

Mathematics for Incompletely Predictable Problems from the Input-White Box-Output Models of A228186, Primes, Composites and Nontrivial zeros of Riemann zeta function

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Abstract

Utilizing Input-White Box-Output (I-WB-O) Modeling, we outline novel applications of Mathematics for Incompletely Predictable Problems (MIPP) to unique sets and subsets from prime and composite numbers, nontrivial zeros of Riemann zeta function, and relevant number sequences from On-Line Encyclopedia of Integer Sequences (OEIS). We show MIPP is valid for selected mathematical functions, equations or algorithms that contain an equality relationship between two expressions. When applied to OEIS number sequence A228186, MIPP is also valid for an inequality. Inclusion-exclusion (I-E) principle from combinatorics removes contributions from over-counted elements in sets and subsets. By invoking I-E principle, arising consequences from MIPP formulations containing I-WB-O Models will provide necessary mathematical arguments for rigorously solving open problems Riemann hypothesis, Polignac's and Twin prime conjectures.

Keywords: Birch and Swinnerton-Dyer conjecture, Carmichael numbers, Polignac's and Twin prime conjectures, Riemann hypothesis

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1. Introduction

Equations and inequalities are mathematical sentences formed by relating two expressions to each other. In an equation, two expressions are deemed equal as indicated by symbol "=" [viz, an equation contains equality relationship]. Mathematics for Incompletely Predictable Problems (MIPP) is valid for various chosen functions, equations or algorithms that contain equality relationship[6]. The eligible functions or algorithms are literally quasi-equations containing this "analogical" equality relationship. For instance: Origin point intercepts \equiv Gram[x=0, y=0] points in Analytically continued *proxy* Dirichlet eta function = {Set of All Nontrivial zeros of Riemann zeta function}. Algorithm *Sieve-of-Eratosthenes* \equiv All Integers greater than 1 with exactly two factors, 1 and the number itself = {Set of All Prime numbers}.

In an inequality, two expressions are not necessarily equal as indicated by symbols ">", "<", " \leq " or " \geq ". As deductively shown in section 2 using proven mathematical arguments, MIPP is also valid for the selected number sequences A100967 and A228186 from On-Line Encyclopedia of Integer Sequences (OEIS) that are precisely defined by inequalities.

We advocate $Input \rightarrow \boxed{Box} \rightarrow Output$ Modeling should result in two descriptive types whereby the " \rightarrow " must in general be replaced by the " \rightleftharpoons " to indicate bidirectional reversibility: Input-White Box-Output (I-WB-O) Model and Input-Black Box-Output (I-BB-O) Model. I-BB-O Model simply refers to I-WB-O Model when its "White Box" is unknown [not explicitly specified], which is traditionally labeled as "Black Box".

1 Basic questions associated with Riemann hypothesis, Polignac's and Twin
2 prime conjectures, and Birch and Swinnerton-Dyer conjecture are *easy to state*
3 *but difficult to resolve or reconcile*. Sir Isaac Newton in 1675 wrote this expres-
4 sion: "If I have seen further [than others], it is by *standing on the shoulders*
5 *of Giants* (Latin: *nani gigantum humeris insidentes*)". This famous metaphor
6 meant *discovering truth by building on previous discoveries*. The inclusion-
7 exclusion (I-E) principle from combinatorics remove all contributions from over-
8 counted elements in sets and subsets. By appropriately invoking I-E principle,
9 arising consequences from relevant MIPP formulations that use I-WB-O Model-
10 ing will provide all necessary correct and complete mathematical arguments for
11 rigorously proving some of these intractable open problems in Number theory.
12

13

14 **2. Hybrid integer sequence A228186, Prime-Composite identifier**
15 **grouping and Inclusion-Exclusion principle**

16

17 Binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2 \cdot 1}$

18

19 $= \prod_{\ell=1}^k \frac{n-\ell+1}{\ell} = \prod_{\ell=0}^{k-1} \frac{n-\ell}{k-\ell}$. Despite having k factors in both numerator and

20

21 denominator of the fraction, it is actually an integer. Denoted by binomial
22 (n, k) , it is equivalent to combination [order does not matter and *without*
23 *repetition*] ${}^n C_k = C(n, k)$, which represents the number of ways to choose k

24

25 items from a set of n distinct items such that the order of selection does not
26 matter. Combination [order does not matter and *with repetition*] is given by

27

28 $\frac{(k+n-1)!}{k!(n-1)!}$. Permutation [order matters and *without repetition*] ${}^n P_k = P(n, k)$

29

30 $= \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}_{k \text{ factors}}$ is 0 when $k > n$, and otherwise is equal

31

32 to $\frac{n!}{(n-k)!}$. It represents the number of ways to choose k items from a set

33

34 of n distinct items such that the order of selection does matter. Permutation
35 [order matters and *with repetition*] is given by $n \times n \times n \times \cdots (k \text{ times}) =$

36

37 n^k . We obtain $C(n, k) = \frac{P(n, k)}{P(k, k)} = \frac{n^k}{k!} = \frac{n!}{(n-k)! k!}$ and note the number of

38

39 permutations will always be greater than the number of combinations.

40

41 *Paired [infinite-length] integer sequences with prestigious connections:*

42

43 A100967+0, which is A100967[4], is precisely defined as "Least k such that
44 binomial($2k+1, k-n-1$) \geq binomial($2k, k$) viz. $(2k+1)!k!k! \geq (2k)!(k-n-1)!(k+n+2)!$ ". The [infinite-length] terms commencing from Position n
45 $= 1, 2, 3, 4, 5, \dots$ of A100967+0 are 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191,
46 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091,

1 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366,
2 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, 3535,....

3 A100967+1 is conveniently defined as "Add 1 to each and every terms
4 from A100967+0". The [infinite-length] terms commencing from Position $n =$
5 1, 2, 3, 4, 5,... of A100967+1 are 4, 10, 19, 30, 45, 62, 82, 105, 131, 160, 192,
6 226, 264, 304, 348, 394, 443, 495, 550, 607, 668, 731, 798, 867, 939, 1014, 1092,
7 1173, 1256, 1343, 1432, 1525, 1620, 1718, 1819, 1923, 2030, 2139, 2252, 2367,
8 2486, 2607, 2731, 2858, 2988, 3120, 3256, 3395, 3536,....

9 A228186[5] is precisely defined as "Smallest natural number k such that
10 $(k + n^* + 1)!(k - n^* - 2)! < 2k!(k - 1)!$ ". It can be alternatively defined as
11 "Greatest natural number $k > n^*$ such that calculated peak values for ratio R
12 $= \frac{\text{Combinations With Repetition}}{\text{Combinations Without Repetition}} = \frac{(k + n^* - 1)!(n^* - k)!}{n^*!(n^* - 1)!}$ belong to the
13 maximal rational numbers < 2 ". (Offset) [infinite-length] terms commencing
14 from Position $n^* = 0, 1, 2, 3, 4, 5, \dots$ of A228186 are 4, 9, 18, 29, 44, 61, 81, 104,
15 130, 159, 191, 226, 263, 304, 347, 393, 442, 494, 549, 607, 667, 731, 797, 866,
16 938, 1013, 1091, 1172, 1256, 1342, 1432, 1524, 1619, 1717, 1818, 1922, 2029,
17 2139, 2251, 2367, 2485, 2606, 2730, 2857, 2987, 3120, 3255, 3394, 3535,....
18

19 As a mathematical curiosity using notation $n^* = n - 1 = 0, 1, 2, 3, 4, 5 \dots$
20 [where $n = n^* + 1 = 1, 2, 3, 4, 5 \dots$] and abbreviation CIS = Countably Infinite
21 Set and CFS = Countably Finite Set; A228186 is an innovative [infinite-length
22 \equiv CIS] "Hybrid integer sequence" that is identical to [infinite-length \equiv CIS]
23 "non-Hybrid integer sequence" A100967+0 except for the interspersed [finite-
24 length \equiv CFS] 21 'exceptional' terms located at Position $n^* = 0, 11, 13, 19,$
25 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with
26 their associated 21 values exactly specified by "non-Hybrid integer sequence"
27 A100967+1 at [corresponding] Position $n = 1, 12, 14, 20, 22, 29, 31, 38, 40,$
28 46, 51, 53, 56, 58, 63, 67, 71, 74, 78, and 82.

29 *Definition 2.1.* With "Entity X" forming a Countably Infinite Set and
30 irrespective of whether "Entity X" are Completely or Incompletely Predictable
31 entities, we consistently define " n^{th} Gap of Entity X" = " $(n+1)^{\text{th}}$ Entity X" -
32 " $(n)^{\text{th}}$ Entity X". This [locational] definition is usually designated for Position
33 $n = 1, 2, 3, 4, 5 \dots$ e.g. n^{th} Prime Gap = $(n+1)^{\text{th}}$ Prime number - $(n)^{\text{th}}$
34 Prime number with using Position $n = 1, 2, 3, 4, 5 \dots$ [whereby we arbitrarily
35 denote *small Prime gaps* to be 2 and 4, and *large Prime gaps* to be ≥ 6]. This
36 definition is equally valid when designated for Position $n^* = 0, 1, 2, 3, 4, 5 \dots$

37 Then for $n^* = 0, 1, 2, 3, 4, 5 \dots$; the formulation is there exist infinitely-
38 many n^{th} A228186 Gaps {5, 9, 11, 15, 17, 20, 23, 30, 29, 32, 35, 37, 41, 43,
39 46, 45, 52, 55, 58, 60, 64, 66, 69, 72, 75, 78, 81, 84, 86, 90, 92, 95, 98, 101, 104,
40 107, 110, 112, 116, 118, 121, 124, 127, 130, 133, 135, 139, 141, 144, 148, 150,
41 153,....}. Our n^{th} A228186 Gaps are Incompletely Predictable entities seen
42

1 to generally manifest fluctuating "cyclical" behavior and progressively increase
2 in an unpredictable constant "linear" manner. For instance, when given a
3 randomly selected $k = 14572$ value [an "unknown" $(n)^{*th}$ A228186 term that
4 satisfies the inequality in A228186], we can only obtain its correct Position n^*
5 $= 99$ by determining all preceding $n^* = 0, 1, 2, 3, 4, 5, \dots, 97, 98$ [total = 99]
6 values that are associated with their corresponding k [total = 99] values.

7 *Remark 2.1.* For $n^* = 0, 1, 2, 3, 4, 5, \dots \equiv Input$; we obtain infinite-length
8 OEIS number sequence A228186 [based on an inequality $\equiv White\ Box$] with its
9 $(n)^{*th}$ A228186 terms $\equiv Output$ and associated n^{*th} A228186 Gaps $\equiv Output$
10 that comply with both I-WB-O Modeling and MIPP from section 3.
11

12 *Definition 2.2.* Formal definition for *Prime-Composite identifier grouping*:
13 We notationally use both i and $n = 1, 2, 3, 4, 5, \dots$ in the following discussions
14 to avoid ambiguity: Let E = even numbers, O = odd numbers, \mathbb{P} = prime
15 numbers, \mathbb{C} = composite numbers, even Prime gap $_i = O-\mathbb{P}_{i+1} - O-\mathbb{P}_i = 2, 4,$
16 $6, 8, 10, 12, \dots$, Composite gap $_i = \mathbb{C}_{i+1} - \mathbb{C}_i = 1, 2$. For even Prime gaps $4, 6,$
17 $8, 10, 12, \dots$, we can generate the orderly consecutive numbers as sequence {Gap
18 $2-E-\mathbb{C}_1, O-\mathbb{P}_i, \text{Gap } 1-E-\mathbb{C}_2, \text{Gap } 1-O-\mathbb{C}_3, \text{Gap } 1-E-\mathbb{C}_4, \text{Gap } 1-O-\mathbb{C}_5, \dots, \text{Gap } 1-E-$
19 $\mathbb{C}_{n-2}, \text{Gap } 1-O-\mathbb{C}_{n-1}, \text{Gap } 2-E-\mathbb{C}_n, O-\mathbb{P}_{i+1}$ }. The cardinality of sub-sequence
20 {Gap $1-E-\mathbb{C}_2, \text{Gap } 1-O-\mathbb{C}_3, \text{Gap } 1-E-\mathbb{C}_4, \text{Gap } 1-O-\mathbb{C}_5, \dots, \text{Gap } 1-E-\mathbb{C}_{n-2}, \text{Gap}$
21 $1-O-\mathbb{C}_{n-1}$ } = even Prime gap $_i - 2 = n - 2$. However for twin primes; this
22 sub-sequence [as an empty set or null set] do not exist with its cardinality = 0
23 since even Prime gap $2 - 2 = 0$. With cardinality of this sub-sequence given by
24 the involved even Prime gap minus 2; we conveniently define **$\mathbb{P}-\mathbb{C}$ identifier**
25 **grouping**[6] as {Gap $2-E-\mathbb{C}_1, O-\mathbb{P}_i, \text{Gap } 1-E-\mathbb{C}_2, \text{Gap } 1-O-\mathbb{C}_3, \text{Gap } 1-E-\mathbb{C}_4,$
26 $\text{Gap } 1-O-\mathbb{C}_5, \dots, \text{Gap } 1-E-\mathbb{C}_{n-2}, \text{Gap } 1-O-\mathbb{C}_{n-1}$ } for Arbitrarily Large Number
27 of even Prime gaps $4, 6, 8, 10, 12, \dots$ with caveat $\mathbb{P}-\mathbb{C}$ identifier grouping for
28 even Prime gap 2 is an exception given by Gap $2-E-\mathbb{C}_1, O-\mathbb{P}_i$.

29 The [decelerating] size of equally distributed Gap $2n-O-\mathbb{P}$ and Gap $2-E-\mathbb{C}$
30 is "inversely proportional" to [accelerating] size of equally distributed Gap $1-$
31 $E-\mathbb{C}$ and Gap $1-O-\mathbb{C}$. Gap $2-E-\mathbb{C}_n$ is now acting as the new Gap $2-E-\mathbb{C}_1$ for
32 $O-\mathbb{P}_{i+1}$ in the following perpetually repeating cycles of $O-\mathbb{P}_i$ to $O-\mathbb{P}_{i+1}$ with
33 a [usually] different even Prime gap $_i$ [except for rare recurring cases of two
34 or more consecutive $O-\mathbb{P}$ having two or more identical consecutive even Prime
35 gaps involving 6 and multiples of 6].

