Revealing a Singularity in Collatz Sequences, Offering New Perspectives for Proving the Conjecture

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Abstract

The Collatz conjecture, first proposed by Lothar Collatz in 1937, has captivated generations of mathematicians due to its deceptive simplicity and its enduring resistance to proof. Also referred to as the 3n+1 problem, the Syracuse problem, the Ulam conjecture, or the Hailstone sequence, it has spread informally across academic communities, often through oral tradition and recreational mathematics. Its basic rule can be explained to a child, yet its resolution has defied the most brilliant minds in mathematics. As Shizuo Kakutani noted in 1960, "For about a month everyone at Yale worked on it, with no result... A joke was made that this problem was part of a conspiracy to slow down mathematical research in the U.S." Paul Erdős, in 1983, famously declared that "Mathematics is not yet ready for such questions." More recently, in 2010, Jeffrey Lagarias described it as "an extraordinarily difficult problem, completely out of reach of present day mathematics. The conjecture sits at the intersection of several mathematical fields, including number theory, dynamical systems, and the study of chaotic behavior. Despite vast numerical evidence and partial results, a general proof remains elusive. In this work, we propose a novel approach the Hidden Order method which reveals many patterns of the Collatz sequences and, more importantly, a singularity that will radically change the understanding of the Collatz dynamic.

1 Introduction

Understanding chaotic dynamical systems often begins with an effort to impose order — not to oversimplify, but to illuminate the hidden structures that chaos may conceal. The Collatz sequence, long considered erratic and unpredictable, is no exception. In this work, we construct a more organized and representative tree structure of the Collatz sequences. This restructured tree is not merely a visual aid; it is a strategic reformulation of the problem that allows underlying patterns to emerge more clearly. The act of ordering brings forth what appears to be a singularity; a critical and recurring structure within the dynamics of the Collatz process that has gone unnoticed in prior formulations. This singularity transforms our understanding of how the sequence behaves and opens entirely new pathways for proving the conjecture. Rather than relying on brute-force computation or stochastic models, this approach highlights deterministic structures encoded within the sequence itself. The Hidden Order method is not just a heuristic; it is a framework that reshapes the way we analyze, visualize, and interpret the Collatz conjecture.

2 Definition of the Collatz Conjecture

2.1 The Collatz Sequence

The Collatz sequence is a recursive sequence defined from a strictly positive natural number.

This sequence is constructed by applying the following rule:

- If the current term is odd, multiply it by **3** and add **1**.
- If the current term is even, divide it by **2**.

Thus, starting from an initial term, we obtain the next term by applying the rule according to the parity of the current term.

2.2 Mathematical Formulation of the Collatz Sequence

We first define the two auxiliary functions:

$$f_2 : 2\mathbb{N} \to \mathbb{N}$$
$$f_2(n) = \frac{n}{2} \quad \text{where } 2\mathbb{N} \text{ is the set of even natural numbers}$$

$$f_3 : \mathbb{N} \to \mathbb{N}$$
$$f_3(n) = 3n + 1$$

The Collatz sequence is then defined as follows:

$$C : \mathbb{N}^* \to \mathbb{N}$$
$$C_0 \in \mathbb{N}^*$$
$$C_{n+1} = \begin{cases} f_3(C_n) & \text{if } C_n \equiv 1 \pmod 2 \\ f_2(C_n) & \text{if } C_n \equiv 0 \pmod 2 \end{cases}$$

2.2.1 Statement of the Collatz Conjecture

The conjecture states that, regardless of the initial natural number $C_0 \in \mathbb{N}^*$, the Collatz sequence always reaches the value **1** after a finite number of iterations.

It can be formally expressed as:

$$\forall C_0 \in \mathbb{N}^*,\ \exists\, k \in \mathbb{N}^* : C_k = 1$$

To this day, the conjecture remains unproven.

3 Properties of the Collatz Sequence

3.1 Trajectory in the Collatz Sequence

A *trajectory* of a strictly positive natural number C_0, refers to the ordered sequence of terms obtained by successively applying the recurrence rule of the Collatz sequence, until the value **1** is reached for the first time, if it ever is. Thus, the trajectory includes the initial term C_0, all the intermediate terms, and stops as soon as **1** is reached for the first time.

3.2 The Trivial Cycle {1,4,2}

Let: $C_0 = 1$, then:

$$C_1 = f_3(C_0) = 3 \times 1 + 1 = 4$$
$$C_2 = f_2(C_1) = \frac{4}{2} = 2$$
$$C_3 = f_2(C_2) = \frac{2}{2} = 1 = C_0$$

Let: $C_0 = 2$, then:

$$\begin{aligned} C_1 &= f_2(C_0) = \frac{2}{2} = 1 \\ C_2 &= f_3(C_1) = 3 \times 1 + 1 = 4 \\ C_3 &= f_2(C_2) = \frac{4}{2} = 2 = C_0 \end{aligned}$$

Let: $C_0 = 4$, then:

$$\begin{aligned} C_1 &= f_2(C_0) = \frac{4}{2} = 2 \\ C_2 &= f_2(C_1) = \frac{2}{2} = 1 \\ C_3 &= f_3(C_2) = 3 \times 1 + 1 = 4 = C_0 \end{aligned}$$

Thus, there exists a cycle $\{1, 4, 2\}$, **called the trivial cycle**.
Since **1** belongs to the cycle, then the set $\{1, 2, 4\}$ satisfies the Collatz conjecture.

Moreover, **4** is the only element of the cycle that can be reached directly from a natural number outside the cycle:

$4 = f_2(8) = \frac{8}{2}$, $8 \notin \{1, 4, 2\}$.

The element 4 is called ***the gateway*** to the trivial cycle.

3.3 Theorem

The Collatz conjecture is valid if and only if the Collatz sequence converges to **4** for every non-zero natural number that does not belong to the trivial cycle, that is, for all $n \in \mathbb{N}^* \setminus \{1, 2, 4\}$. This can be expressed as:

$$\forall\, C_0 \in \mathbb{N}^*,\ \exists\, k \in \mathbb{N}^* \ :\ C_k = 1 \quad \Leftrightarrow \quad \forall\, C_0 \in \mathbb{N}^* \setminus \{1, 4, 2\}, \exists\, k \in \mathbb{N}^* \ :\ C_k = 4.$$

3.4 Property

It can be shown that every non-zero natural number converges to **4** before reaching **1**. To demonstrate this, we define the *inverse Collatz sequence* C^{-1}.

