)^k}\right) + \frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\csc\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - \mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) - \tan\frac{(k+1)\pi}{2}\,\Im\left(\frac{1}{2(-b)^k} + \sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right) \tag{21}$$

5 Addition formulae

Like before, this section aims to organize better the calculations of the system of equations obtained by adding the reflection formulae together.

5.1 The real part

The real part for the addition is,

$$\zeta(k,b) + \Re(\zeta(k,-b)) = \frac{1}{2\,b^k} + \Re\left(\frac{1}{2(-b)^k}\right) + \frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) + \mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) - \frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{(k+1)\pi}{2}\int_0^1\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u) + e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-k+1,u) - \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - \mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du \tag{22}$$

5.2 The imaginary part

Like in the analogous section, this equation can be simplified as,

$$\frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\sin\frac{(k+1)\pi}{2}\int_0^1\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u) + e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-k+1,u) - \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - \mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du = \boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^k} + \sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right) + \frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\sin\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) + \mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) \tag{23}$$

5.3 The closed-form

When the closed-form of the integral from (23) is replaced in equation (22), the below is obtained,

$$\zeta(k,b)+\Re(\zeta(k,-b))=\frac{1}{2\,b^k}+\Re\left(\frac{1}{2(-b)^k}\right)$$
$$+\frac{(2\pi)^k}{2\,\Gamma(k)}\cos\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$\boldsymbol{i}\cot\frac{(k+1)\pi}{2}\left(\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^k}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right)+\frac{(2\pi)^k}{2\,\Gamma(k)}\boldsymbol{i}\sin\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\right)$$

This one can also be simplified through the below identity,

$$\cos\frac{k\,\pi}{2}+\boldsymbol{i}\cot\frac{(k+1)\pi}{2}\,\boldsymbol{i}\sin\frac{k\,\pi}{2}=\sec\frac{k\,\pi}{2},\tag{24}$$

which then gives this final relation for the addition,

$$\zeta(k,b)+\Re(\zeta(k,-b))=\frac{1}{2\,b^k}+\Re\left(\frac{1}{2(-b)^k}\right)$$
$$+\frac{(2\pi)^k}{2\,\Gamma(k)}\sec\frac{k\,\pi}{2}\left(\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$-\cot\frac{(k+1)\pi}{2}\,\Im\left(\frac{1}{2(-b)^k}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right)\tag{25}$$

6 Generalized Riemann functional equation

To obtain a simpler relation, formulae (21) and (25) are combined, with a few of the parts cancelling out.

The final relation is further simplified by means of the following identities,

$$\sec\frac{k\,\pi}{2}+\boldsymbol{i}\csc\frac{k\,\pi}{2}=2\,\boldsymbol{i}\,\frac{e^{-k\,\boldsymbol{i}\,\pi/2}}{\sin k\,\pi},$$
$$\sec\frac{k\,\pi}{2}-\boldsymbol{i}\csc\frac{k\,\pi}{2}=-2\,\boldsymbol{i}\,\frac{e^{k\,\boldsymbol{i}\,\pi/2}}{\sin k\,\pi},$$
$$\tan\frac{(k+1)\pi}{2}+\cot\frac{(k+1)\pi}{2}=-2\csc k\,\pi$$

which gives the relation,

$$\zeta(k,b) = \frac{1}{2\,b^k} + \frac{1}{\sin k\,\pi}\,\Im\left(\frac{1}{2(-b)^k} + \sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right) + \frac{\boldsymbol{i}\,(2\pi)^k}{2\,\Gamma(k)\,\sin k\,\pi}\left(e^{-k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right),$$

Finally, Euler's reflection formula,

$$\frac{1}{2\,\Gamma(k)\,\sin k\,\pi} = \frac{\Gamma(1-k)}{2\pi},$$

leads to a relation that resembles the literature more closely, as seen in equation (1),

$$\zeta(k,b) = \frac{1}{2\,b^k} + \frac{1}{\sin k\,\pi}\,\Im\left(\frac{1}{2(-b)^k} + \sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^k}\right) + \boldsymbol{i}\,(2\pi)^{k-1}\,\Gamma(1-k)\left(e^{-k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right), \quad (26)$$

which holds for all real k and all positive real b, except for the singularities at integer k (all of which can be worked around with limits, except $k = 1$).

The Riemann functional equation is a particular case of this more general functional equation, as it can be verified by making $b = 1$ (the relation also holds for positive integer b, as long as the singularity that appears in the imaginary part function is removed).

The generalized functional equation from the literature is also a particular case of this new broader relation. The proof requires (or implies) the below identity,

$$\frac{1}{2\,b^k} + \frac{1}{\sin k\,\pi}\,\Im\left(\frac{1}{2(-b)^k}\right) = 0, \text{ if } 0 < b \leq 1 \quad (27)$$

References

[1] Risomar Sousa, Jose *Generalized Harmonic Numbers, eprint arXiv:1810.07877,* 2018.

[2] Risomar Sousa, Jose *Generalized Harmonic Progression, eprint arXiv:1811.11305,* 2018.

[3] Risomar Sousa, Jose *The Hurwitz Zeta Function at the Positive Integers, eprint arXiv:1902.06885,* 2019.

[4] Risomar Sousa, Jose *Lerch's* Φ *and the Polylogarithm at the Positive Integers, eprint arXiv:2109.15306,* 2020.

[5] Risomar Sousa, Jose *Lerch's* Φ *and the Polylogarithm at the Negative Integers, eprint arXiv:2006.08406,* 2021.