36 Abbreviations: CFS = Countably Finite Set, CIS = Countably Infinite Set,
37 UIS = Uncountably Infinite Set. From Remark 3.2 in section 3 on associating
38 "thin set" with "decelerating CIS" and "thick set" with "accelerating CIS",
39 we provide the following insightful deductions when analyzing $\mathbb{P}-\mathbb{C}$ identifier
40 grouping. **Subset** {Gap $2-E-\mathbb{C}$ } and **Subsets** {Gap $2n-O-\mathbb{P}$ } are identical
41 "thin sets" and "decelerating CIS". With even Prime gaps { $2, 4, 6, 8, 10, \dots$ }
42

$\frac{1}{2}$ = (**cardinality** {Gap 1-E-C} + **cardinality** {Gap 1-O-C} + 2) and whereby
 $\frac{2}{2}$ **cardinality** {Gap 1-E-C} [as null set] = **cardinality** {Gap 1-O-C} [as null
 $\frac{3}{2}$ set] = 0 in even Prime gap = 2 for Twin primes; the combined **Subsets** {Gap
 $\frac{4}{2}$ 1-E-C + Gap 1-O-C} are "thick sets" and "accelerating CIS". However as two
 $\frac{5}{2}$ CIS with exact same cardinality, **Subsets** {Gap 1-E-C} and **Subsets** {Gap
 $\frac{6}{2}$ 1-O-C} in isolation by themselves are neither "thin sets" nor "thick sets".
 $\frac{7}{2}$ There must be complete presence of both **Subsets** {Gap 1-E-C} + **Subsets**
 $\frac{8}{2}$ {Gap 1-O-C} as a [combined] "thick set" that contain $n = 0, 2, 4, 6, 8, 10, \dots$
 $\frac{9}{2}$ elements [\equiv even Prime gaps $(n + 2) = 2, 4, 6, 8, 10, 12, \dots$] with [combined]
 $\frac{10}{2}$ cardinality "accelerating CIS" \implies Polignac's and Twin prime conjectures is
 $\frac{11}{2}$ true. Obeying the Addition-Subtraction Laws of even \pm even = even; even \pm
 $\frac{12}{2}$ odd = odd; and odd \pm odd = even and Multiplication Laws of even \times even =
 $\frac{13}{2}$ even; even \times odd = even; and odd \times odd = odd for all $n = 0, 2, 4, 6, 8, 10, \dots$
 $\frac{14}{2}$ elements \equiv all even Prime gaps $(n + 2) = 2, 4, 6, 8, 10, 12, \dots$: Corresponding
 $\frac{15}{2}$ consecutive $\sum n$ terms as $\frac{n}{2}$ Even numbers + $\frac{n}{2}$ Odd numbers = [alternating]
 $\frac{16}{2}$ even, odd, even, odd, even, odd, ...; Corresponding consecutive $\prod n$ terms as $\frac{n}{2}$
 $\frac{17}{2}$ Even numbers $\times \frac{n}{2}$ Odd numbers = [same] even, even, even, even, even, even, ...

Synopsis on Product (Multiplication) of Integers and Complex numbers:

$\frac{20}{2}$ An integer can be either zero, a nonzero natural number, or minus a nonzero
 $\frac{21}{2}$ natural number. The product of zero and another integer is always zero. The
 $\frac{22}{2}$ product of two nonzero integers is determined by the product of their positive
 $\frac{23}{2}$ amounts, combined with the sign derived from the following rule [which is a
 $\frac{24}{2}$ consequence of the distributivity of multiplication over addition, and is not an
 $\frac{25}{2}$ additional rule]:

$\frac{26}{2}$ A positive number multiplied by a positive number is positive,

$\frac{27}{2}$ A positive number multiplied by a negative number is negative,

$\frac{28}{2}$ A negative number multiplied by a positive number is negative,

$\frac{29}{2}$ A negative number multiplied by a negative number is positive.

$\frac{30}{2}$ The rule for Product of two complex numbers is that two complex numbers
 $\frac{31}{2}$ can be multiplied by the distributive law and the fact that $i^2 = -1$:

$$\begin{aligned}
 \frac{32}{2} (a + bi) \cdot (c + di) &= a \cdot c + a \cdot di + bi \cdot c + b \cdot d \cdot i^2 \\
 \frac{33}{2} &= (a \cdot c - b \cdot d) + (a \cdot d + b \cdot c) i
 \end{aligned}$$

$\frac{34}{2}$ Geometrically, complex multiplication is understood by rewriting complex
 $\frac{35}{2}$ numbers in polar coordinates: $a + bi = r \cdot (\cos(\varphi) + i \sin(\varphi)) = r \cdot e^{i\varphi}$

$\frac{36}{2}$ Furthermore, $c + di = s \cdot (\cos(\psi) + i \sin(\psi)) = s \cdot e^{i\psi}$, from which one obtains
 $\frac{37}{2}$ $(a \cdot c - b \cdot d) + (a \cdot d + b \cdot c) i = r \cdot s \cdot e^{i(\varphi+\psi)}$. The geometric meaning is that
 $\frac{38}{2}$ the magnitudes are multiplied and the arguments are added.
 $\frac{39}{2}$

$\frac{40}{2}$ **Definition 2.3. Formal definition for two unique Subsets of Odd**
 $\frac{41}{2}$ **Primes that have either 'Even Parity' or 'Odd Parity' resulting in**
 $\frac{42}{2}$

1 the postulated "**Theorem of Nil Predilection for Even-Odd Parity in**
2 **Odd Primes associated with Carmichael numbers**".

3 For $n = 0, 2, 4, 6, 8, 10, \dots$; Addition of consecutive $\frac{n}{2}$ E-C and $\frac{n}{2}$ O-C
4 is dependent on n giving alternating even numbers manifesting "Even Parity"
5 and odd numbers manifesting "Odd Parity" [but their Multiplication is not
6 dependent on n]. For $i = 1, 2, 3, 4, 5, \dots$; Set of all Odd Primes are derived
7 from Set of all even Prime gaps $2i = \{2, 4, 6, 8, 10, \dots\}$ whereby we classify
8 Subset of Odd Primes derived from even Prime gaps $(4i - 2) = \{2, 6, 10, 14,$
9 $18, 22, 26, \dots\}$ to have "*Even Parity*" and Subset of Odd Primes derived from
10 even Prime gaps $(4i) = \{4, 8, 12, 16, 20, 24, 28, \dots\}$ to have "*Odd Parity*".

11 THEOREM 2.1. *Theorem of Nil Predilection for Even-Odd Parity in Odd*
12 *Primes associated with Carmichael numbers.*

14 **Proof.** List of first few Carmichael numbers with their locations specified
15 by Odd Primes and Prime gaps that manifest both "Even Parity" and "Odd
16 Parity": 561 [connected to Odd Prime 557 of "Even Parity" and having Prime
17 gap 6], 1105 [connected to Odd Prime 1103 of "Even Parity" and having Prime
18 gap 6], 1729 [connected to Odd Prime 1723 of "Even Parity" and having Prime
19 gap 10], 2465 [connected to Odd Prime 2459 of "Odd Parity" and having Prime
20 gap 8], 2821 [connected to Odd Prime 2819 of "Even Parity" and having Prime
21 gap 14], 6601 [connected to Odd Prime 6599 of "Odd Parity" and having Prime
22 gap 8], 8911 [connected to Odd Prime 8893 of "Even Parity" and having Prime
23 gap 30], 10585 [connected to Odd Prime 10567 of "Even Parity" and having
24 Prime gap 22], 15841 [connected to Odd Prime 15823 of "Odd Parity" and
25 having Prime gap 36], 29341 [connected to Odd Prime 29339 of "Odd Parity"
26 and having Prime gap 8], etc.

27 The even Prime gaps $(4i - 2) = \{2, 6, 10, 14, 18, 22, 26, \dots\}$ manifest "*Even*
28 *Parity*" (parity 0) and even Prime gaps $(4i) = \{4, 8, 12, 16, 20, 24, 28, \dots\}$
29 manifest "*Odd Parity*" (parity 1). Observe both types of even Prime gaps have
30 equal $\sim 20\%$ (rotating) Probability of getting their last-digit to end in 0, 2, 4, 6
31 or 8. In keeping with Odd Primes-Prime gaps constraints from Axiom 6.1 [and
32 its **List of eligible Last digit of Odd Primes**] on applying Prime number
33 theorem for Arithmetic Progression to statistically confirm Polignac's and Twin
34 prime conjectures to be true, we deduce our "*Theorem of Nil Predilection for*
35 *Even-Odd Parity in Odd Primes associated with Carmichael numbers*" must
36 be true whereby the "thin set" and "decelerating CIS" of Carmichael numbers
37 in the long run implies these rare but ubiquitous numbers are statistically
38 associated with two kinds of Odd Primes that manifest $\sim 50\%$ "Even Parity"
39 and $\sim 50\%$ "Odd Parity". **The proof is now complete for Theorem 2.1**□.

40 *Useful (colloquial) mathematical statements:* We simply have no choice
41 but to accept "There is zero probability that appearances of P-C identifier
42

1 **grouping** when computed as Cardinality 0 for Gap 2-Twin primes, Cardinality
2 2 for Gap 4-Cousin primes, Cardinality 4 for Gap 6-Sexy primes, etc should ever
3 stop or terminate in a discriminatory manner over the large range of integer
4 numbers, thus confirming Modified Polignac's and Twin prime conjectures to
5 be true". Here the word *Modified* denote the use of more appropriate term
6 "decelerating CIS" [that represent "thin sets" and "thin subsets"] instead of just
7 "CIS". Similarly for Riemann zeta function via *proxy* Dirichlet eta function, we
8 simply have no choice but to accept "The solitary $\sigma = \frac{1}{2}$ -critical line location
9 for all nontrivial zeros thus confirming Riemann hypothesis to be true".

10 In combinatorics [that deals with counting and arrangements], the inclusion-
11 exclusion principle is a counting technique which generalizes the familiar method
12 of obtaining number of elements in union of two or more sets when these sets
13 may have overlaps. In essence, this principle removes all contributions from
14 over-counted elements in sets and subsets. Instead of using raw *cardinality*
15 from Pure Set theory when / if relevant, we should selectively use Measure
16 theory such as length, area, probability (or proportion), Natural density (a.k.a.
17 Asymptotic density, used when [for example] there is no uniform probability
18 distribution over Natural numbers), and Dirichlet density (useful analytic tool
19 for thin sets like set of Prime numbers that do not have well-defined Natural
20 density; and with deep connections to Riemann zeta function, prime distribu-
21 tion and analytic number theory). Only under strict convergence conditions
22 that any resultant infinite alternating series converge absolutely, this principle
23 is valid for CFS, CIS or UIS [irrespective of whether there are finite or infinite
24 number of these CFS, CIS or UIS]. We shall succinctly adapt or adopt this
25 principle into various relevant mathematical arguments, lemmas, propositions,
26 corollaries, axioms or theorems in this paper.

27 An example based on Measure theory: (Step 1) Define and measure two
28 lengths as two UIS of Set A and Set B using two intervals of Real numbers on
29 number line; viz, two individual "continuous lengths" are both [quantitatively]
30 infinite $\mu(A) = 2 - 0 = 2$ and $\mu(B) = 4 - 1 = 3$. (Step 2) Measure the
31 intersection as $A \cap B = [1, 2] \implies \mu(A \cap B) = 2 - 1 = 1$. (Step 3) Apply
32 inclusion-exclusion principle $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) = 2 +$
33 $3 - 1 = 4 \implies$ total "discrete length" covered by both intervals is 4; viz,
34 total "size" is [qualitatively] finite. Mainly based on Number theory, Mobius
35 function $\mu(n)$ that connects deeply with Euler's totient function, zeta functions,
36 and multiplicative number theory also gives a powerful compact formula for
37 inclusion-exclusion principle over divisibility conditions.

38 Let A, B, C, \dots be finitely large sets or infinitely large sets, and $|S|$
39 indicates the cardinality of a set S (\equiv 'number of elements' for set S). For
40 CFS e.g. *Set* of even Prime number = $\{2\}$ with cardinality = 1, *Set* of odd
41 Prime number with last-digit ending in 5 = $\{5\}$ with cardinality = 1; CIS
42

1 e.g. Set of odd Prime numbers = $\{3, 5, 7, 11, 13, 17, 19, \dots\}$ with cardinality
 2 = \aleph_0 ; and UIS e.g. Set of Real numbers with cardinality = \mathfrak{c} (*cardinality of*
 3 *the continuum*). The inclusion-exclusion principle for three sets is given by
 4 $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
 5 This formula expresses the fact that sum of sizes for these three sets may be too
 6 large since some elements may be counted twice (two times) or thrice (three
 7 times). General formula for a finite number of sets [with alternating signs +,
 8 -, +, -, ... that depends on number of sets in the intersection] is $\left| \bigcup_{i=1}^n A_i \right| =$
 9
 10 $\sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|$.
 11
 12

13 In Probability theory, this formula for a finite number of sets is $\mathbb{P} \left(\bigcup_{i=1}^n A_i \right) =$
 14
 15 $\sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} \mathbb{P} \left(\bigcap_{i=1}^n A_i \right)$.
 16
 17

18 In closed form, this formula is $\mathbb{P} \left(\bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n \left((-1)^{k-1} \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} \mathbb{P}(A_I) \right)$,
 19
 20

21 where the last sum runs over all subsets I of indices $1, \dots, n$ which contain
 22 exactly k elements, and $A_I := \bigcap_{i \in I} A_i$ denotes intersection of all those A_i with
 23 index in I . This formula for an infinite number of sets [strict convergence for
 24

25 infinite alternating series] is $\mathbb{P} \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{k=1}^{\infty} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{1, \dots, \infty\} \\ |I|=k}} \mathbb{P}(A_I) \right)$.
 26
 27

28 For a general measure space (S, Σ, μ) and measurable subsets A_1, A_2, \dots, A_n
 29 [or $A_1, A_2, \dots, A_n, A_{n+1}, A_{n+2}, \dots, A_{\infty}$] of finite [or infinite measure], the above
 30 identity also hold when probability measure \mathbb{P} is replaced by the measure μ .
 31

3. Mathematics for Incompletely Predictable Problems and Input-White Box-Output Modeling

32
 33
 34
 35 *Definition 3.1.* Where all infinitely-many prime [and composite] numbers
 36 are classified as Pseudo-random entities, so must all nontrivial zeros be classified
 37 as such. Pseudo-random entities are Incompletely Predictable entities. Largely
 38 based on p. 18 of [6], we provide formal definitions for three types of [infinitely-
 39 many] entities as Countably Infinite Sets in a succinct manner.