3.4.1 Inverse Functions

- $f_2^{-1} : \mathbb{N} \to \mathbb{N}$

$$f_2^{-1}(n) = 2n.$$

- $f_3^{-1} : \{n \in \mathbb{N}^* \mid n \equiv 1 \mod 3\} \to \mathbb{N}$

$$f_3^{-1}(n) = \frac{n-1}{3}$$

3.4.2 Inverse Collatz Sequence

$$C_0^{-1} \in \mathbb{N},$$
$$C_{n+1}^{-1} = \begin{cases} f_3^{-1}(C_n^{-1}) & \text{if } C_n^{-1} \equiv 1 \mod 3 \text{ and } C_n^{-1} \equiv 0 \mod 2 \\ f_2^{-1}(C_n^{-1}) \end{cases}$$

Proof:

Let $C_0 \in \mathbb{N}^* \setminus \{1, 2, 4\}$
If C_0 satisfies the Collatz conjecture, then:

$$\exists\, k \in \mathbb{N}^* : \quad C_k = 1 \quad \Rightarrow \quad C_{k-1} = f_2^{-1}(C_k) = 2 \times 1 = 2,$$
$$\Rightarrow \quad C_{k-2} = f_2^{-1}(C_{k-1}) = 2 \times 2 = 4.$$

Therefore:

$$\exists\, m \in \mathbb{N}^* : \quad C_m = 4 \quad \text{with} \quad m < k$$

3.4.3 Lemma

Every natural number $C_0 \in \mathbb{N}^* \setminus \{1, 4, 2\}$, if it converges to **1**, then it first converges to **4**.

Formal statement:

$$\forall\, C_0 \in \mathbb{N}^* \setminus \{1, 4, 2\}, \quad \exists\, k \in \mathbb{N}^* : C_k = 1 \quad \Rightarrow \quad \exists\, m = k - 2 \;:\; C_m = 4 \quad \text{with} \quad m < k$$

3.5 Property

The Collatz conjecture is false in the following cases:

1$^{\text{st}}$ Case : If there exists a natural number $n \in \mathbb{N}^*$ such that the sequence starting at **n** does not converge to **1**, that is, it either diverges to infinity or remains in a non-periodic behavior, never reaching **1**.

2nd Case : If there exists a non-trivial cycle in the Collatz sequence, that is, another cycle distinct from the trivial cycle $\{1, 4, 2\}$.

We now aim to show why the existence of a non-trivial cycle implies that the Collatz conjecture is false.

1st Case : Suppose the existence of a non-trivial cycle completely disjoint from the trivial cycle $\{1, 4, 2\}$ (Figure 1), This implies the existence of a finite set

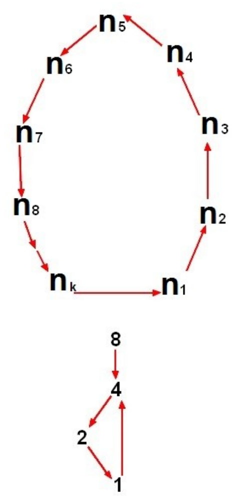

Figure 1: Disjoint Non-Trivial Cycle

$$\text{Let } \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}^* \quad \text{such that} \quad \{n_1, n_2, \ldots, n_k\} \cap \{1, 4, 2\} = \varnothing$$

and such that:

$$n_2 = \begin{cases} f_2(n_1) \\ f_3(n_1) \end{cases} \quad , \quad n_3 = \begin{cases} f_2(n_2) \\ f_3(n_2) \end{cases} \quad , \quad \ldots , \quad n_1 = \begin{cases} f_2(n_k) \\ f_3(n_k) \end{cases}$$

Then, the Collatz sequence will never reach **1**, because it will be trapped in the cycle $\{n_1, n_2, \ldots, n_k\}$, which contains neither **1** nor any of the elements from the trivial cycle $\{1, 4, 2\}$.

2nd Case : Let us assume the existence of a non-trivial cycle that does contain elements from the trivial cycle $\{1, 4, 2\}$. This type of cycle is not completely disjoint from the trivial cycle (Figure2).

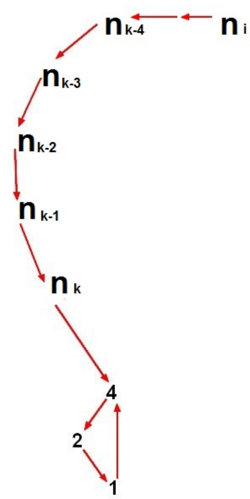

Figure 2: Non-Trivial Cycle with Junction

It has already been shown that **4** is the only gateway out of the trivial cycle $\{1, 4, 2\}$, Therefore, any possible junction must occur at the level of element **4**.

Let $\{4, n_1, n_2, \ldots, n_k\}$ be a non-trivial cycle which connects to the trivial cycle $\{1, 4, 2\}$ at the level of **4**.

We set $4 = C_0$, hence :

$$C_1 = f_2(C_0) = \frac{C_0}{2} = \frac{4}{2} = 2 \Rightarrow C_2 = f_2(C_1) = \frac{C_1}{2} = \frac{2}{2} = 1$$
$$\Rightarrow C_3 = f_3(C_2) = 3 \times C_2 + 1 = 3 \times 1 + 1 = 4$$

$$\Rightarrow C_4 = f_2(C_3) = \frac{C_3}{2} = \frac{4}{2} = 2 = C_1$$

So, starting from an element C_i of a supposed non-trivial cycle, we always reach the element $C_0 = 4$, and then we enter the trivial cycle with no possibility of exiting. Therefore, we can no longer return to the starting point C_i.

This means that it is impossible to have a non-trivial cycle that joins the trivial cycle.

3.6 Property

- $\forall\ C_0 \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have :

$$C_k \equiv 1 \pmod 2 \quad \Rightarrow \quad C_{k+1} \equiv 0 \pmod 2$$

That is to say, every odd term is followed by an even term.

