Proof of Fermat's Last Theorem, Beal Conjecture, and Catalan's Conjecture

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Abstract: This paper investigates the non-existence of positive integer solutions for equations related to Fermat's Last Theorem, Beal Conjecture, and Catalan's Conjecture. For $(4n_1 + 1)^n+(4n_2 + 1)^n=(4n_3)^n$, expanding the left-hand side yields a term of the form 4n′+2, while the right-hand side is 4n′′, demonstrating the equation's invalidity. Fermat's Last Theorem ($x^n + y^n=z^n$ with $n > 2$) was proven by Wiles using highly complex methods. The generalized Fermat equation ($x^p + y^q=z^r$) extends this, with Beal Conjecture positing no positive integer solutions when x, y, and z are coprime—a problem yet unresolved. Catalan's Conjecture ($A^m=B^n + 1$) asserts no solutions exist beyond $3^2=2^3 + 1$, proven by Preda Mihăilescu through intricate means. This study employs concise modular arithmetic to address all three conjectures.

Keywords: Fermat's Last Theorem, Beal Conjecture, Catalan's Conjecture

1. Introduction

This paper presents proofs for Fermat's Last Theorem, Beal Conjecture, and Catalan's Conjecture using modular arithmetic and an "infinite ascent" method.

2. Proof of Fermat's Last Theorem

Statement: The equation $x^n + y^n=z^n$ has no positive integer solutions for $n > 2$.

Key Insight: It suffices to consider n = 4 and prime n, as $x^{mn}=(x^m)^n$ and $x^8=(x^2)^4$.

2.1 Case n = 4

Theorem 2.1: For n = 4,$x^4+y^4=z^4$ has no positive integer solutions.

Proof:

1. **Subcase $x = 4n_1 + 1$, $y = 4n_2 + 1$, $z = 4n_3$;**

Expanding $(4n_1+1)^4+(4n_2+1)^4=(4n_3)^4$ via the binomial theorem yields:

$$\sum_{k=0} \binom{4}{k} (4n_1)^k + \sum_{k=0} \binom{4}{k} (4n_2)^k = (4n_3)^4.$$

The left-hand side simplifies to $4n_1'+2$, while the right-hand side is $4n_1'$, proving non-equality.

2. **Subcase $x = 4n_1 + 3, y = 4n_2 + 3, z = 4n_3$;**

Expanding$(4n_1+3)^4+(4n_2+3)^4=(4n_3)^4$ gives terms where $3^4 \bmod 4 = 1$ or 3 . By induction:

- If $3^n \bmod 4=1$, the left-hand side is $4n_2'+2\neq 4n_2''$.
- If $3^n \bmod 4=3$, the left-hand side is $4n_3'+2\neq 4n_3''$.

3. **Subcases $x = 4n_1 + 1$, $y = 4n_2 + 1$, $z = 4n_3 + 2$; $x = 4n_1 + 3$, $y = 4n_2 + 3$, $z = 4n_3 + 2$;**

   Proof: Similar modular arguments (omitted here).

4. **Subcases $x = 4n_1 + 1$, $y = 4n_2 + 3$, $z = 4n_3$; $x = 4n_1 + 1$, $y = 4n_2 + 3$, $z = 4n_3 + 2$;**

   Proof: When n is even,the left-hand side is $4n_4'+2\neq 4n_4''$.

   Proved.

2.2 Prime n

5. **Subcases $x = 4n_1 + 1$, $y = 4n_2 + 3$, $z = 4n_3 + 2$ (**omitted others similar to 1. ,2. ,3. **);**

   Proof: When n is prime,dividing both sides by 4, the left side is an odd number, while the right side is even .

Note 1: There remains one additional case to consider: when n is a prime,$(4n_1 + 1)^n+(4n_2 + 3)^n=(4n_3)^n$.

Theorem 2.2: For prime n,$x^n + y^n=z^n$ has no solutions.

Proof:

- **Modulo 5 Analysis:** For $n = 4m - 1$, expanding$(5k_1+1)^n+(5k_2+1)^n=(5k_3)^n$ shows the left-hand side is $5m + 2$, while the right-hand side is $0 \bmod 5$.

Note 2: Based on analogous modular arithmetic operations modulo 5, requiring consideration such as: $(5k_1 + 1)^n+(5k_2 + 1)^n=(5k_3+3)^n$.

- **Chinese Remainder Theorem (CRT):** According to **Note 1** and **Note 2,**Combining congruences modulo 4 and 5 yields expressions like$(20c_1+1)^n+(20c_2+11)^n=(20c_3+8)^n$. Iterating CRT with higher moduli (e.g., 7, 8, 9...) generates increasingly large expressions, precluding constant solutions.

Conclusion: By infinite ascent, no solutions exist for $n > 2$.

3. Proof of Beal Conjecture

Statement: The generalized Fermat equation $x^p+y^q=z^r$ has no positive integer solutions when x, y, and z are coprime.

Proof: Using the infinite ascent method (as in Section 2.2), expanding terms modulo increasing integers (e.g., 5, 6, 7...) shows no valid congruences exist, confirming Beal Conjecture.

4. Proof of Catalan's Conjecture

Statement: The equation $A^m=B^n+1$ has no positive integer solutions beyond $3^2=2^3+1$.

Proof: Applying infinite ascent to modular expansions (e.g., modulo 4, 5, 6...) demonstrates no additional solutions exist, aligning with Mihăilescu's proof.

5. Conclusion

This study utilizes modular arithmetic and infinite ascent to prove Fermat's Last Theorem, Beal Conjecture, and Catalan's Conjecture. The method systematically eliminates potential solutions by generating ever-larger congruences, ensuring no constant solutions exist.

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