40
 41 *Completely Unpredictable* (non-deterministic) entities are [the statistically]
 42 defined as entities that are actually random and DO behave like one e.g. [true]

¹ random number generator that supply sequences of entities (as non-distinct
² Sets of numbers) that are not reproducible; viz, these entities do not contain
³ any repeatable spatial or temporal patterns. We work in base-10 system (a.k.a.
⁴ decimal system) that represent numbers using ten unique digits $\{0, 1, 2, 3, 4,$
⁵ $5, 6, 7, 8, 9\}$. For $n = \{1, 2, 3, 4, 5, \dots\}$, and with our [true] random number
⁶ generator also utilizing these ten unique digits to supply n^{th} Entity as $n \rightarrow \infty$;
⁷ then Probability (P) of independently obtaining each digit is $P(0) = P(1) =$
⁸ $P(2) = P(3) = P(4) = P(5) = P(6) = P(7) = P(8) = P(9) \simeq \frac{1}{10} \simeq 0.1 \simeq 10\%$.
⁹

¹⁰ *Completely Predictable* (deterministic) entities are defined as entities that
¹¹ are actually not random and DO NOT behave like one e.g. non-overlapping
¹² distinct Set of Even numbers $\{0, 2, 4, 6, 8, 10, \dots\}$ and Set of Odd numbers $\{1,$
¹³ $3, 5, 7, 9, 11, \dots\}$; viz, these entities are reproducible. Chosen "Even [or Odd]
¹⁴ Gap", as [non-varying] integer number value 2 between any two adjacent Even
¹⁵ [or Odd] numbers, always consist of a fixed value. The distinct Sets of trivial
¹⁶ zeros from various L-functions [as infinitely-many negative integers] are other
¹⁷ examples of Completely Predictable entities. Both Riemann zeta function and
¹⁸ its *proxy* Dirichlet eta function have simple zeros at each even negative integer
¹⁹ $s = -2n$ where $n = 1, 2, 3, 4, 5, \dots$; viz, $s = -2, -4, -6, -8, -10, \dots$. In
²⁰ addition, the factor $1 - 2^{1-s}$ in Dirichlet eta function adds an infinite number
²¹ of [Completely Predictable] complex simple zeros, located at equidistant points
²² on the line $\Re(s) = 1$, at $s_n = 1 + \frac{2n\pi i}{\ln(2)}$ whereby $n = \dots, -3, -2, -1, 1, 2, 3, \dots$ is
²³ any nonzero integer and i is the imaginary unit satisfying equation $i^2 = -1$.
²⁴

²⁵ *Incompletely Predictable* [or *Pseudo-random*] (deterministic) entities are
²⁶ defined as entities that are actually not random but DO behave like one e.g.
²⁷ non-overlapping distinct Set of Prime numbers $\{2, 3, 5, 7, 11, 13, 17, 19, 23,$
²⁸ $29, 31, \dots\}$ and Set of Composite numbers $\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18,$
²⁹ $20, 21, 22, 24, 25, 26, 27, 28, \dots\}$; viz, these entities are reproducible. Chosen
³⁰ "Prime [or Composite] Gaps", as [varying] integer number values between any
³¹ two adjacent Prime [or Composite] numbers, will never consist of a fixed value.
³² Examples: Set of Prime Gaps = $\{1, 2, 2, 4, 2, 4, 2, 4, 2, 4, 6, 2, \dots\}$ and Set of
³³ Composite Gaps = $\{2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1, 1, 1, 2, \dots\}$. The
³⁴ (odd) Prime Gap 1 indicates the only (solitary) even Prime number 2. All
³⁵ infinitely many (odd) Prime numbers have (even) Prime Gaps 2, 4, 6, 8, 10, ...
³⁶ [to infinitely large size or, more precisely, to an arbitrarily large number] at
³⁷ sufficiently large integer range. Again, one can conveniently and arbitrarily
³⁸ classify small Prime Gaps to be 2 and 4, and large Prime Gaps to be ≥ 6 . Only
³⁹ two finite integer number values $\{1, 2\}$ represent Composite Gaps. Occurrences
⁴⁰ of (even) Composite Gap 2 in the specific Composite even number that always
⁴¹ follow an (odd) Prime number are thus associated with appearances of all (odd)
⁴² Prime numbers. The cardinality of consecutive (odd) Composite Gap 1 \propto size of

1 (even) Prime Gaps; viz, cardinality of consecutive Gap 1-Composite numbers
2 = even Prime Gap - 2 with cardinality of Gap 1-Composite even numbers
3 = $\frac{\text{even Prime Gap} - 2}{2}$ and cardinality of Gap 1-Composite odd numbers =
4 $\frac{\text{even Prime Gap} - 2}{2}$. Note that Gap 2-Prime numbers (twin primes) do not
5 have Gap 1-Composite numbers. The inclusion-exclusion principle for two sets
6 $|A \cup B| = |A| + |B| - |A \cap B|$. $|\text{All Even numbers} \cap \text{All Prime numbers}| =$
7 1, which represent the only even Prime number 2. All Prime numbers [with
8 exception of even Prime number 2] are (almost totally) constituted by Odd
9 numbers. All odd Prime numbers are (totally) constituted by Odd numbers
10 [although the majority of Odd numbers are not odd Prime numbers].
11

12 Apart from integers, Incompletely Predictable entities can also be consti-
13 tuted from other number systems e.g. distinct Sets of t -valued transcendental
14 numbers that faithfully represent infinitely-many nontrivial zeros (spectrum) of
15 dual or self-dual L-functions. Geometrically, all nontrivial zeros of L-functions
16 are simply the "Origin point intercepts". L-functions [e.g. from the Genus
17 1 elliptic curves representing self-dual L-functions] can have Analytic rank 0,
18 1, 2, 3, 4, 5,... [to an arbitrarily large number]; viz, have "solitary" (zero)
19 Analytic rank and "all other" (nonzero) Analytic rank. Thus, it seems that
20 most L-functions should "qualitatively" have MORE (nonzero) Analytic rank
21 = 1, 2, 3, 4, 5,... and LESS (zero) Analytic rank = 0. Only (zero) Analytic
22 rank L-functions, such as from Genus 0 (non-elliptic) Riemann zeta function
23 [and its *proxy* Dirichlet eta function] and selected (Analytic rank 0) Genus 1
24 elliptic curves, DO NOT HAVE first nontrivial zeros with t value = 0 [viz, an
25 algebraic number]. Then, all (nonzero) Analytic rank L-functions DO HAVE
26 first nontrivial zeros with t value = 0 [viz, an algebraic number].
27

28 In Riemann hypothesis or Generalized Riemann hypothesis, all nontrivial
29 zeros are conjecturally *only* located on $\Re(s) = \frac{1}{2}$ -Critical line or Analytically
30 normalized $\Re(s) = \frac{1}{2}$ -Critical line. Chosen "Nontrivial Zero Gaps", as [varying]
31 transcendental number values, between any two adjacent nontrivial zeros never
32 consist of a fixed value. All infinitely-many nontrivial zeros are Incompletely
33 Predictable entities. Note the infinitely-many digits after decimal point of
34 each (algebraic) or (transcendental) irrational number are also Incompletely
35 Predictable entities whereby individual irrational number has greater precision
36 or accuracy when it is computed as having increasing number of digits.
37

38 *Remark 3.1.* We use abbreviations: CP = Completely Predictable, IP =
39 Incompletely Predictable, CFS = Countably finite sets, CIS = Countably in-
40 finite sets, UIS = Uncountably infinite sets. We compare and contrast Sets,
41 Subsets, Even k -tuple and Prime k -tuple when derived from CP entities versus
42

1 IP entities. There is only one mathematical possibility for CIS having CP or
2 IP entities: Cardinality of *different or changing values* denoting the "Gaps"
3 between any two adjacent elements in CIS with CP entities must be CFS.
4 Cardinality of *different or changing values* denoting the "Gaps" between any
5 two adjacent elements in CIS with IP entities must be CIS. Broadly applying
6 inclusion-exclusion principle to two or more [mutually exclusive] cardinalities:

7 We can never obtain CIS having both CP entities and IP entities. In a
8 similar manner, irrespective of having CP entities or IP entities, a given set
9 must simply be UIS, CIS or CFS [and cannot be a mixture of UIS, CIS and/or
10 CFS]. Subsets of CP entities are "non-unique and overlapping" e.g. Derived
11 from Set of Gap 2-Even numbers (Twin Even numbers) = $\{0, 2, 4, 6, 8, 10, \dots\}$:
12 Subset of Gap 4-Even numbers (Cousin Even numbers) = $\{0, 4, 8, 12, 16,$
13 $20, \dots\}$, Subset of Gap 6-Even numbers (Sexy Even numbers) = $\{0, 6, 12, 18,$
14 $24, 30, \dots\}$, etc. Subsets of IP entities are "unique and non-overlapping" e.g.
15 Derived from Set of All Prime numbers = $\{2, 3, 5, 7, 11, 13, \dots\}$: Subset of
16 Gap 2-Prime numbers (Twin Primes) = $\{3, 5, 11, 17, 29, 41, \dots\}$, Subset of Gap
17 4-Prime numbers (Cousin Primes) = $\{7, 13, 19, 37, 43, 67, \dots\}$, Subset of Gap
18 6-Prime numbers (Sexy Primes) = $\{23, 31, 47, 53, 61, 73, \dots\}$, etc.

19 For $k = 2, 3, 4, 5, 6, \dots$ and $n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots$; the
20 diameter of a Prime k -tuple is difference of its largest and smallest elements.
21 Note the special case of $k = 2$ simply corresponds to Gap 2-prime numbers
22 (Twin primes). An admissible Prime k -tuple with smallest possible diameter
23 d (among all admissible Prime k -tuples) is a Prime constellation \equiv Prime k -
24 tuple. Prime constellations manifest the Incompletely Predictable property
25 whereby certain prime numbers are "non-unique and overlapping" represented
26 e.g. When $k = 3, d = 6$: Constellation $(0, 2, 6) \equiv$ [smallest] prime numbers $(5,$
27 $7, 11)$ with chosen $n = 5$; Constellation $(0, 4, 6) \equiv$ [smallest] prime numbers
28 $(7, 11, 13)$ with chosen $n = 7$. When $k = 4, d = 8$: Constellation $(0, 2, 6, 8)$
29 \equiv [smallest] prime numbers $(5, 7, 11, 13)$ with chosen $n = 5$. For all $n \geq k$
30 this will always produce consecutive Primes. Recall from above that all n are
31 integers for which values $(n+a, n+b, n+c, \dots)$ are prime numbers. This means
32 that, for large n : $p_{n+k-1} - p_n \geq d$ where p_n is the n^{th} prime number.

33 We intuitively infer from above synopsis in previous two paragraphs that
34 only by analyzing non-overlapping Subsets of even Prime gaps $2, 4, 6, 8, 10, \dots$
35 [instead of analyzing overlapping Prime k -tuples or Prime k -tuplets] would we
36 obtain the rigorous proofs for Polignac's and Twin prime conjectures.

37 For $k = 2, 3, 4, 5, 6, \dots$ and $n = 0, 2, 4, 6, 8, 10, 12, 14, 16, \dots$; the diameter of
38 an Even k -tuple is difference of its largest and smallest elements. An admissible
39 Even k -tuple with smallest possible diameter d (among all admissible Even k -
40 tuples) is an Even constellation \equiv Even k -tuple. Even constellations manifest
41 the Completely Predictable property whereby even numbers are "unique and
42

1 non-overlapping" represented e.g. When $k = 4, d = 2(k - 1) = 6$: Constellation
2 $(0, 2, 4, 6) \equiv$ [smallest] even numbers $(0, 2, 4, 6)$ with chosen $n = 0$ or [using
3 larger] even numbers $(102, 104, 106, 108)$ with arbitrarily chosen $n = 102$.
4 For all n [as fully obtained from $n < k$ and $n \geq k$], this will always produce
5 consecutive even numbers. Recall from above that all n are integers for which
6 values $(n + 2, n + 4, n + 6, \dots)$ are even numbers. This means that, for all n :
7 $E_{n+k-1} - E_n = d$ where E_n is the n^{th} even number. Observe we could instead
8 use odd numbers that will also produce the same equally valid deductions.

9 Gram's rule says there is exactly one nontrivial zero (NTZ) \equiv Gram[x=0,
10 y=0] point in Riemann zeta function between any two Gram points \equiv Gram[y=0]
11 points. A Gram block is an interval bounded by two "good" Gram points such
12 that all Gram points between them are "bad". Rosser's rule says Gram blocks
13 often have the expected number of NTZ in them [viz, NTZ is "conserved" and
14 is the same as the number of Gram intervals], even though some individual
15 Gram intervals in the block may not have exactly one NTZ in them [viz, some
16 of the individual Gram intervals in the block violate Gram's rule]. Both Gram's
17 rule and Rosser's rule say in some sense NTZ do not stray too far from their ex-
18 pected positions. Violations of Gram's rule equate to intermittently observable
19 geometric variants of two consecutive (+ve first and then -ve) Gram points [\equiv
20 missing NTZ] that is alternately followed by two consecutive NTZ [\equiv extra
21 NTZ]. The rarer violations of Rosser's Rule equate to intermittently observable
22 geometric variants of reduction in expected number of x-axis intercept points.
23 They both fail infinitely many times in a +ve proportion of cases. We expect
24 in $\sim 66\%$ one NTZ is enclosed by two successive Gram points, but in $\sim 17\%$ no
25 NTZ and in $\sim 17\%$ two NTZ are in such a Gram interval on the long run.

26 The success and failures of both Gram's rule and Rosser's rule occur in
27 Dirichlet eta function [*proxy* for Riemann zeta function] on $\sigma = \frac{1}{2}$ -Critical line.
28 An insightful inference with deep connection to Riemann hypothesis: Only by
29 analyzing non-overlapping Subset of "One NTZ" = $\sim 66\%$, Subset of "Zero
30 NTZ" = $\sim 17\%$, and Subset of "Two NTZ" = $\sim 17\%$ as precisely derived from
31 Set of "All NTZ" = "conserved" 100% [instead of analyzing various overlapping
32 Gram blocks and Gram intervals containing "good" or "bad" Gram points,
33 missing NTZ or extra NTZ] can we rigorously prove Riemann hypothesis.