- $\forall\ C_0^{-1} \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have:

$$C_k^{-1} \equiv 1 \pmod 2 \quad \Rightarrow \quad C_{k+1}^{-1} \equiv 0 \pmod 2$$

  That is to say, every odd term in the inverse Collatz sequence is followed by an even term.

- $\forall\ C_0^{-1} \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have:

$$C_k^{-1} \equiv 0 \pmod 2 \quad \Rightarrow \quad \begin{cases} C_{k+1}^{-1} = f_2^{-1}(C_k^{-1}) \equiv 0 \pmod 2 \\ C_{k+1}^{-1} = f_3^{-1}(C_k^{-1}) \equiv 1 \pmod 2 \end{cases}$$

  That is to say, every even term in the inverse Collatz sequence is followed by an even or odd term.

- $\forall\ C_0 \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have:

$$C_k \equiv 0 \pmod 2 \quad \Rightarrow \quad \begin{cases} C_{k+1} = f_2(C_k) \equiv 0 \pmod 2 \\ C_{k+1} = f_2(C_k) \equiv 1 \pmod 2 \end{cases}$$

  Every even term in the Collatz sequence is followed by either an even or an odd term.

3.6.1 Remark:

The validity of the Collatz conjecture for C_0, C_k, C_{k+1}, C_0^{-1}, C_k^{-1}, C_{k+1}^{-1} is not a condition for the property to hold.

4 Construction of a More Representative Collatz Tree

4.1 Division of $\mathbb{N}^*$ into Subsets

According to the fundamental theorem of arithmetic, every strictly positive natural number can be written as a product of prime numbers in a unique way.

- If the factorization of the natural number $n \in \mathbb{N}^*$ gives factors that are all equal to **2**, then n belongs to the subset of natural numbers that are powers of **2** :

$$\mathcal{A} = \left\{ n \in \mathbb{N}^* \,\middle|\, \exists\ k \in \mathbb{N},\ n = 2^k \right\}$$

  Here, **1** is included in $\mathcal{A}$, because $1 = 2^0$.

- If the factorization of $n \in \mathbb{N}^*$, $n > 1$, gives at least one prime factor that is not equal to **2**, and if $n \equiv 1 \pmod 2$, then n belongs to the subset:

$$\mathcal{B} = \{ n \in \mathbb{N}^* \,|\, n > 1,\ n \equiv 1 \pmod 2 \} \quad \text{with} \quad \mathcal{A} \cap \mathcal{B} = \emptyset$$

- If the factorization of $n \in \mathbb{N}^*$ gives at least one factor (but not all) different from **2**, then **n** belongs to the subset:

$$\mathcal{M} = \left\{ n \in \mathbb{N}^* \,\middle|\, n = p \cdot 2^k,\ p > 1,\ p \equiv 1 \pmod 2,\ k \geq 1 \right\}$$

$$\text{with} \quad \mathcal{M} \cap \mathcal{A} = \mathcal{M} \cap \mathcal{B} = \emptyset$$

  Thus, we have the following partition:

$$\mathbb{N}^* = \mathcal{A} \cup \mathcal{B} \cup \mathcal{M} \quad \text{and} \quad \mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{M} = \mathcal{B} \cap \mathcal{M} = \emptyset$$

4.2 Description of the Collatz Tree to Be Constructed

The Collatz tree we are about to construct represents the strictly positive natural numbers that satisfy the Collatz conjecture. The representation of these numbers will be based on the **rank** of the integer.

The **rank** of a natural number **n**, denoted rank(n), is defined as the number of iterations required by the inverse function f_3^{-1} to reach a value equal to **n**, using the inverse Collatz sequence C^{-1}, and starting from $C_0^{-1} = 1$ as the initial term.

To determine the rank of a natural number and its representation in the Collatz tree, it is not possible to deduce it directly from **n** using C^{-1}, Instead, it is necessary to:

- Verify whether **n** satisfies the Collatz conjecture (otherwise, it cannot be represented in the tree);

- Determine the rank of **n**, rank(n), as well as the number of successive f_2 iterations required to go from one odd number to the next odd number.

According to the Collatz sequence, for each term, there is only one possible next term. This implies that every natural number has a unique trajectory in the Collatz sequence **C**. Therefore, every natural number that satisfies the Collatz conjecture can only have one unique representation in the Collatz tree.

4.2.1 Properties of the Elements in Subset $\mathcal{A}$ (Powers of 2)

Figure 3: Trunk of the Collatz Tree

$$\forall\ C_0 \in \mathcal{A}:\ C_0 > 1 \quad \text{then} \quad C_0 = 2^k : \quad k \in \mathbb{N}^*$$
$$\Rightarrow \quad C_1 = f_2(C_0) = \frac{C_0}{2} = \frac{2^k}{2} = 2^{k-1}$$
$$\Rightarrow \quad C_k = \frac{2^k}{2^k} = 1$$

Therefore, for any $C_0 \in \mathcal{A} : C_0 > 1$, C_0 satisfies the Collatz conjecture.

For $C_0 = 1$, we have previously shown that **1** satisfies the Collatz conjecture.

Thus:

$$\forall\ C_0 \in \mathcal{A}, \quad C_0 \text{ satisfies the Collatz conjecture.}$$

The elements of subset $\mathcal{A}$ will be represented by a vertical line, which we will call the ***trunk*** of the Collatz tree (Figure 3).

4.2.2 Remark:

The elements of subset $\mathcal{A}$ have rank zero.

Let $n \in \mathcal{M}$

$$\text{We have: } \mathcal{M} = \left\{ n \in \mathbb{N}^* \,\middle|\, n = p \cdot 2^k,\ p > 1,\ p \equiv 1 \pmod 2,\ k \geq 1 \right\}$$

Thus,

$$f_2^k(n) = f_2^k(p \cdot 2^k) = p$$

where f_2 is the division by 2 function.

$p \in \mathcal{B}$, so any natural number $n \in \mathcal{M}$ will be transformed into an element of $\mathcal{B}$ in the Collatz sequence by applying f_2 k times.

Therefore, if all the elements of $\mathcal{B}$ satisfy the Collatz conjecture, then all the elements of $\mathcal{M}$ also satisfy the conjecture.