34 $INPUT \rightarrow$ *White Box* or *Black Box* $\rightarrow OUTPUT$. White Box (or
35 Black Box) is a system where its [unique] inner components or logic are (or are
36 not) available for inspection. Key ideas for computer & mathematical systems
37 as White Box or Black Box $INPUT \rightarrow$ (unique) CP vs IP *Information processor*
38 & *Mathematical function, equation or algorithm* \rightarrow (reproducible) CP vs IP
39 $OUTPUT$. Examples of Mathematical function, equation or algorithm:
40
41
42

1 CP CIS k^{th} Even numbers are Integers $\{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \dots\}$ [\equiv
 2 *OUTPUT*] faithfully given by equation $n = \pm 2k$ [\equiv *White Box*], where k are
 3 Integers $\{0, 1, 2, 3, 4, 5, \dots\}$ [\equiv *INPUT*]. Since Even n are integrally divisible
 4 by 2, congruence $n = 0 \pmod{2}$ holds for Even n . The generating function of
 5 Even numbers is $\frac{2x}{(x-1)^2} = 2x^1 + 4x^2 + 6x^3 + 8x^4 + \dots$
 6

7 CP CIS k^{th} Odd numbers are Integers $\{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \dots\}$ [\equiv
 8 *OUTPUT*] faithfully given by equation $n = \pm(2k-1)$ [\equiv *White Box*], where k
 9 are Integers $\{1, 2, 3, 4, 5, \dots\}$ [\equiv *INPUT*]. Since Odd n when divided by 2 leave
 10 a remainder 1, congruence $n = 1 \pmod{2}$ holds for Odd n . The generating
 11 function of Odd numbers is $\frac{x(1+x)}{(x-1)^2} = 1x^1 + 3x^2 + 5x^3 + 7x^4 + \dots$. The
 12 oddness of a number is called its parity, so an Odd number has parity 1 (Odd
 13 Parity), while an Even number has parity 0 (Even Parity). The product of an
 14 Even number and an Odd number is always Even, as can be seen by writing
 15 $(2k)(2l+1) = 2[k(2l+1)]$, which is divisible by 2 and hence is Even.
 16

17 IP "decelerating"-CIS k^{th} Prime numbers are Integers $\{\pm 2, \pm 3, \pm 5, \pm 7,$
 18 $\pm 11, \pm 13, \pm 17, \pm 19, \pm 23, \pm 29, \pm 31, \pm 37, \dots\}$ [\equiv *OUTPUT*] faithfully given
 19 by algorithm \pm "*Sieve-of-Eratosthenes*" [\equiv *White Box*], where k are Integers
 20 $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots\}$ [\equiv *INPUT*]. A Prime number is an
 21 Integer greater than 1 with exactly two factors, 1 and the number itself.
 22

23 IP "accelerating"-CIS k^{th} Composite numbers are Integers $\{\pm 4, \pm 6, \pm 8,$
 24 $\pm 9, \pm 10, \pm 12, \pm 14, \pm 15, \pm 16, \pm 18, \pm 20, \pm 21, \dots\}$ [\equiv *OUTPUT*] faithfully
 25 given by algorithm \pm "*Complement-Sieve-of-Eratosthenes*" [\equiv *White Box*], where
 26 k are Integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots\}$ [\equiv *INPUT*]. A Composite
 27 number is an Integer greater than 1 with more than two factors (including 1
 and the number itself).
 28

29 IP CIS k^{th} Nontrivial zeros are Complex numbers $s = \sigma \pm it = \frac{1}{2} \pm it$ that
 30 are traditionally denoted by t -valued Transcendental numbers $\{\pm 14.13, \pm 21.02,$
 31 $\pm 25.01, \pm 30.42, \pm 32.93, \pm 37.58, \dots\}$ [\equiv *OUTPUT*] as faithfully satisfied by
 32 equation "*Riemann zeta function* $\zeta(s) = 0$ " / "*Dirichlet eta function* $\eta(s) =$
 33 0 " [\equiv *White Box*], where k are Integers $\{1, 2, 3, 4, 5, 6, \dots\}$ [\equiv *INPUT*]. All
 34 nontrivial zeros are proposed in 1859 Riemann hypothesis to be only located
 35 on $\sigma = \frac{1}{2}$ -Critical Line.
 36

37 For $i = 1, 2, 3, \dots, \infty$; let i^{th} Even number = E_i and i^{th} Odd number = O_i .
 38 We can precisely, easily and independently calculate e.g. $E_5 = (2 \times 5) = 10$ and
 39 e.g. $O_5 = (2 \times 5) - 1 = 9$. A generated CP number is *locationally defined* as a
 40 number whose i^{th} position is independently determined by simple calculations
 41 without needing to know related positions of all preceding numbers - this is a
 42 "reproducible" Universal Property. The congruence $n \equiv 0 \pmod{2}$ holds for

$\frac{1}{2}$ positive even numbers (n). The congruence $n \equiv 1 \pmod{2}$ holds for positive
 $\frac{2}{2}$ odd numbers (n). Then the zeroeth Even number $E_0 = (2 \times 0) = 0$ must exist.

$\frac{3}{4}$ For $i = 1, 2, 3, \dots, \infty$; let i^{th} Prime number = P_i and i^{th} Composite number
 $\frac{4}{5}$ = C_i . We can precisely, tediously and dependently compute e.g. $C_6 = 12$ and
 $\frac{5}{6}$ $P_6 = 13$: 2 is 1st prime, 3 is 2nd prime, 4 is 1st composite, 5 is 3rd prime, 6 is
 $\frac{6}{7}$ 2nd composite, 7 is 4th prime, 8 is 3rd composite, 9 is 4th composite, 10 is 5th
 $\frac{7}{8}$ composite, 11 is 5th prime, 12 is 6th composite, 13 is 6th prime, etc. Our desired
 $\frac{8}{9}$ integer 12 is the 6th composite and integer 13 is the 6th prime. A generated IP
 $\frac{9}{10}$ number is *locationally defined* as a number whose i^{th} position is dependently
 $\frac{10}{11}$ determined by complex calculations with needing to know related positions of
 $\frac{11}{12}$ all preceding numbers - this is a "reproducible" Universal Property. Observe
 $\frac{12}{13}$ that integers $\{0, 1\}$ are neither prime nor composite.

$\frac{13}{14}$ **Remark 3.2. Natural and Dirichlet density in Thin and Thick set:**

$\frac{15}{16}$ Natural density of a Set $A \subseteq \mathbb{N}$ is: $d(A) = \lim_{n \rightarrow \infty} \frac{|A \cap \{1, 2, 3, \dots, n\}|}{n}$. If this limit
 $\frac{16}{17}$ exist, it measure how "large" a subset of the set of natural numbers is. It relies
 $\frac{17}{18}$ chiefly on the probability of encountering members of the desired subset when
 $\frac{18}{19}$ combing through the interval $[1, n]$ as n grows large. We have $0 \leq d(A) \leq 1$: If
 $\frac{19}{20}$ $d(A) = 1$, the set is thick or co-dense (almost everything is in A). If $d(A) = 0$,
 $\frac{20}{21}$ the set is thin or sparse.

$\frac{21}{22}$ Let $A \subseteq \mathbb{P}$ be a subset of prime numbers. Dirichlet density of A is defined

$\frac{22}{23}$ as: $\delta(A) = \lim_{s \rightarrow 1^+} \frac{\sum_{p \in A} \frac{1}{p^s}}{\sum_p \frac{1}{p^s}}$ provided this limit exist. Since the prime zeta function
 $\frac{24}{25}$

$\frac{25}{26}$ [an analogue of Riemann zeta function] $\sum_p \frac{1}{p^s} \sim \log_e\left(\frac{1}{s-1}\right)$ as $s \rightarrow 1^+$, we
 $\frac{27}{28}$

$\frac{28}{29}$ also have $\delta(A) = \lim_{s \rightarrow 1^+} \frac{\sum_{p \in A} \frac{1}{p^s}}{\log_e\left(\frac{1}{s-1}\right)}$. This expression is usually the order of the
 $\frac{30}{31}$

$\frac{31}{32}$ "pole" of $\prod_{p \in A} \frac{1}{1 - \frac{1}{p^s}}$ at $s = 1$, (though in general it is not really a pole as
 $\frac{33}{34}$

$\frac{34}{35}$ it has non-integral order), at least if this function is a holomorphic function
 $\frac{35}{36}$ times a (real) power of $s - 1$ near $s = 1$. Dirichlet density is useful when
 $\frac{36}{37}$ Natural density is undefined or hard to compute. It is especially effective for
 $\frac{37}{38}$ sets of primes in arithmetic progression. It bridges combinatorics with complex
 $\frac{38}{39}$ analysis through the zeta and L -functions. If Natural density exists, Dirichlet
 $\frac{39}{40}$ density also exists, and they are equal [but not the opposite way]. If A is the
 $\frac{40}{41}$
 $\frac{41}{42}$

1 set of all primes, it is the Riemann zeta function which has a pole of order 1 at
2 $s = 1$, so the set of all primes has Dirichlet density 1.

3 Set A is a *thin set* if it has zero Natural density, and is sparse or rare
4 among Natural numbers; viz, it becomes vanishingly small compared to Natural
5 numbers as you go to infinity. Example: The [earliest] ancient Euclid's Proof
6 of the infinitude of \mathbb{P} (c. 300 BC) utilize *reductio ad absurdum*. Set of all
7 (odd) Prime numbers \mathbb{P} is a "decelerating CIS" and a "thin set". Let $\mathbb{P}\text{-}\pi(n)$
8 be the Prime counting function defined as number of primes $\leq n$. Prime
9 number theorem tells us $\mathbb{P}\text{-}\pi(n) \sim \frac{n}{\log_e n}$. With Prime Gaps = Set of $\mathbb{E} =$
10 $\{2, 4, 6, 8, 10, \dots\}$ being Arbitrarily Large in Number as you go to infinity,
11 the Natural density for All odd \mathbb{P} Set [= $\sum_{n \in \mathbb{E}} \text{Gap } n\text{-}\mathbb{P}$] that "decelerates to an
12

13 infinitesimal small number value just above zero" is given by $\lim_{n \rightarrow \infty} \frac{\mathbb{P}\text{-}\pi(n)}{n} =$

15 $\lim_{n \rightarrow \infty} \frac{1}{\log_e n} = 0$. We recognize that Gap 2- \mathbb{P} Subset, Gap 4- \mathbb{P} Subset, Gap
16 6- \mathbb{P} Subset, Gap 8- \mathbb{P} Subset, Gap 10- \mathbb{P} Subset, ... being proposed to all consist
17 of "decelerating CIS" and "thin sets" would imply the 1849-dated Polignac's
18 conjecture [regarding all even Prime Gaps 2, 4, 6, 8, 10...] and the 1846-dated
19 Twin prime conjecture [regarding "subset" even Prime Gaps 2] to both be true.
20 We ultimately observe *Dimensional analysis homogeneity* when "decelerating
21 CIS" and "thin set" properties are uniformly applicable to all quantities from
22 both sides of the equation: All odd \mathbb{P} Set = Gap 2- \mathbb{P} Subset + Gap 4- \mathbb{P} Subset
23 + Gap 6- \mathbb{P} Subset + Gap 8- \mathbb{P} Subset + Gap 10- \mathbb{P} Subset + \dots
24

25 A thick set is a set of integers that contains arbitrarily long intervals;
26 viz, long blocks of consecutive integers [even if it also skips large chunks else-
27 where]. Given a thick set A , for every $p \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that
28 $\{n, n + 1, n + 2, \dots, n + p\} \subset A$. Trivially Natural numbers \mathbb{N} , as Completely
29 Predictable entities having "Natural Gap" = non-varying integer number value
30 1, is a thick set with Natural density being exactly 1. Other well-known sets
31 that are thick include non-primes and non-squares. Thick sets can also be
32 sparse, e.g. $\bigcup_{n \in \mathbb{N}} \{x : x = 10^n + m : 0 \leq m \leq n\}$. Thus a thick set has Natural
33

34 density which can be 0 or > 0 ; viz, can be sparse or dense overall. It must
35 always have long intervals (large chunks) but its sparsity can be low [or can
36 be high]; viz, having Natural density close to 1 [or close to 0]. Set of Incom-
37 pletely Predictable Composite numbers \mathbf{C} is both thick and dense. Let $\mathbf{C}\text{-}\pi(n)$
38 be the Composite counting function defined as number of composites $\leq n$.
39 Analogical "Composite number theorem" tells us $\mathbf{C}\text{-}\pi(n) \approx n - \mathbb{P}\text{-}\pi(n)$. Since
40 $\mathbb{P}\text{-}\pi(n) \sim \frac{n}{\log_e n}$, we get $|\mathbf{C} \cap [1, n]| \approx n - \frac{n}{\log_e n} \implies \frac{|\mathbf{C} \cap [1, n]|}{n} \rightarrow 1$ as
41 $n \rightarrow \infty$; viz, Composite numbers as an "accelerating CIS" and "thick set" that
42

1 "accelerate to an infinitesimal small number value just below one" have Natural
2 density 1. An exception is specific subset of Gap 2-Composite even numbers
3 that follow, and are associated with, every odd Prime numbers: This unique
4 subset is "decelerating CIS", and is a "thin set" with Natural density 0.

5 Both the Completely Predictable sets of Even numbers \mathbb{E} and Odd numbers
6 \mathbb{O} are neither *thin set* nor *thick set*. There are never any arbitrarily long blocks
7 of consecutive \mathbb{E} or $\mathbb{O} \implies$ both sets are not thick set. Natural density of both
8 \mathbb{E} or \mathbb{O} is $\frac{1}{2}$ [viz, $\neq 0$] \implies both sets are not thin set.
9

10 *Remark 3.3.* Yitang Zhang proved a landmark result [announced on April
11 17, 2013]: There are infinitely many pairs of (odd) Prime numbers that dif-
12 fer by unknown even number $N \leq 70$ million[7]; viz, there is a "decelerating
13 CIS" and "thin set" of Gap N -Prime numbers with unknown even number N
14 ≤ 70 million. This solitary N value as an existing "privileged" but unknown
15 even Prime gap must, without exception, comply with the imposed Odd Prime-
16 Prime Gap constraint on "eligible last digit of Odd Primes" as per Axiom 6.1.
17 Aesthetically, this N value by itself is insufficient since its generated "deceler-
18 ating CIS" Odd Primes simply cannot exist alone amongst the large range of
19 prime numbers. Always as finite [but NOT infinite] length, we observe as a side
20 note that two or more consecutive Odd Primes can validly and rarely be consti-
21 tuted by [same] even Prime gap of 6 or multiples of 6. Hence there must be at
22 least two, if not three, existing even Prime gaps generating their corresponding
23 "decelerating CIS" Odd Primes. Polignac's and Twin prime conjectures refers
24 to all even Prime gaps 2, 4, 6, 8, 10... generating corresponding "decelerating
25 CIS" Odd Primes [which are, *by default*, all "thin sets"].

26 Polymath8a "Bounded gaps between primes" (4 June 2013 – 17 Novem-
27 ber 2014) was a project to improve N by developing the techniques of Zhang
28 [viz, constructing an "admissible k -tuple" whose diameter was bounded by 70
29 million]. This project concluded with obtaining $N = 4,680$.

30 Polymath8b "Bounded intervals with many primes" (19 November 2013 –
31 19 June 2014) was a project to further improve N by combining Polymath8a
32 results with the techniques of James Maynard [viz, introducing a refinement
33 of GPY sieve method for studying prime k -tuples and small gaps between
34 primes which establishes that "a positive proportion of admissible m -tuples
35 satisfy the prime m -tuples conjecture for every m "]. This project concluded
36 with a bound of $N = 246$; and by assuming Elliott-Halberstam conjecture and
37 its generalized form further lower N to 12 and 6, respectively. Regarded as
38 "Zhang's optimized result", these lowering of N involve studying *overlapping*
39 k -tuples. But maximally lowering N to 2 will likely require clever breakthrough
40 concepts that involve studying *non-overlapping* even Prime gaps.
41
42

1 *Remark 3.4.* The notion of *thin set* and *thick set* typically apply to sub-
2 sets of [discrete] \mathbb{N} (Natural numbers), or more generally, [discrete] \mathbb{Z} (integer
3 numbers). Forming a CIS, the Set of Incompletely Predictable [discrete] non-
4 trivial zeros from e.g. Riemann zeta function $\zeta(s)$ are derived from complex
5 solutions to $\zeta(s) = 0$, whereby $s = \sigma \pm it$. Traditionally given as $\pm t$ -valued
6 transcendental numbers; nontrivial zeros conceptually form a "discrete" and
7 "sparse" ("small") set in [continuous] 1-dimensional $\sigma = \frac{1}{2}$ -Critical Line [viz,
8 constituted by \mathbb{R} of infinite length] or in [continuous] 2-dimensional Complex
9 plane [viz, constituted by \mathbb{C} of infinite area]. CIS nontrivial zeros has zero
10 Natural density in UIS \mathbb{R} or UIS \mathbb{C} , and do not form dense clusters or intervals.
11 We intuitively and meaningfully interpret Set of nontrivial zeros as a *thin set*.
12

14 4. Incompletely Predictable Carmichael numbers in Base-10

15 In number theory, an n -Knodel number for a given positive integer n is a
16 composite number m with the property that each $i < m$ coprime to m satisfies
17 $i^{m-n} \equiv 1 \pmod{m}$. The set of all n -Knodel numbers is denoted K_n . The
18 special case K_1 is the Carmichael numbers. There are infinitely many n -Knodel
19 numbers for a given n . Due to Euler's theorem every composite number m is
20 an n -Knodel number for $n = m - \varphi(m)$ where φ is Euler's totient function.
21

22 Fermat's little theorem states that if p is a prime number, then for any
23 integer a , the number $a^p - a$ is an integer multiple of p . In the notation of
24 modular arithmetic, this is expressed as $a^p \equiv a \pmod{p}$. For example, if $a =$
25 2 and $p = 7$, then $2^7 = 128$, and $128 - 2 = 126 = 7 \times 18$ is an integer multiple
26 of 7 . If a is not divisible by p , that is, if a is coprime to p , then Fermat's little
27 theorem is equivalent to the statement that $a^{p-1} - 1$ is an integer multiple of
28 p , or in symbols: $a^{p-1} \equiv 1 \pmod{p}$. For example, if $a = 2$ and $p = 7$, then 2^6
29 $= 64$, and $64 - 1 = 63 = 7 \times 9$ is a multiple of 7 .

30 A Carmichael number can be defined as a composite number n which in
31 modular arithmetic satisfies congruence relation $b^n \equiv b \pmod{n}$ for all integers
32 b . The relation may also be expressed in the form $b^{n-1} \equiv 1 \pmod{n}$ for all
33 integers b that are relatively prime to n . They are infinitely many Carmichael
34 numbers. They constitute the comparatively rare instances where the strict
35 converse of Fermat's Little Theorem does not hold. This fact precludes the use
36 of that theorem as an absolute test of primality.

37 Derived from prime and composite numbers, the following are well-defined
38 Incompletely Predictable entities that faithfully comply with inclusion-exclusion
39 principle: All Odd Primes [as a subset] derived from All Prime numbers [as a
40 set] are selected prime numbers that constitute a "decelerating CIS" and "thin
41 set" with Natural density 0. Gap 2-Composite even numbers always precede
42 Odd Primes. All Gap 2-Composite even numbers [as a subset] derived from

1 All Composite numbers [as a set] are highly selective composite numbers that
2 constitute a "decelerating CIS" and "thin set" with Natural density 0. The n -
3 Carmichael numbers [constituting the entire smaller subsets of All Carmichael
4 numbers] will always be Gap 1-Composite odd numbers that have n prime
5 factors with $n \geq 3$ e.g. 3-Carmichael numbers [as a subset] have three prime
6 factors, 4-Carmichael numbers [as a subset] have four prime factors, etc. All
7 Carmichael numbers [as a subset] derived from All Composite numbers [as a
8 set] are highly selective composite numbers that constitute a "decelerating CIS"
9 and "thin set" with Natural density 0.

10 For Carmichael numbers with exactly three prime factors, once one of
11 the primes has been specified, there are only a finite number of Carmichael
12 numbers which can be constructed. Indeed, for Carmichael numbers with n
13 prime factors, there are only a finite number with the least $n - 2$ specified.

14 Chernick's construction of Carmichael numbers is an extended way to
15 obtain even smaller subsets of n -Carmichael numbers: $M_k(m) = (6m + 1)$

16 $(12m + 1) \prod_{i=1}^{k-2} (9 \cdot 2^i m + 1)$, $k \geq 3$, with the condition that each of the factors
17

18 are prime and that m is divisible by 2^{k-4} . For example, Chernick Carmichael
19 numbers as a subset of 3-Carmichael numbers: If, for a natural number m , the
20 three numbers $6m + 1$, $12m + 1$ and $18m + 1$ are prime numbers, the product
21 $M_k(m) = (6m + 1) (12m + 1) (18m + 1)$ is a 3-Carmichael number. This con-
22 dition can only be satisfied if the number m ends with digits 0, 1, 5 or 6 in
23 base 10 (i.e. m is congruent to 0 or 1 modulo 5). The first few 3-Carmichael
24 numbers that correspond to $m = 1, 6, 35, 45, 51, 55, 56, 100, 121, \dots$ are 1729,
25 294409, 56052361, 118901521, 172947529, 216821881, 228842209, 1299963601,
26 2301745249,.... An equivalent formulation of Chernick's construction is that if
27 p , $2p - 1$ and $3p - 2$ are prime numbers congruent to 1 modulo 6, then the
28 product $p (2p - 1) (3p - 2)$ is a 3-Carmichael number. Incidentally, the Hardy-
29 Ramanujan number, e.g. $1729 = 7 \cdot 13 \cdot 19$, is the third Carmichael number and
30 the first Chernick Carmichael number.

31 Let $C(x)$ denote the number of Carmichael numbers less than x . Then, for
32 all sufficiently large x , $C(x) > x^{2/7}$, which proves that there are infinitely many
33 Carmichael numbers. The upper bound $C(x) < x \exp\left(-\frac{\log x \log \log \log x}{\log \log x}\right)$
34 has also been proved.
35

36 Bertrand's postulate is the theorem that for any integer $n > 3$, there exists
37 at least one prime number p with $n < p < 2n - 2$. A less restrictive formulation
38 is: for every $n > 1$, there is always at least one prime p such that $n < p < 2n$.
39 Another formulation, where p_n is the n -th prime, is: for $n \geq 1$ $p_{n+1} < 2p_n$.

40 Daniel Larsen in 2021[1] proved an analogue of Bertrand's postulate for
41 Carmichael numbers first conjectured by Alford, Granville, and Pomerance in
42

1994. Using techniques developed by Yitang Zhang and James Maynard [see Remark 3.3] to establish results concerning small gaps between primes, his work yielded the much stronger statement that, for any $\delta > 0$ and sufficiently large x in terms of δ , there will always be at least $\exp\left(\frac{\log x}{(\log \log x)^{2+\delta}}\right)$ Carmichael numbers between x and $x + \frac{x}{(\log x)^{\frac{1}{2+\delta}}}$.

Carmichael numbers have the following properties:

1. If a prime p divides Carmichael number n , then $n \equiv 1 \pmod{p-1}$ implies that $n \equiv p \pmod{p(p-1)}$.
2. Every Carmichael number is squarefree.
3. An odd composite squarefree number n is a Carmichael number iff n divides the denominator of Bernoulli number B_{n-1} .

Theorem 2.1 in section 2 contains the required proof for "*Theorem of Nil Predilection for Even-Odd Parity in Odd Primes associated with Carmichael numbers*". The "thin set" and "decelerating CIS" of very rare Carmichael numbers are statistically associated with Odd Primes manifesting $\sim 50\%$ "Even Parity" and $\sim 50\%$ "Odd Parity". This deduction validly lend strong support for Polignac's and Twin prime conjectures to be true.

5. Incompletely Predictable Primes with restricted digits in Base-10

Let $a_0 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. James Maynard in 2016 elegantly show there are infinitely many prime numbers which do not have the digit a_0 in their decimal expansion [viz, there are infinitely many Primes with restricted digits]. The proof is an application of the Hardy-Littlewood circle method to a binary problem, and rests on obtaining suitable 'Type I' and 'Type II' arithmetic information for use in Harman's sieve to control the minor arcs. Throughout in his paper[3] that publish the required proof for this problem, $f \asymp g$ means that there are absolute constants $c_1, c_2 > 0$ such that $c_1 f < g < c_2 f$. In relation to (i) Set of All Prime numbers $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\}$, (ii) Subset of All Odd Primes $\{3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\}$, and (iii) ten Subsets of Restricted Primes missing one digit $\{0 \text{ or } 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6 \text{ or } 7 \text{ or } 8 \text{ or } 9\}$ [when written in base 10]; all these Set and Subsets will consist of "decelerating CIS" and "thin set" whereby the phrase "infinitely many" should more concisely be replaced by "Arbitrarily Large Number (ALN)".

Maynard then consider the same problem in bases other than 10, and with more than one excluded digit. The set of numbers less than X missing s digits in base q has $\asymp X^c$ elements, where $c = \log(q-s)/\log q$. For fixed s , the density becomes larger as q increases, and so the problem becomes easier. His methods are not powerful enough to show the existence of infinitely many primes with two digits not appearing in their decimal expansion, but they can show that

1 there are infinitely many primes with s digits excluded in base q provided q is
 2 large enough in terms of s . Moreover, if the set of excluded digits possesses
 3 some additional structure this can apply to very thin sets formed in this way.

4 THEOREM 5.1. *With nine excluded digits in the decimal expansion of Odd*
 5 *Primes [in base 10 with $a_0 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$], there is a theoretical*
 6 *inherent limitation on the number eligibility of these particular Odd Primes.*
 7

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9

Primes having only one digit a_0 in their decimal expansion			
Digit a_0	Prime numbers	No. of Primes	Properties
0	nil	0	Not applicable
1	11, 111111111111111111, 11111111111111111111,...	ALN	Repunit Primes (extremely rare)
2	2 [Only even Prime number]	1	odd Prime gap 1
3	3	1	As 1-digit Prime
4	nil	0	Always composite
5	5	1	As 1-digit Prime
6	nil	0	Always composite
7	7	1	As 1-digit Prime
8	nil	0	Always composite
9	nil	0	Always composite

23 Table 1. Restricted Primes with 9 digits excluded in base 10.
 24 ALN = Arbitrarily Large Number, $a_0 \in \{0, 1, 2, \dots, 7, 8, 9\}$
 25

26

27 A repdigit, or sometimes monodigit, is a natural number composed of
 28 repeated instances of the same digit in a positional number system (that is
 29 often implicitly decimal). A repunit in base-10 is a number like 1, 11, 111,
 30 1111,... that contains only the digit 1 – a more specific type of repdigit. A
 31 repunit prime is a repunit that is also a prime number.

32 The base- b repunits are defined as (where this b can be either positive or
 33 negative) $R_n^{(b)} \equiv 1 + b + b^2 + \dots + b^{n-1} = \frac{b^n - 1}{b - 1}$ for $|b| \geq 2, n \geq 1$. Thus, the
 34 number $R_n^{(b)}$ consists of n copies of the digit 1 in base- b representation. The first
 35 two repunits base- b for $n = 2$ are $R_1^{(b)} = \frac{b - 1}{b - 1} = 1$ and $R_2^{(b)} = \frac{b^2 - 1}{b - 1} = b + 1$
 36 for $|b| \geq 2$. In particular, the decimal (base-10) repunits that are often referred
 37 to as simply repunits are defined as $R_n \equiv R_n^{(10)} = \frac{10^n - 1}{10 - 1} = \frac{10^n - 1}{9}$ for $n \geq 1$.
 38

39 Thus, the number $R_n = R_n^{(10)}$ consists of n copies of digit 1 in base-10
 40 representation. The sequence of repunits base-10 starts with 1, 11, 111, 1111,
 41
 42

1 11111, 111111, 1111111,.... For decimal (base-10) repunit primes, R_n is prime
2 for $n = 2, 19, 23, 317, 1031, 49081, 86453, 109297, \dots$

3 The prime repunits are a trivial subset of the permutable primes, i.e.,
4 primes that remain prime after any permutation of their digits. Particular
5 properties of decimal (base-10) repunit primes are:

- 6 • The remainder of R_n modulo 3 is equal to the remainder of n modulo
7 3. Using $10^a \equiv 1 \pmod{3}$ for any $a \geq 0$, $n \equiv 0 \pmod{3} \Leftrightarrow R_n \equiv 0$
8 $\pmod{3} \Leftrightarrow R_n \equiv 0 \pmod{R_3}$, $n \equiv 1 \pmod{3} \Leftrightarrow R_n \equiv 1 \pmod{3} \Leftrightarrow$
9 $R_n \equiv R_1 \equiv 1 \pmod{R_3}$, $n \equiv 2 \pmod{3} \Leftrightarrow R_n \equiv 2 \pmod{3} \Leftrightarrow R_n \equiv$
10 $R_2 \equiv 11 \pmod{R_3}$. Therefore, $3 \mid n \Leftrightarrow 3 \mid R_n \Leftrightarrow R_3 \mid R_n$.
- 11 • The remainder of R_n modulo 9 is equal to the remainder of n modulo
12 9. Using $10^a \equiv 1 \pmod{9}$ for any $a \geq 0$, $n \equiv r \pmod{9} \Leftrightarrow R_n \equiv r$
13 $\pmod{9} \Leftrightarrow R_n \equiv R_r \pmod{R_9}$, for $0 \leq r < 9$. Therefore, $9 \mid n \Leftrightarrow 9 \mid$
14 $R_n \Leftrightarrow R_9 \mid R_n$.