This is expressed as:

$$\forall\, n \in \mathcal{B},\ \exists\, k \in \mathbb{N}^*,\ C_k = 1 \quad \Rightarrow \quad \forall\, m \in \mathcal{M},\ \exists\, j \in \mathbb{N}^*,\ C_j = 1$$

4.2.3 Conclusion

We have shown that:

- All elements of the subset $\mathcal{A}$ satisfy the Collatz conjecture.
- If all elements of the subset $\mathcal{B}$ satisfy the conjecture, then all elements of the subset $\mathcal{M}$ also satisfy the conjecture.

And since:

$$\mathcal{A} \cup \mathcal{B} \cup \mathcal{M} = \mathbb{N}^*$$

we derive the following lemma:

4.2.4 Lemma

The validity of the Collatz conjecture is equivalent to its validity over the subset $\mathcal{B}$ of odd natural numbers strictly greater than **1**.

4.2.5 Lemma

We have; then subset $\mathcal{B}$ is included in the set of non-zero natural numbers and does not contain any elements from the trivial cycle $\{1, 4, 2\}$.

Then, according to **lemma 3.4.3**, for every odd initial term C_0 belonging to $\mathcal{B}$, the Collatz sequence always reaches a term $C_k = 4$ before reaching **1**. Therefore, for every $C_0 \in \mathcal{B}$, there exists a term:

$$C_j = 4 \cdot 2^p = 2^{2+p} \quad \text{with} \quad p \in \mathbb{N},$$

and then the term:

$$C_k = 4 = f_2^p(C_j),$$

after p iterations via the function f_2.

4.2.6 Representation of Elements from $\mathcal{B}$ and $\mathcal{M}$

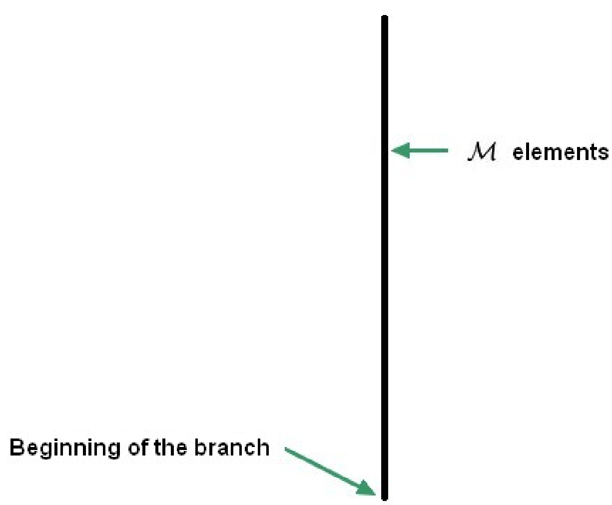

Figure 4: Diagram of a Branch

Each element $n \in \mathcal{B}$ that satisfies the Collatz conjecture will be represented by a distinct point on the Collatz tree.

All elements $m \in \mathcal{M}$ Reducible to an $n \in \mathcal{B}$ which means $m = n{\cdot}2^k$ with $n > 1,\ n \equiv 1 \pmod 2,\ k \geq 1$, will be represented on a half-line starting from it's original point in **n**.

This half-line is called a ***Branch*** (Figure 4) and the origin point **n** is called the **Beginning of a branch**.
A branch can be:

- A **mother branch** (emerging directly from the trunk or the root);
- Or a **daughter branch** (emerging from a mother branch or from the trunk).

4.2.7 Exception

For the number **1**, it will be represented on the trunk, even if we can consider that rang(1) = 1.

4.2.8 Branch Generator

A **branch generator** (Figure 5) is an even natural number that belongs to a mother branch or the trunk, which means, it is an element of $\mathcal{A}$ or $\mathcal{M}$.

A non-zero natural number **g** is a branch generator if and only if $f_3^{-1}(g) = n$, with n being a branch beginning.

$$f_3^{-1}(g) = n \quad \Rightarrow \quad n = \frac{g-1}{3} \quad \Rightarrow \quad g \equiv 1 \pmod 3 \quad \text{and} \quad g \equiv 0 \pmod 2$$

Therefore, **g** is a branch generator if and only if:

$$\exists\ n \in \mathcal{B}, g = 3n + 1$$

We represent the connection between a branch and its mother branch (or the trunk) by an inclined **line segment**, running from the top (the branch generator) down to the bottom (the branch's beginning).

4.2.9 Remark:

The validity of the Collatz conjecture for the beginning of the mother branch is not a condition for defining the generators of daughter branches.

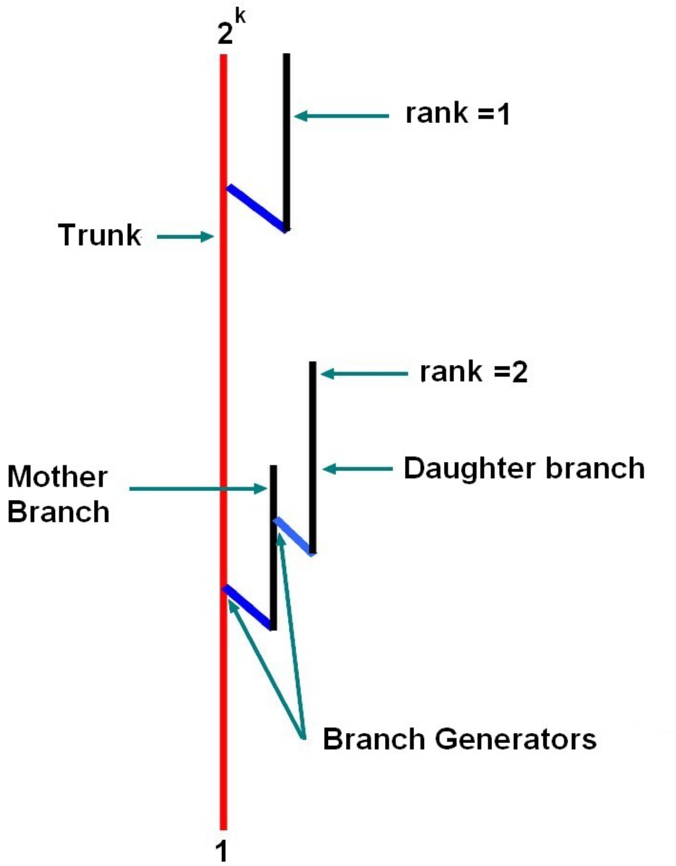

Figure 5: Branch Generator

4.2.10 Sister Branches

Two branches are **sister branches** (Figure 6) if and only if they share the same mother branch or they are daughter branches of the trunk.