15 Similarly, repunits base-2 are defined as $R_n^{(2)} = \frac{2^n - 1}{2 - 1} = 2^n - 1$ for $n \geq 1$.
16

17 The number $R_n^{(2)}$ consists of n copies of digit 1 in base-2 representation. The
18 base-2 repunits are the well-known Mersenne numbers $M_n = 2^n - 1$, they start
19 with 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767,
20 65535,.... Primes that are repunits in base-2 are Mersenne primes.
21

22 Useful known properties of Repunits:

- 23 • Any repunit in any base having a composite number of digits is neces-
24 sarily composite. Only repunits (in any base) having a prime number
25 of digits can be prime [whereby there are an ALN or decelerating CIS
26 or thin set of prime numbers]. This is a necessary but not sufficient
27 condition as discussed next. It is easy to show that if n is divisible by
28 a , then $R_n^{(b)}$ is divisible by $R_a^{(b)}$: $R_n^{(b)} = \frac{1}{b-1} \prod_{d \mid n} \Phi_d(b)$, where $\Phi_d(x)$
29

30 is the d^{th} cyclotomic polynomial and d ranges over the divisors of n .

31 For p prime, $\Phi_p(x) = \sum_{i=0}^{p-1} x^i$, which has the expected form of a repunit
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33 when x is substituted with b . For example, 9 is divisible by 3, and thus
34 R_9 is divisible by R_3 - in fact, $111111111 = 111 \cdot 1001001$. The corre-
35 sponding cyclotomic polynomials $\Phi_3(x)$ and $\Phi_9(x)$ are $x^2 + x + 1$ and
36 $x^6 + x^3 + 1$, respectively. Thus, for R_n to be prime, n must necessarily
37 be prime, but it is not sufficient for n to be prime. For example, $R_3 =$
38 $111 = 3 \cdot 37$ is not prime. Except for this case of R_3 , p can only divide
39 R_n for prime n if $p = 2kn + 1$ for some k .

- 40 • If p is an odd prime, then every prime q that divides $R_p^{(b)}$ must be either
41 1 plus a multiple of $2p$, or a factor of $b - 1$. For example, a prime factor
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of R_{29} is $62003 = 1 + 2 \cdot 29 \cdot 1069$. The reason is that the prime p is the smallest exponent greater than 1 such that q divides $b^p - 1$, because p is prime. Therefore, unless q divides $b - 1$, p divides the Carmichael function of q , which is even and equal to $q - 1$.

- Any positive multiple of the repunit $R_n^{(b)}$ contains at least n nonzero digits in base- b . Any number x is a two-digit repunit in base $x - 1$. The only known numbers that are repunits with at least 3 digits in more than one base simultaneously are 31 (111 in base-5, 11111 in base-2) and 8191 (111 in base-90, 1111111111111 in base-2). Goormaghtigh conjecture says there are only these two cases.
- Using the pigeon-hole principle it can be easily shown that for relatively prime natural numbers n and b , there exists a repunit in base- b that is a multiple of n . To see this consider repunits $R_1^{(b)}, \dots, R_n^{(b)}$. Because there are n repunits but only $n - 1$ non-zero residues modulo n there exist two repunits $R_i^{(b)}$ and $R_j^{(b)}$ with $1 \leq i < j \leq n$ such that $R_i^{(b)}$ and $R_j^{(b)}$ have the same residue modulo n . It follows that $R_j^{(b)} - R_i^{(b)}$ has residue 0 modulo n , i.e. is divisible by n . Since $R_j^{(b)} - R_i^{(b)}$ consists of $j - i$ ones followed by i zeroes, $R_j^{(b)} - R_i^{(b)} = R_{j-i}^{(b)} \times b^i$. Now n divides the left-hand side of this equation, so it also divides the right-hand side, but since n and b are relatively prime, n must divide $R_{j-i}^{(b)}$.
- The Feit-Thompson conjecture is that $R_q^{(p)} R_q(p)$ never divides $R_p^{(q)}$ for two distinct primes p and q .
- Using the Euclidean Algorithm for repunits definition: $R_1^{(b)} = 1$; $R_n^{(b)} = R_{n-1}^{(b)} \times b + 1$, any consecutive repunits $R_{n-1}^{(b)}$ and $R_n^{(b)}$ are relatively prime in any base- b for any n .
- If m and n have a common divisor d , $R_m^{(b)}$ and $R_n^{(b)}$ have the common divisor $R_d^{(b)}$ in any base- b for any m and n . That is, the repunits of a fixed base form a strong divisibility sequence. As a consequence, If m and n are relatively prime, $R_m^{(b)}$ and $R_n^{(b)}$ are relatively prime. The Euclidean Algorithm is based on $\gcd(m, n) = \gcd(m - n, n)$ for $m > n$. Similarly, using $R_m^{(b)} - R_n^{(b)} \times b^{m-n} = R_{m-n}^{(b)}$, it can be easily shown that $\gcd(R_m^{(b)}, R_n^{(b)}) = \gcd(R_{m-n}^{(b)}, R_n^{(b)})$ for $m > n$. Therefore, if $\gcd(m, n) = d$, then $\gcd(R_m^{(b)}, R_n^{(b)}) = R_d^{(b)}$.

We concentrate on Repunit Primes in base-10: In keeping with Odd Primes-Prime gaps constraints from Axiom 6.1 [and its **List of eligible Last digit of Odd Primes**] on applying Prime number theorem for Arithmetic Progression to statistically confirm Polignac's and Twin prime conjectures to be true, we deduce that all Repunit Primes must be associated with their Prime gaps with last digit ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40... = ~25%] or 2

1 [viz, Gap 2, Gap 12, Gap 22, Gap 32... = ~25%] or 6 [viz, Gap 6, Gap 16, Gap
2 26, Gap 36... = ~25%] or 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38... = ~25%]
3 but NOT 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34... = 0%]. Repunit Primes
4 have probabilistically an ALN of Prime gaps to choose from. Therefore, we can
5 statistically state that the extremely rare ≥ 2 -digit Repunit Primes belong to
6 "decelerating CIS" and "thin set", whereby they seem to occur roughly as often
7 as the Prime number theorem would predict: the exponent of the N^{th} repunit
8 prime is generally around a fixed multiple of the exponent of the $(N - 1)^{th}$.

9 Table 1 depict "Primes having only one digit a_0 in their decimal expansion"
10 to overall be the rarest: (i) non-existing when $a_0 = \{0, 4, 6, 8 \text{ or } 9\}$, (ii) solitary
11 when $a_0 = \{2, 3, 5 \text{ or } 7\}$, and (iii) ALN as "decelerating CIS" or "thin set"
12 when $a_0 = \{1\}$. By logical extrapolation, Repunit Primes with 9 digits excluded
13 in base-10 should statistically be much rarer than any other Restricted Primes
14 with lesser $\{8 \text{ or } 7 \text{ or } 6 \text{ or } 5 \text{ or } 4 \text{ or } 3 \text{ or } 2 \text{ or } 1\}$ digits excluded in base-10 \implies
15 when based on Axiom 6.1, these Restricted Primes with lesser digits excluded in
16 base-10 must inevitably also be "decelerating CIS" and "thin sets" that validly
17 lend strong support for Polignac's and Twin prime conjectures to be true.

18 ***The proof is now complete for Theorem 5.1***□.

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22 6. Prime number theorem for Arithmetic Progressions

23 There is solitary odd Prime gap 1 for the only even prime number 2.
24 Abbreviations: ALN = Arbitrarily Large Number, CFS = Countably Finite
25 Set, CIS = Countably Infinite Set, CP = Completely Predictable, IP = Incom-
26 pletely Predictable, FL = Finite Length, IL = Infinite Length. For $i = 1, 2, 3,$
27 $4, 5, \dots, n$; relevant algorithm and sub-algorithms from Sieve of Eratosthenes for
28 initial even Prime gaps 2, 4 and 6 that represent odd Twin primes, odd Cousin
29 primes and odd Sexy primes are given below.

30 (a) For IP IL algorithm [Gap 2, 4, 6, 8, 10...]-Sieve of Eratosthenes $p_{n+1} =$
31 $3 + \sum_{i=1}^n g_i$ [where $n = \text{ALN}$] that faithfully generates all Odd $\mathbb{P} \{3, 5, 7, 11,$
32 $13, 17, 19 \dots\}$ with cardinality \aleph_0 -decelerating, the n^{th} even Prime gap between
33 two successive Odd \mathbb{P} is denoted by $g_n = (n + 1)^{st}$ Odd $\mathbb{P} - (n)^{th}$ Odd \mathbb{P} , i.e.
34 $g_n = p_{n+1} - p_n = 2, 2, 4, 2, 4, 2 \dots$
35

36 (b) For CP FL sub-algorithm [Gap 1]-Sieve of Eratosthenes $p_{n+1} = 2 + \sum_{i=1}^n g_i$
37 [where $n = 1$ and not ALN] that faithfully generates the first and only Even
38 $\mathbb{P} \{2\} \equiv$ first and only paired Even $\mathbb{P} \{(2,3)\}$ with cardinality CFS of 1, the
39 solitary n^{th} odd prime gap between two successive primes is denoted by $g_n =$
40 $(n + 1)^{st}$ Odd $\mathbb{P} - (n)^{th}$ Even \mathbb{P} , i.e. $g_n = p_{n+1} - p_n = 3 - 2 = 1$.
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2 (c) For IP IL sub-algorithm [Gap 2]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^n g_i$
3 [where $n = \text{ALN}$] that faithfully generates all Odd twin $\mathbb{P} \{3, 5, 11, 17, 29, 41,$
4 $59\dots\} \equiv$ all paired Odd twin $\mathbb{P} \{(3,5), (5,7), (11,13), (17,19), (29,31), (41,43),$
5 $(59,61)\dots\}$ with cardinality \aleph_0 -decelerating, the n^{th} even Prime gap between
6 two successive Odd twin \mathbb{P} is denoted by $g_n = (n + 1)^{\text{st}}$ Odd twin $\mathbb{P} - (n)^{\text{th}}$
7 Odd twin \mathbb{P} , i.e. $g_n = p_{n+1} - p_n = 2, 6, 6, 12, 12, 18\dots$
8

9 (d) For IP IL sub-algorithm [Gap 4]-Sieve of Eratosthenes $p_{n+1} = 7 + \sum_{i=1}^n g_i$
10 [where $n = \text{ALN}$] that faithfully generates all Odd cousin $\mathbb{P} \{7, 13, 19, 37,$
11 $43, 67\dots\} \equiv$ all paired Odd cousin $\mathbb{P} \{(7,11), (13,17), (19,23), (37,41), (43,47),$
12 $(67,71)\dots\}$ with cardinality \aleph_0 -decelerating, the n^{th} even Prime gap between
13 two successive Odd cousin \mathbb{P} is denoted by $g_n = (n + 1)^{\text{st}}$ Odd cousin $\mathbb{P} - (n)^{\text{th}}$
14 Odd cousin \mathbb{P} , i.e. $g_n = p_{n+1} - p_n = 6, 6, 8, 6, 24\dots$
15

16 (e) For IP IL sub-algorithm [Gap 6]-Sieve of Eratosthenes $p_{n+1} = 23 + \sum_{i=1}^n g_i$
17 [where $n = \text{ALN}$] that faithfully generates all Odd sexy $\mathbb{P} \{23, 31, 47, 53, 61,$
18 $73, 83\dots\} \equiv$ all paired Odd sexy $\mathbb{P} \{(23,29), (31,37), (47,53), (53,59), (61,67),$
19 $(73,79), (83,89)\dots\}$ with cardinality \aleph_0 -decelerating, the n^{th} even Prime gap
20 between two successive Odd sexy \mathbb{P} is denoted by $g_n = (n + 1)^{\text{st}}$ Odd sexy $\mathbb{P} -$
21 n^{th} Odd sexy \mathbb{P} , i.e. $g_n = p_{n+1} - p_n = 8, 16, 6, 8, 12, 10\dots$
22

23 A number base, consisting of any whole number greater than 0, is the
24 number of digits or combination of digits that a number system uses to represent
25 numbers e.g. decimal number system or base 10, binary number system or
26 base2, octal number system or base 8, hexa-decimal number system or base 16.
27 As $x \rightarrow \infty$, various derived properties of Prime counting function, Prime- $\pi(x)$
28 [= number of primes up to x] occur in, for instance, Prime number theorem
29 for Arithmetic Progressions, Prime- $\pi(x; b, a)$ [= number of primes up to x with
30 last digit of primes given by a in base b]. For any choice of digit a in base b with
31 $\text{gcd}(a, b) = 1$: Prime- $\pi(x; b, a) \sim \frac{\text{Prime-}\pi(x)}{\phi(b)}$. Here, Euler's totient function
32 $\phi(n)$ is defined as the number of positive integers $\leq n$ that are relatively prime
33 to (i.e., do not contain any factor in common with) n , where 1 is counted as
34 being relatively prime to all numbers. Then each of the last digit of primes given
35 by digit a in base b as $x \rightarrow \infty$ is equally distributed between the permitted
36 choices for digit a with this result being valid for, and is independent of, any
37 chosen base b .
38

39 Numbers with their last digit ending in (i) 1, 3, 7 or 9 [which can be either
40 primes or composites] constitute $\sim 40\%$ of all integers; and (ii) 0, 2, 4, 5, 6 or 8
41 [which must be composites] constitute $\sim 60\%$ of all integers. We validly ignore
42

1 the only single-digit even prime number 2 and odd prime number 5. We note
2 ≥ 2 -digit Odd Primes can only have their last digit ending in 1, 3, 7 or 9 but
3 not in 0, 2, 4, 5, 6 or 8. **List of eligible Last digit of Odd Primes:**

- 4 • The last digit of Odd Primes having their Prime gaps with last digit
5 ending in 2 [viz, Gap 2, Gap 12, Gap 22, Gap 32...] can only be 1, 7 or
6 9 [but not (5) or 3] as three choices.
- 7 • The last digit of Odd Primes having their Prime gaps with last digit
8 ending in 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34...] can only be 3, 7 or
9 9 [but not (5) or 1] as three choices.
- 10 • The last digit of Odd Primes having their Prime gaps with last digit
11 ending in 6 [viz, Gap 6, Gap 16, Gap 26, Gap 36...] can only be 1, 3 or
12 7 [but not (5) or 9] as three choices.
- 13 • The last digit of Odd Primes having their Prime gaps with last digit
14 ending in 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38...] can only be 1, 3 or
15 9 [but not (5) or 7] as three choices.
- 16 • The last digit of Odd Primes having their Prime gaps with last digit
17 ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40...] can only be 1, 3,
18 7 or 9 [but not (5)] as four choices.

20 AXIOM 6.1. *Applying Prime number theorem for Arithmetic Pro-*
21 *gressions confirm Modified Polignac's and Twin prime conjectures,*
22 *and support generalized and ordinary Riemann hypothesis.*

23 **Proof.** We use decimal number system (base $b = 10$), and ignore the only
24 single-digit even prime number 2 and odd prime number 5. For $i = 1, 2, 3,$
25 4, 5...; the last digit of all Gap $2i$ -Odd Primes can only end in 1, 3, 7 or 9
26 that are each proportionally and equally distributed as $\sim 25\%$ when $x \rightarrow \infty$,
27 whereby this result is consistent with Prime number theorem for Arithmetic
28 Progressions. The 100%-Set of, and its derived four unique 25%-Subsets of,
29 Gap $2i$ -Odd Primes based on their last digit being 1, 3, 7 or 9 must all be
30 decelerating CIS. "Different Prime numbers literally equates to different Prime
31 gaps" is a well-known intrinsic property. Since the ALN of Gap $2i$ as fully
32 represented by all Prime gaps with last digit ending in 0, 2, 4, 6 or 8 are
33 associated with various permitted combinations of last digit in Gap $2i$ -Odd
34 Primes being 1, 3, 7 and/or 9 as three or four choices [as per **List of eligible**
35 **Last digit of Odd Primes**]; then these ALN unique subsets of Prime gaps
36 based on their last digit being 0, 2, 4, 6 or 8 together with their correspondingly
37 derived ALN unique subsets constituted by Gap $2i$ -Odd Primes having last
38 digit 1, 3, 7 or 9 must also all be decelerating CIS. The Probability (any Gap
39 $2i$ abruptly terminating as $x \rightarrow \infty$) = Probability (any Gap $2i$ -Odd Primes
40 abruptly terminating as $x \rightarrow \infty$) = 0. Thus Modified Polignac's and Twin
41 prime conjectures is confirmed to be true. With ordinary Riemann hypothesis
42

$\frac{1}{2}$ being a special case, generalized Riemann hypothesis formulated for Dirichlet
 $\frac{2}{2}$ L-function holds once $x > b^2$, or base $b < \frac{1}{2}$ as $x \rightarrow \infty$. **The ["statistical"
 $\frac{3}{2}$ or "probabilistic"] proof is now complete for Axiom 6.1.**
 $\frac{4}{2}$
 $\frac{5}{2}$

$\frac{6}{2}$ 7. Conclusions

$\frac{7}{2}$ (Modified) Polignac's and Twin prime conjectures are proven to be true in
 $\frac{8}{2}$ this paper by mainly using statistical arguments. With having *Analytic rank 0*
 $\frac{9}{2}$ as common overlapping "component" between them, both Riemann hypothesis
 $\frac{10}{2}$ (RH), and Birch and Swinnerton-Dyer (BSD) conjecture involve proving the
 $\frac{11}{2}$ unexpected presence of certain [overall] "macro-properties". Profound State-
 $\frac{12}{2}$ ment: *Irrespective of L-function sources and always with the one [unique] set*
 $\frac{13}{2}$ *of nontrivial zeros as OUTPUTS from each L-function, all the infinitely-many*
 $\frac{14}{2}$ *nontrivial zeros as [well-defined] Incompletely Predictable entities are ONLY*
 $\frac{15}{2}$ *located on (Analytically normalized) $\sigma = \frac{1}{2}$ -Critical Line.*

$\frac{16}{2}$ With respecting Remark A.2, the above statement insightfully describes
 $\frac{17}{2}$ intractable open problem in Number theory of (Generalized) RH. Graphs of Z-
 $\frac{18}{2}$ function from LMFDB[2] on Genus 1 elliptic curves with nonzero Analytic rank
 $\frac{19}{2}$ 1, 2, 3, 4, 5... have trajectories that intersect Origin point. Graphs of Z-function
 $\frac{20}{2}$ from LMFDB[2] on Genus 1 elliptic curves with Analytic rank 0 [viz, having
 $\frac{21}{2}$ zero independent basis point (with infinite order) associated with either finitely
 $\frac{22}{2}$ many or zero $E(\mathbb{Q})$ solutions] DO NOT have trajectories that intersect Origin
 $\frac{23}{2}$ point. *Ditto* for Graph of Z-function on Genus 0 (non-elliptic) Riemann zeta
 $\frac{24}{2}$ function / Dirichlet eta function [in Figure 1] with Analytic rank 0 [viz, it DOES
 $\frac{25}{2}$ NOT have trajectory that intersect Origin point]. *This implies the "simplest*
 $\frac{26}{2}$ *version" of BSD conjecture to be true; and also simultaneously implies the*
 $\frac{27}{2}$ *"simplest version" of RH to be true (with its Geometrical-Mathematical proof*
 $\frac{28}{2}$ *in [6] and its Algebraic-Transcendental proof in Appendix A).*

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 $\frac{33}{2}$

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Appendix A. Algebraic-Transcendental proof for Riemann hypothesis using Algebraic-Transcendental theorem

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14

15 Proposed by German mathematician Bernhard Riemann (17 September

16 1826 – 20 July 1866) in 1859, Riemann hypothesis states that all infinitely-

17 many nontrivial zeros (NTZ), as a "thin set", of $\zeta(s)$ are located on its $\sigma = \frac{1}{2}$ -

18 Critical line. L-function associated to Genus 0 (non-elliptic) curve of $\zeta(s)$ is

19 known to admit an analytic continuation and satisfy a functional equation via

20 its *proxy* $\eta(s)$; viz, we do not need to assume Hasse-Weil conjecture. In [6], we

21 have provided Geometrical-Mathematical proof for Riemann hypothesis.

22

23 The success and failures of both Gram's rule and Rosser's rule only occur

24 in Dirichlet eta function [*proxy* for Riemann zeta function] on $\sigma = \frac{1}{2}$ -Critical

25 line. To solve Riemann hypothesis, one must analyze non-overlapping Subset

26 of "One NTZ" = ~66%, Subset of "Zero NTZ" = ~17%, and Subset of "Two

27 NTZ" = ~17% as precisely derived from Set of "All NTZ" = ("conserved")

28 100% [instead of analyzing various overlapping Gram blocks and Gram intervals

29 containing "good" or "bad" Gram points, missing NTZ or extra NTZ, etc].

30

31 Transcendental functions \gg Algebraic functions with the Uncountably In-

32 finite Set of Transcendental irrational numbers \gg Countably Infinite Set of

33 Algebraic irrational numbers. From selected mathematical arguments, we formally

34 derive Algebraic-Transcendental theorem which supports the Statement:

35 *Algebraic functions must form a subset of the broader class of Transcendental*

36 *functions.* We now supply a non-exhaustive list of Algebraic-Transcendental

37 links. This will suffice for our purpose to create Algebraic-Transcendental

38 theorem required to complete the deceptively simple Algebraic-Transcendental

39 proof for Riemann hypothesis.

40 LEMMA A.1. *We outline relevant Algebraic-Transcendental connections*

41 *when based on algebraic functions and algebraic numbers, and transcendental*

42 *functions and transcendental numbers.*

Proof. An algebraic function is a function often defined as root of an irreducible polynomial equation. The algebraic functions are usually algebraic expressions using a *finite number of terms*, involving only algebraic operations addition (+), subtraction (-), multiplication (\times), division (\div), and raising to a fractional power. Examples of pure algebraic function are: $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, $f(x) = \frac{\sqrt{1+x^3}}{x^{3/7} - \sqrt{7}x^{1/3}}$, Golden ratio $\phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\dots$ [that is the most irrational number because it's hard to approximate with a rational number], etc. Algebraic functions usually cannot be defined as finite formulas of elementary functions, as shown by the example of Bring radical $f(x)^5 + f(x) + x = 0$ (this is the Abel-Ruffini theorem).

A transcendental function is an analytic function that does not satisfy a polynomial equation whose coefficients are functions of independent variable written using only the basic operations of addition, subtraction, multiplication, and division (without the need of taking limits). Examples of pure transcendental functions are: logarithm function $\ln x$ or $\log_e x$, exponential function e^x , trigonometric functions $\sin x$ and $\cos x$, hyperbolic functions $\sinh x$ and $\cosh x$, generalized hypergeometric functions, class of numbers called Liouville numbers [that can be more closely approximated by rational numbers than can any irrational algebraic number], etc. Equations over these expressions are called transcendental equations. A transcendently transcendental function or hypertranscendental function is transcendental analytic function which is not the solution of an algebraic differential equation with coefficients in integers \mathbb{Z} and with algebraic initial conditions; e.g. zeta functions of algebraic number fields, in particular, Riemann zeta function $\zeta(s)$ and gamma function $\Gamma(s)$ (cf. Holder's theorem).

The indefinite integral of many algebraic functions is transcendental. For example, integral $\int_{t=1}^x \frac{1}{t} dt$ turns out to equal logarithm function $\log_e(x)$. Similarly, the limit or the infinite sum of many algebraic function sequences is transcendental. Example, $\lim_{n \rightarrow \infty} (1 + x/n)^n$ converges to exponential function e^x , and infinite sum $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ turns out to equal hyperbolic cosine function $\cosh x$. In fact, it is impossible to define any transcendental function in terms of algebraic functions without using some such "limiting procedure" (integrals, sequential limits, and infinite sums are just a few).

A function that is not a transcendental function must logically be an algebraic function. This implies every algebraic function is algebraic solution to a polynomial equation but transcendental functions are not solutions to any such equation. Stated in another way: The output values of an algebraic function

1 (for specific input values of x) are algebraic numbers. This is because algebraic
2 function itself is defined as a solution to an algebraic equation, and any solution
3 to such an equation is [and must be] an algebraic number.

4 While transcendental functions often produce transcendental numbers as
5 outputs, they also have solutions as algebraic numbers. The composition of
6 transcendental functions in $f(x) = \cos \arcsin x = \sqrt{1-x^2}$ will give an alge-
7 braic function. Outputs from transcendental functions as algebraic numbers:
8 Equation $e^x = 1$ has solution $x = 0$, an algebraic number (since 0 is algebraic).
9 Equation $\sin(x) = 0$ has solutions $x = n\pi$, where $n = 0, 1, 2, 3, 4, 5, \dots$ are alge-
10 braic numbers (since integers are algebraic). Equation $\ln(x) = 0$ has solution
11 $x = 1$, an algebraic number (since 1 is algebraic). Outputs from transcendental
12 function as transcendental numbers: Equation $e^x = 2$ has solution $x = \ln(2)$,
13 which is transcendental, since $\ln(2)$ is a transcendental number.

14 *Remark A.1.* Two trigonometric functions in equation $\sin(x) = \cos(x)$
15 $= \frac{\pi}{4}$ have identical solution $x = \frac{1}{\sqrt{2}}$. This "sweet-spot" property is due to
16 17 sine-cosine complementary angle relationship for isosceles triangle. The $\frac{1}{\sqrt{2}} \approx$
18 19 0.70710678 is (algebraic) irrational number and $\frac{\pi}{4}$ is (transcendental) irrational
20 21 number. Then $\frac{\pi}{4}$ radian ≈ 0.785398 radian $\equiv 45^\circ$.

22 ***The proof is now complete for Lemma A.1***□.
23

24 PROPOSITION A.2. *Algebraic functions never give rise to transcendental*
25 *numbers as outputs unless we start involving transcendental functions.*

26 **Proof.** Algebraic numbers = {Integers + Rational numbers + Roots of In-
27 tegers (or Algebraic irrational numbers) as Algebraic (non-complex) numbers}
28 + { $z = a + bi$ as Algebraic (complex) numbers where a, b must be Integers or
29 Rational numbers}. Thus certain algebraic functions may involve more com-
30 plex operations such as roots or radicals, giving complicated outputs that are
31 still algebraic. We deduce from mathematical arguments in Lemma A.1: While
32 a given *de novo* function itself is algebraic [viz, a pure algebraic function], it
33 will never give rise to transcendental numbers unless we involve transcendental
34 functions [viz, create a mixed algebraic-transcendental function].

35 ***The proof is now complete for Proposition A.2***□.
36

37 COROLLARY A.3. *Any outputs as transcendental numbers from a given*
38 *function must involve transcendental functions, which can be given as either*
39 *pure transcendental functions or mixed algebraic-transcendental functions.*

40 **Proof.** Pure algebraic functions always give outputs that are algebraic
41 [but never transcendental]. Both pure transcendental functions and mixed
42

1 algebraic-transcendental functions give outputs as transcendental numbers \pm
2 algebraic numbers [but never as outputs that are all algebraic numbers].

3 Examples of mixed algebraic-transcendental functions: $f(x) = x^2 + e^x$ that
4 involves both algebraic and transcendental terms; and $f(x) = e^{\sqrt{x}}$ that involves
5 transcendental operation on algebraic number.

6 **The proof is now complete for Corollary A.3**□.

7
8 AXIOM A.4. *Nontrivial zeros (spectrum) computed [e.g. using Hardy Z-
9 function as $Z(t)$ plots] for any L-function involve transcendental functions in one
10 form or another, and are inherently given as t -valued transcendental numbers.*

11 **Proof.** It is precisely the case that since all infinitely-many nontrivial
12 zeros (spectrum) computed [e.g. using Hardy Z-function as $Z(t)$ plots] for any
13 given L-function will involve transcendental functions in one form or another
14 [often as mixed algebraic-transcendental functions]; then it is simply a math-
15 ematical impossibility that nontrivial zeros as outputs will not be given as
16 t -valued transcendental numbers. This deduction is completely consistent with
17 our Proposition A.2 and Corollary A.3.

18 A particular solution with '0' (zero) value from a given function may imply
19 that function to be a pure algebraic function, a pure transcendental function
20 or a mixed algebraic-transcendental function. Integer 0 is an algebraic number
21 (since 0 is algebraic). L-functions are usually mixed algebraic-transcendental
22 functions: [1] Analytic rank 0 L-functions will never have their 1st nontrivial
23 zero being endowed with algebraic 0 value. [2] Analytic rank 1, 2, 3, 4, 5...
24 (viz, non-zero ≥ 1) L-functions will always have their 1st nontrivial zero being
25 endowed with algebraic 0 value.