4.2.11 Remark:

The validity of the Collatz conjecture for the beginning of the mother branch **is not** a condition for defining sister branches.

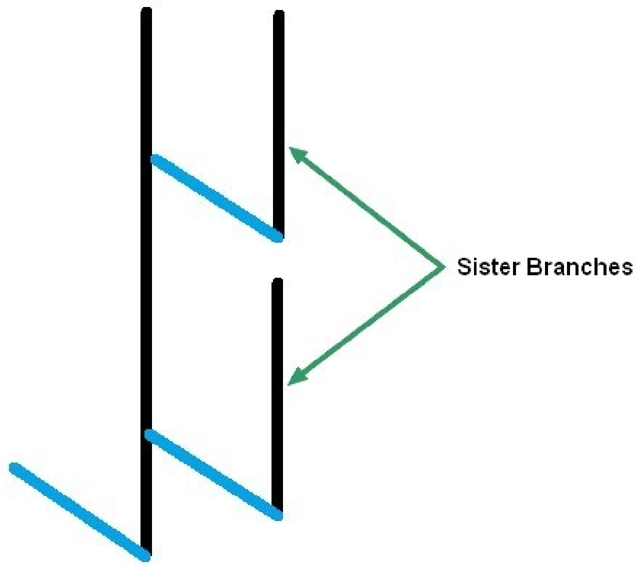

Figure 6: sister branches

5 Recognition of Structural Patterns in the Dynamics of Collatz

5.1 The Form of a Branch Generator

Let $g \in \mathbb{N}^*$

g is a branch generator if and only if:

$$\exists\ k \in \mathbb{N}^* \quad \text{such that} \quad k \equiv 1 \pmod{2} \quad \text{and} \quad g = 3k + 1$$

$$k > 0 \Rightarrow g \geq 4$$
$$\text{and } k \equiv 1 \pmod{2} \Rightarrow g \equiv 0 \pmod{2}$$

5.1.1 Lemma

A natural number **g** is a branch generator if and only if:

- $g \geq 4$,

- $g \equiv 0 \pmod 2$,
- $g - 1 \equiv 0 \pmod 3$.

5.1.2 Remark:

The validity of the Collatz conjecture for **g** is not a condition.

5.2 The First Branch Generator

Now we ask the question: how many times must we multiply by **2** a branch beginning or the beginning of the trunk to obtain the first branch generator?

Let $n \in 2\mathbb{N}+1$ be the beginning of a branch or the trunk.

$$\text{Then:} \quad \exists\, k \in \mathbb{N},\ n = \begin{cases} 3k, & k \text{ is odd} \\ 3k+1, & k \text{ is even} \\ 3k+2, & k \text{ is odd} \end{cases}$$

$2\mathbb{N}+1$ is the set of odd natural numbers.

1^{st} Case : $n = 3k$, k odd

Let $p \in \mathbb{N}^*$ such that $2^p \cdot n$ is the first branch generator. So $2^p \cdot n = (3k) \cdot 2^p$ with k odd.

Then the beginning of the branch that is generated from $(3k) \cdot 2^p$ is:

$$d = f_3^{-1}(3k \cdot 2^p) = \frac{(3k \cdot 2^p) - 1}{3} = k \cdot 2^p - \frac{1}{3}$$

d cannot be a natural number. Thus, a branch beginning with the form $3k$ where k is odd cannot generate a branch generator.

We call a ***dead branch*** (Figure 7) any branch that has a beginning of the form $3k$ where **k** is odd.

2^{nd} Case : $n = 3k+1$, k even

If we multiply **n** once by **2**, we obtain $2n$.

So:

$$d = f_3^{-1}(2n) = \frac{2n-1}{3} = \frac{2(3k+1)-1}{3} = 2 \cdot k + \frac{1}{3}$$

d cannot be a natural number.

If we multiply **n** twice by **2**, we get $n \cdot 2^2$.

Therefore:

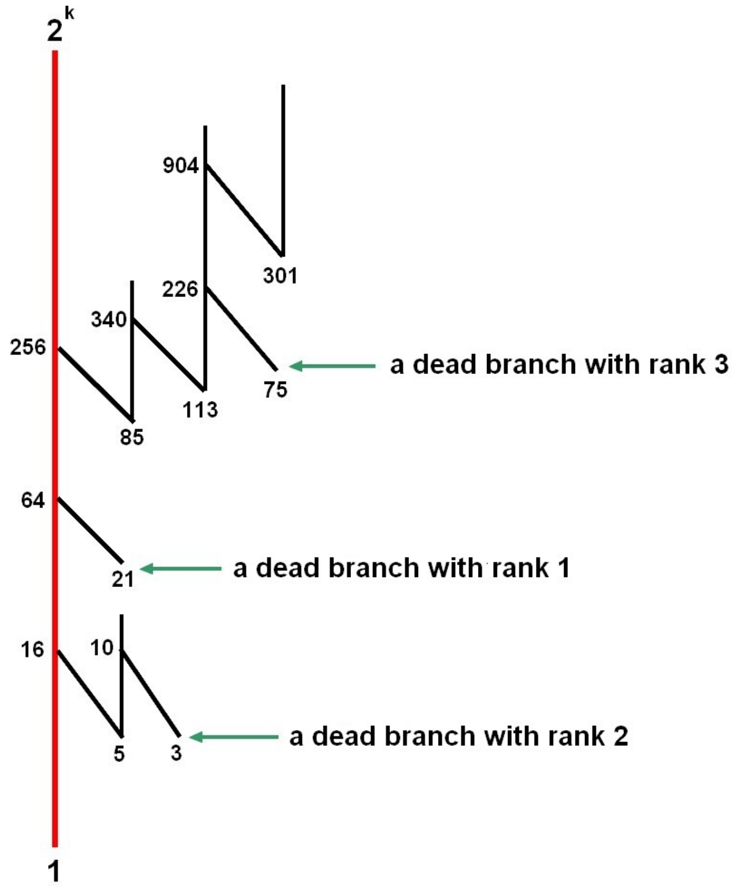

Figure 7: Dead Branches

$$d = f_3^{-1}(2^2 n) = \frac{n \cdot 2^2 - 1}{3} = \frac{(3k+1) \cdot 2^2 - 1}{3} = \frac{12k+3}{3} = 4k+1$$

d is odd, so $n \cdot 2^2$ is the first branch generator.