26 **The proof is now complete for Axiom A.4**□.

27
28 THEOREM A.5. *We can categorically formulate Algebraic-Transcendental
29 theorem which states that all infinitely-many nontrivial zeros (spectrum) from
30 Riemann zeta function must be located on its $\sigma = \frac{1}{2}$ -Critical Line [as was
31 originally proposed by the 1859-dated Riemann hypothesis].*

32
33 **Proof.** Being a self-dual L-function, Riemann zeta function as a Genus
34 0 curve admits an analytic continuation and satisfy a functional equation via
35 proxy Dirichlet eta function. By its very definition, geometrical $\sigma = \frac{1}{2}$ -Origin
36 point \equiv mathematical $\sigma = \frac{1}{2}$ -Critical line. All infinitely-many Origin intercept
37 points \equiv All infinitely-many Nontrivial zeros are proposed to lie on this Critical
38 line \implies Geometrical-Mathematical proof for Riemann hypothesis as outlined
39 in [6]. Consistent with Axiom A.4 is the fact that all infinitely-many nontrivial
40 zeros from Dirichlet eta function [proxy function for Riemann zeta function] are
41
42

1 always given as t -valued transcendental (irrational) numbers. From previous
 2 mathematical arguments in section 3 on properties for Incompletely Predictable
 3 entities, there are two occurrences of these entities in nontrivial zeros: (i) The
 4 integer numbers representing each and every one of infinitely-many nontrivial
 5 zeros, and (ii) Infinitely-many digital numbers after decimal point in each and
 6 every one of infinitely-many nontrivial zeros.

7 $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ [as example involving infinite sum of infinitely-many algebraic
 8 functions] turns out to equal hyperbolic cosine function $\cosh x$ [which is a tran-
 9 scendental function]. We can now conceptually represent a transcendental
 10 function as infinite sum of infinitely-many algebraic function sequences [viz,
 11 an 'infinite series']. Thus when based on inclusion-exclusion principle, we
 12 validly deduce that $\sigma = \frac{1}{2}$ -Dirichlet eta function is an unique $\sigma = \frac{1}{2}$ -mixed-
 13 algebraic-transcendental function [that contains all nontrivial zeros] AND $\sigma \neq$
 14 $\frac{1}{2}$ -Dirichlet eta functions are infinitely-many non-unique $\sigma \neq \frac{1}{2}$ -mixed-algebraic-
 15 transcendental functions [that cannot contain nontrivial zeros]. We now have
 16 the mutually exclusive statement based on $\sigma = \frac{1}{2}$ -Dirichlet eta function and
 17 $\sigma \neq \frac{1}{2}$ -Dirichlet eta functions being completely different 'infinite series': *{It is*
 18 *a mathematical impossibility for any nontrivial zeros to be located away from*
 19 *Critical line.}* \equiv *{It is a mathematical certainty for all nontrivial zeros to be*
 20 *located on Critical line.}*

21 Euler formula can be stated as $e^{in} = \cos n + i \cdot \sin n$. Applying this famous
 22 formula to $\sigma = \frac{1}{2}$ -Dirichlet eta function results in simplified $\sigma = \frac{1}{2}$ -Dirichlet
 23 eta function that faithfully contains all t -valued nontrivial zeros [whereby this
 24 simplified function will clearly identify itself as representing a mixed-algebraic-
 25 transcendental function involving both algebraic and transcendental functions]:
 26 The simplified $\sigma = \frac{1}{2}$ -Dirichlet eta function

$$27 = \sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4}\pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n-1) + \frac{1}{4}\pi)$$

28 ***The proof is now complete for Theorem A.5***□.

29 *Remark A.2.* Hasse-Weil zeta function is a global L-function defined as an
 30 Euler product of local zeta functions. Hasse-Weil conjecture states that Hasse-
 31 Weil zeta function attached to an algebraic variety V defined over an algebraic
 32 number field K should *admit an meromorphic continuation* for all complex s
 33 and *satisfy a functional equation*. In, for instance, Genus 2 curves over totally
 34 real fields, they have non-regular Hodge numbers and the Taylor-Wiles method
 35 that was successful in proving this conjecture for Genus 1 curves (for example)

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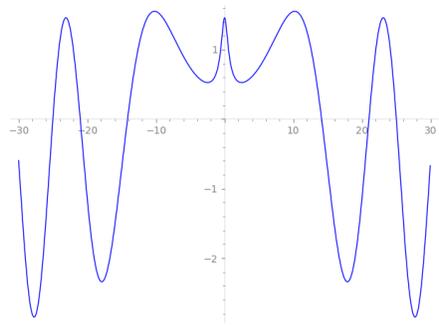


Figure 1. Graph of Z -function along $\Re(s) = \frac{1}{2}$ -critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet eta function $\eta(s)$ of degree 1 over $K = \mathbb{Q}$ as *Analytic continuation* of Riemann zeta function $\zeta(s)$. Line Symmetry vertical y -axis, trajectory DO NOT intersect Origin point, and manifest $Z(t)$ positivity as part of Sign normalization by L-functions and modular forms database (LMFDB)[2]. Integral basis 1. An integral basis of a number field K is a \mathbb{Z} -basis for ring of integers of K . It is a \mathbb{Q} -basis for K . Initial +ve nontrivial zeros: 14.13, 21.02, 25.01, 30.42, 32.93, 37.58,... The gaps between any two adjacent nontrivial zeros never consist of a fixed value \implies all infinitely-many nontrivial zeros must be Incompletely Predictable entities.

breaks down in several places. Many of the L-functions we consider in this paper (including those associated to curves of Genus > 1), are not known to admit an analytic continuation or satisfy a functional equation. To properly discuss nontrivial zeros on the Critical Line and in the Hardy Z -function; we therefore need to, at least, assume this conjecture.

Taking Remark A.2 into perspective consideration; all the correct and complete mathematical arguments in this paper are assumed to comply with two conditions below [that have "Analytic rank 0" component present in both]:

Condition 1. Generalized Riemann hypothesis (RH): All the nontrivial zeros (spectrum) of General [or Generic] L-functions from Genus 0, 1, 2, 3, 4, 5... curves with Analytic rank 0, 1, 2, 3, 4, 5... lie on the $\sigma = \frac{1}{2}$ -Critical Line or the Analytically normalized $\sigma = \frac{1}{2}$ -Critical Line. The 'special case' (*simplest*) RH[6] refers to the [Analytic rank 0] Genus 0 non-elliptic curve called Riemann zeta function / Dirichlet eta function.

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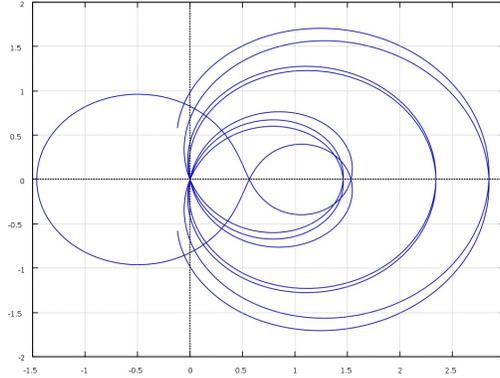


Figure 2. OUTPUT at $\sigma = \frac{1}{2}$ -Critical Line. Polar graph of $\zeta(\frac{1}{2} + it) / \eta(\frac{1}{2} + it)$ plotted for real values t between -30 and $+30$ from $s = \sigma \pm it$. Horizontal axis: $Re\{\eta(\frac{1}{2} + it)\}$. Vertical axis: $Im\{\eta(\frac{1}{2} + it)\}$. Indicating Riemann hypothesis, Origin intercept points \equiv nontrivial zeros are present. Manifesting perfect Mirror (Line) symmetry about horizontal x-axis.

Condition 2. Generalized Birch and Swinnerton-Dyer (BSD) conjecture: All Generic L-functions from Genus 0, 1, 2, 3, 4, 5... curves satisfy Algebraic (Mordell-Weil) rank = Analytic rank [for even Analytic rank 0, 2, 4, 6, 8, 10... and odd Analytic rank 1, 3, 5, 7, 9, 11...]. The 'special case' (*simplest*) BSD conjecture refers to Genus 1 elliptic curves; expressed as *weak form* and *strong form* of BSD conjecture.

Analogy for (Generalized) Riemann hypothesis: Let $\delta = \frac{1}{\infty}$ [which represents an infinitesimal small number value], Geometrical 0-dimensional $\sigma = \frac{1}{2}$ -Origin point \equiv Mathematical 1-dimensional $\sigma = \frac{1}{2}$ -Critical Line, and Origin intercept points \equiv nontrivial zeros. [Using sine-cosine complementary angle relationship $\sin(\theta) = \cos(\theta - \frac{\pi}{2}) \equiv \cos(\theta) = \sin(\theta - \frac{\pi}{2}) \equiv$ "always having complete set of nontrivial zeros" as alternative analogical explanation: Riemann hypothesis is uniquely denoted by $\theta = \frac{\pi}{4}$ with $\sin(\theta) = \cos(\theta) = \frac{1}{\sqrt{2}}$ whereby all (100%) nontrivial zeros are "conserved" despite success / failure of Gram's rule and Rosser's rule. Then Generalized Riemann hypothesis are non-uniquely denoted by $\theta \neq \frac{\pi}{4}$ with $\sin(\theta) \neq \cos(\theta) \neq \frac{1}{\sqrt{2}}$.]

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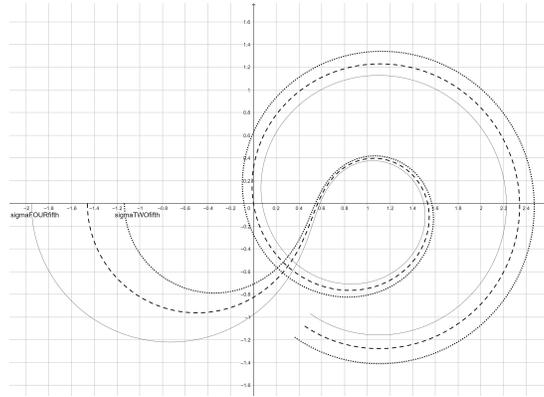


Figure 3. Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$. Horizontal axis: $Re\{\zeta(\sigma + it)\} / Re\{\eta(\sigma + it)\}$, and vertical axis: $Im\{\zeta(\sigma + it)\} / Im\{\eta(\sigma + it)\}$. Presence of Origin intercept points at [static] Origin point. Presence of virtual Origin intercept points as additional negative virtual Gram[y=0] points on x-axis (e.g. using $\sigma = \frac{2}{5}$ value) at [infinitely-many varying] virtual Origin points; viz, these negative virtual Gram[y=0] points on x-axis cannot exist at Origin point since two trajectories form co-linear lines (or co-lines) [viz, two parallel lines that never cross over near Origin point].

Proposition: Always having Origin point intercept $\Leftrightarrow \sin x = \cos(Ax - \frac{C\pi}{2})$ uniquely when $C = 1$.

Corollary: Never having Origin point intercept $\Leftrightarrow \sin x \neq \cos(Ax - \frac{C\pi}{2})$ non-uniquely when $C = 1 \pm \delta$.

Assigned values for A is "inconsequential" in the sense that the solitary $A = 1$ value \implies 'special case' Riemann hypothesis [on Genus 0 curve], and the multiple $A \neq 1$ values \implies Generalized Riemann hypothesis [on Genus 1, 2, 3, 4, 5... curves].

Remark A.3. Geometrical-Mathematical proof[6] for Riemann hypothesis is exemplified by Figure 1, Figure 2 and Figure 3. Let $\delta = \frac{1}{\infty}$ [an infinitesimal small number value] in reference to Figure 3. Then the plotted trajectories arising from inputting $\sigma = \frac{1}{2} + \delta$ and $\sigma = \frac{1}{2} - \delta$ into Riemann zeta function/Dirichlet eta function will always result in two co-linear lines being

1 located (approximately) an infinitesimal small δ distance, respectively, just to
2 right and left of Origin point [but never touching Origin point \equiv Critical line].

3 *Remark A.4. Proof by induction for Riemann hypothesis using*
4 *plotted co-linear lines [that conceptually comply with inclusion-exclusion*
5 *principle].* For $n = 0, 1, 2, 3, 4, 5, \dots, \infty$ in reference to $-\infty < t < +\infty$ when
6 inputting $\sigma = \frac{1}{2} + n\delta$ [\equiv "To Right of Origin Point"] and $\sigma = \frac{1}{2} - n\delta$ [\equiv "To Left
7 of Origin Point"] into Riemann zeta function/Dirichlet eta function, there are
8 infinitely-many [self-similar] plotted trajectories as co-linear lines using Polar
9 graph in, and to cover, entire $0 < \sigma < 1$ -Critical strip.

11 *Proving the Base case when $n = 1$:* At $n = 0$ [\equiv "On the Origin Point"]
12 in Figure 2 using either $\sigma = \frac{1}{2} + n\delta$ or $\sigma = \frac{1}{2} - n\delta$, this will always represent
13 the Polar graph at $\sigma = \frac{1}{2}$ -Critical line with having all (100%) nontrivial zeros,
14 thus implying Riemann hypothesis to be true. At $n = 1$ [\equiv "To Right of Origin
15 Point"] using $\sigma = \frac{1}{2} + n\delta$, this will always represent the Polar graph at $\sigma \neq \frac{1}{2}$ -
16 Non-critical line without having any (0%) nontrivial zeros. At $n = 1$ [\equiv "To
17 Left of Origin Point"] using $\sigma = \frac{1}{2} - n\delta$, this will always represent the Polar
18 graph at $\sigma \neq \frac{1}{2}$ -Non-critical line without having any (0%) nontrivial zeros.

22 *Induction step:* Suppose $\sigma = \frac{1}{2} + k\delta \equiv$ "To Right of Origin Point" or
23 $\sigma = \frac{1}{2} - k\delta \equiv$ "To Left of Origin Point" for some $k > 0$ [viz, $k = 1, 2, 3, 4,$
24 $5, \dots, \infty$]. Based on deviation property "Increasing distance away from Origin
25 Point as k becomes larger", we correctly claim both scenario are valid for next
26 case $k + 1$ that always represent Polar graph at $\sigma \neq \frac{1}{2}$ -Non-critical line without
27 having any (0%) nontrivial zeros. We now establish the truth of this statement
28 for all natural numbers $k \geq 1$, thus implying Riemann hypothesis to be true.

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