3^rd Case : $n = 3k+2$**,** k **odd**

If we multiply **n** once by **2**, we obtain $2n$.

So:

$$d = f_3^{-1}(2n) = \frac{2n-1}{3} = \frac{2(3k+2)-1}{3} = \frac{6k+3}{3} = 2k+1$$

So the first branch generator is $n \cdot 2$.

5.2.1 Lemma

For any beginning **n** of a branch or of the trunk:

- If **n** is of the form $3k$ with k odd, then **n** is the beginning of a dead branch that has no daughter branch generator.
- If **n** is of the form $3k+1$ with k even, then then the first daughter branch generator is $n \cdot 2^2$.
- If **n** is of the form $3k + 2$ with k odd, then the first daughter branch generator is $n \cdot 2^1$.

5.2.2 Remark:

The validity of the Collatz conjecture for **n** is not a condition.

5.3 The smallest successor of a branch generator

Let **g** be a branch generator. Then $g \geq 4$, $g \equiv 0 \pmod 2$, $g \equiv 1 \pmod 3$.

Let $d = f_3^{-1}(g)$ be the beginning of the branch generated by g with $g = 3d + 1$ and d odd.

- If we multiply **g** once by **2** we get:

  $$2g = 2(3d + 1) = 6d + 2$$

  Let **d'** be the beginning of the branch generated by $2g$.

  $$d' = \frac{2g-1}{3} = \frac{2(3d+1)-1}{3} = \frac{6d+1}{3} = \ 2d - \frac{1}{3}$$

  which is not a natural number.

- If we multiply **g** twice by **2**, we get:

  $$g \cdot 2^2 = 4(3d + 1) = 12d + 4$$

  Let **d'** be the beginning of the branch generated by $g2^2$.

  $$d' = \frac{4g-1}{3} = \frac{4(3d+1)-1}{3} = \frac{12d+3}{3} = 4d + 1$$

  with d odd.

Therefore, the smallest successor of a branch generator **g** is $g \cdot 2^2$.

5.3.1 Lemma

The smallest successor of a branch generator $\boldsymbol{g}$ *is* $g \cdot 2^2$.

5.3.2 Remark:

The validity of the Collatz conjecture for **g** is not a condition.

5.4 Expression of the beginning of the sister branch that follows a branch

Let d_2 be the beginning of the sister branch that follows the sister branch whose beginning is d_1, g_1 is the generator of d_1 and g_2 the generator of d_2 (Figure 8).

We will express d_2 as a function of d_1.

We have:

$$d_2 = \frac{g_2 - 1}{3} \quad \text{and} \quad g_2 = 2^2 \cdot g_1 \quad (\textbf{from Lemma 5.3.1})$$

We also have:

$$g_1 = 3d_1 + 1$$

Therefore:

$$g_2 = (3d_1+1)\cdot 2^2 \quad \Rightarrow \quad d_2 = \frac{g_2 - 1}{3} = \frac{(3d_1 + 1) \cdot 2^2 - 1}{3} = \frac{12d_1 + 3}{3} = 4d_1+1$$

5.4.1 Remark:

The validity of the Collatz conjecture for g_1, g_2, d_1, d_2 is not a condition.

5.4.2 Lemma

The beginning of the sister branch that follows a branch whose beginning is d_1 *is* $4d_1 + 1$.

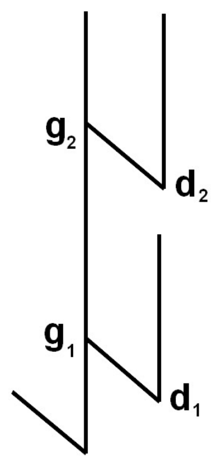

Figure 8: Diagram of sister branches

5.5 The ternary form of the beginnings of branches and generated branch beginnings

Let $d_1 = 3k + 1$ with k even, be the beginning of a mother branch. Then, the first generator of the daughter branch is:

$$g = 2^2 \cdot d_1 = 4(3k + 1)$$

and the beginning of the daughter branch is:

$$d_2 = \frac{g-1}{3} = \frac{4(3k+1)-1}{3} = \frac{12k+3}{3} = 4k + 1 \quad \text{with} \quad k \text{ even.}$$

Exemples

$d_1 = 3k + 1$	$g = d_1 \cdot 2^2$	$d_2 = 4k + 1$
7	28	$9 \equiv 0 \pmod 3$
13	52	$17 \equiv 2 \pmod 3$
19	76	$25 \equiv 1 \pmod 3$

Let $d_1 = 3k + 2$, k odd, be the beginning of a mother branch. This implies that the first generator of the daughter branch is, according to **lemma 5.2.1**:

$$g = 2^1 \cdot d_1 = 2(3k + 2)$$

and the beginning of the daughter branch is :

$$d_2 = \frac{g - 1}{3} = \frac{2(3k + 2) - 1}{3} = \frac{6k + 3}{3} = 2k + 1$$

Example

$d_1 = 3k + 2$	$g = 2^1 \cdot d_1$	$d_2 = 2k + 1$
5	10	$3 \equiv 0 \pmod 3$
11	22	$7 \equiv 1 \pmod 3$
17	34	$11 \equiv 2 \pmod 3$

Let d_1 be the beginning of a mother branch, not necessarily the first daughter branch.

1^{st} case : $d_1 = 3k + 1$, k even

What will be the form of d_2, the beginning of the sister branch that follows the branch having d_1 as its beginning?

From **lemma 5.4.2** we have:

$$d_2 = 4d_1 + 1 = 4(3k + 1) + 1 = 12k + 5 = 3(4k + 1) + 2$$

We observe that $4k + 1$ is odd.

Let $4k + 1 = k'$, then:

$$d_2 = 3k' + 2$$

$\mathbf{2^{nd}}$: $d_1 = 3k + 2$, k odd

According to **lemma 5.4.2** :

$$d_2 = 4d_1 + 1 = 4(3k + 2) + 1 = 12k + 9 = 3(4k + 3)$$

We observe that $4k + 3$ is odd. Let $4k + 3 = k'$, then:

$$d_2 = 3k'$$

with k' odd.

$\mathbf{3^{rd}}$: $d_1 = 3k$, k odd

According to **lemma 5.4.2** :

$$d_2 = 4d_1 + 1 = 4(3k) + 1 = 12k + 1 = 3(4k + 1) + 1$$

We observe that $4k + 1$ is odd. so let $4k + 1 = k'$, then:

$$d_2 = 3k' + 1$$

5.5.1 Lemma

Let $d_1 = 3k + a$ with $a \in \{0, 1, 2\}$ be the beginning of a branch, Then the beginning of the sister branch that succeeds it is of the form $3k + b$ such that:

$$a + 1 \equiv b \pmod 3$$

5.5.2 Remark:

The validity of the Collatz conjecture for d_1 is not a condition.

5.6 Consequence of lemma 5.5.1

Whatever the rank **r**>0, may be, there is always a branch of rank **r**, that is not a dead branch and can therefore generate daughter branches. Thus, one can always have branches of rank **r+1** regardless of the rank **r** of a branch.

5.6.1 Theorem

$$\forall\, r \in \mathbb{N}, \quad \exists\, d \in \mathcal{B} \cup \mathcal{A} \quad \text{such that} \quad \text{rang}(d) = r$$

where d is the beginning of a branch or of the trunk that satisfies the Collatz conjecture.

$\mathcal{B} \cup \mathcal{A}$: the union of the two subsets of $\mathbb{N}^*$.

5.7 Second Consequence of Lemma 5.5.1

Whatever the beginning of a branch or the trunk, one can always generate a daughter branch whose beginning is a multiple of **3**. That is to say, any beginning of a branch or of the trunk belonging to the trajectory of an initial value which is an odd multiple of **3**. Therefore, the assertion that all elements of subset $\mathcal{B}$ which are multiples of **3** satisfy the Collatz conjecture is equivalent to the assertion that all elements of $\mathcal{B}$ satisfy the Collatz conjecture. Thus, according to **lemmas 5.5.1** and **4.2.4**, we have the following lemma:

5.7.1 Lemma

The validity of the Collatz conjecture is equivalent to its validity for every $n \in \mathcal{B}$ such that $n \equiv 0 \pmod 3$.

$\mathcal{B}$ is the subset of $\mathbb{N}^*$ consisting of strictly greater-than **1** odd natural numbers.

5.8 The formula for branch generators on the trunk

According to **lemma 5.1.1** : A natural number **g** is a branch generator if and only if:

$$g \geq 4, \quad g \equiv 0 \pmod 2, \quad g \equiv 1 \pmod 3$$

Therefore, the first number to test is **4**.

We have : $4 \geq 4$, $4 \equiv 0 \pmod 2$ et $4 \equiv 1 \pmod 3$.

Thus **4** is the first branch generator on the trunk.

According to **lemma 5.3.1**, the branch generator that succeeds the trunk branch generator $g_1 = 4$ is:

$$g_2 = g_1 \cdot 2^2 = 4 \cdot 2^2 = 2^2 \cdot 2^2 = 2^{2(1+1)}$$

The third branch generator on the trunk is:

$$g_3 = g_2 \cdot 2^2 = 2^{2(1+1)} \cdot 2^2 = 2^{2(1+1+1)}$$

So, we deduce that the **n**-th branch generator on the trunk is 2^{2n}, but we must exclude the first branch generator **4**, because if we generate the beginning of the branch $\frac{4-1}{3} = 1$, there would be a trunk that regenerates itself again as if it is rank **1**.

Thus, the **n**-th branch generator on the trunk is:

$$g_n = 2^{2(n+1)}$$

5.8.1 Lemma

The n-th branch generator on the trunk is $2^{2(n+1)}$, with $n \in \mathbb{N}^*$.

5.9 The form of first-rank branch beginnings

Let d_n be the **n**-th beginning of a rank **1**, branch, which is generated by the **n**-th branch generator on the trunk g_n such that:

$$g_n = 2^{2(n+1)}$$
$$d_n = \frac{g_n - 1}{3}$$

We start with $n = 1$, so the first beginning of a rank **1** branch is:

$$d_1 = \frac{g_1 - 1}{3} = \frac{2^{2(1+1)} - 1}{3} = \frac{2^4 - 1}{3} = \frac{16 - 1}{3}$$

Expanding in binary:

$$g_n - 1 = (16 - 1) = (10000)_2 - (1)_2 = (1111)_2$$

Thus:

$$(1111)_2 = (101)_2 + (1010)_2$$
$$= (101)_2 + (101)_2 \cdot (10)_2$$
$$= (101)_2 \cdot (11)_2$$

Hence:

$$d_1 = \frac{16-1}{3} = \frac{(101)_2 \cdot (11)_2}{(11)_2} = (101)_2$$

So:

$$d_1 = 2^0 + 2^2 = \sum_{i=0}^{1} 2^{2i}$$

We propose the following formula to be proven by induction, which gives the **n**-th beginning of a rank **1** branch::

$$d_n = \sum_{i=0}^{n} 2^{2i} \quad \text{to be proven.}$$

We now formulate the recurrence property $P(n)$:

$$P(n) \Leftrightarrow d_n = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}^*$$

Initialization

For the initial index $n_0 = 1$, we have:

$$d_1 = \sum_{i=0}^{1} 2^{2i}$$

This is true because:

$$d_1 = 5 = 1 + 4 = 2^0 + 2^2 = \sum_{i=0}^{1} 2^{2i}$$

So $P(1)$ is true.

Induction

We suppose that $P(k)$ is true for some $k \in \mathbb{N}^*$, that is:

$$d_k = \sum_{i=0}^{k} 2^{2i}.$$

We will now show that this implies $P(k+1)$ is also true, that is:

$$d_{k+1} = \sum_{i=0}^{k+1} 2^{2i}.$$

According to **lemma 5.4.2** we have : $d_{k+1} = 4d_k + 1$

So:

$$d_{k+1} = 4 \cdot \left(\sum_{i=0}^{k} 2^{2i} \right) + 1 = 2^2 \cdot \left(\sum_{i=0}^{k} 2^{2i} \right) + 2^0$$
$$= \left(\sum_{i=0}^{k} 2^{2i+2} \right) + 2^0 = \left(\sum_{i=0}^{k} 2^{2(i+1)} \right) + 2^0$$

We set j=i+1, then we have:

$$d_{k+1} = \left(\sum_{j=1}^{k+1} 2^{2j} \right) + 2^{2\times 0} = \left(\sum_{j=0}^{k+1} 2^{2j} \right)$$

Thus: $P(k+1)$ is true.

Conclusion

By the principle of mathematical induction, we have thus proven that for all $n \in \mathbb{N}^*$:

$$d_n = \sum_{i=0}^{n} 2^{2i}$$

5.9.1 Theorem

The **n**-th beginning of a rank **1** branch is given by:

$$d_n = \sum_{i=0}^{n} 2^{2i} : \quad n \in \mathbb{N}^*.$$

5.10 Singularity of the Collatz sequence

According to **lemma 4.2.5** and the **lemma 5.8.1**, any natural number $C_0 \in \mathcal{B}$ chosen as the first term of the Collatz sequence **C**, and which satisfies the Collatz conjecture, will always yield a term $C_k = 2^{2(p+1)}$, $p \in \mathbb{N}^*$, before reaching the term $C_i = 4$, and then the term $C_j = 1$.

So all the elements of $\mathcal{B}$ that satisfy the Collatz conjecture will be transformed by the Collatz sequence into a branch generator located on the trunk:

$$C_k = 2^{2(p+1)}, \quad p \in \mathbb{N}^*$$

But before reaching the term C_k, we will always have the term:

$$C_{k-1} = \sum_{j=0}^{p} 2^{2j} \quad \text{(beginning of a rank } \mathbf{1} \text{ branch).}$$

Having the first term C_0 or reaching a term that is the beginning of a rank **1** branch is a **singularity of the Collatz sequence**, because:

- Only the branches beginnings of rank **1** are directly transformed into a power of **2**, $2^{2(p+1)} \in \mathcal{A}$, by a single iteration via function f_3, whereas all other branch beginnings are transformed through iterations via f_3 into an element of a subset $\mathcal{M}$.

5.10.1 Lemma

All branch beginnings $C_0 \in \mathcal{B}$ that satisfy the Collatz conjecture, if they are not of rank 1, will be transformed into the beginning of a rank **1** branch by the Collatz sequence. And all beginnings of rank 1 branches will be transformed, via f_3, into a power of 2 (branch generator on the trunk), and the only one that allows reaching power of 2 via a single iteration of f_3. having the first term or reaching a term that is the beginning of a rank 1 branch, is a **singularity of the Collatz sequence**.

5.11 Consequence of Lemmas 5.10.1 and 5.7.1

According to **lemma 5.7.1**, the Collatz conjecture is valid if and only if it is validated for every $C_0 \in \mathcal{B}$ such that $C_0 \equiv 0 \pmod 3$.

And according to *lemma 5.10.1*, for every $C_0 \in \mathcal{B}$, if C_0 satisfies the Collatz conjecture, then:

$$\exists\ k \in \mathbb{N},\ C_k = \sum_{i=0}^{n} 2^{2i}$$

That is C_k is a beginning of a rank 1 branch.

5.11.1 Theorem

The Collatz conjecture is true **if and only if**:

$$\forall C_0 \in \mathcal{B}: \quad C_0 \equiv 0 \pmod 3$$

then either $C_0 = \sum_{i=0}^{n} 2^{2i}$, with $n \in \mathbb{N}^*$, meaning that C_0 is directly transformed by the Collatz sequence into a term that is:

$$\sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}^*$$

that is, transformed into a beginning of a rank 1 branch.

5.11.2 The Significance of the Binary Form of Rank 1 Branch Beginnings

We have that any beginning of a rank 1 branch can be written in the form:

$$d = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}^*$$

where d is the sum of powers of 2, 2^{2i} with even exponents.

We know that every natural number n can be written in binary as a sequence of bits:

$$n = (b_k, b_{k-1}, \ldots, b_1, b_0)_2$$

where each bit $b_i \in \{0, 1\}$. and b_0 is the least significant bit (furthest to the right).

We can write:

$$n = b_k 2^k + b_{k-1} 2^{k-1} + \ldots + b_1 2^1 + b_0 2^0$$

with $b_k = 1$, Thus, for d, we have:

$$d = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}^* \implies d = (1_{2n}, 0, 1_{2(n-1)}, 0, \ldots 1_{2\times 2}, 0, 1_{2\times 1}, 0, b_{2\times 0})_2$$

And therefore, in the binary representation of the beginnings of rank 1, branches, all the bits with an **even index** are equal to **1**, and all the bits with an **odd index** are equal to **0**.

5.11.3 Theorem

A natural number $d \in \mathcal{B}$ is a beginning of a rank 1, branch, that is:

$$d = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}^*$$

if and only if in its binary representation:

- all bits with an even index are equal to **1** and
- and all bits with an odd index are equal to **0**.

6 Conclusion

The Hidden Order approach introduced in this work offers a fresh structural perspective on the Collatz problem. By reordering the landscape through a refined Collatz tree, we have uncovered a singularity — a recurring pattern deeply embedded in the dynamics — that challenges the prevailing view of the sequence as chaotic and patternless. This discovery not only reframs our understanding of Collatz sequences but also opens new methodological directions for tackling the conjecture. Currently, i am exploring several additional structural patterns, aiming to further expose the underlying deterministic framework. These patterns may provide the necessary scaffolding to move beyond empirical observation and toward formal proof. The results so far are promising, and I believe that a clearer understanding of these hidden regularities will play a key role in resolving this long-standing mathematical enigma.