

Proof of Legendre's Conjecture

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Abstract

A proof of Legendre's conjecture is obtained by establishing the following regularity: In any interval between the squares of two consecutive positive integers, the number of odd integers of the form $6a \mp 1$ strictly exceeds the number of integers of the same form that can be represented as

$$6b \mp 1 = (6m \mp 1)(6(m + x) \mp 1).$$

Keywords: prime numbers, Legendre's conjecture, squares, intervals, arithmetic progression.

1 Introduction

Legendre's conjecture was formulated by Adrien-Marie Legendre in 1808. It later became known as Landau's third problem, since at the International Congress of Mathematicians in 1912, Edmund Landau presented four fundamental open problems concerning prime numbers, including Legendre's conjecture. These problems were characterized in his address as "insurmountable in the present state of mathematics" [1, 2].

The conjecture states:

Legendre's Conjecture. *For every natural number n , there exists at least one prime number between n^2 and $(n + 1)^2$.*

2 Main Result

Theorem 1. *Let*

$$K = \{k \in \mathbb{Z}_{>0} \mid k = 6a \mp 1, n^2 < k < (n+1)^2\},$$

$$Q = \{q \in \mathbb{Z}_{>0} \mid q = 6b \mp 1 = (6m \mp 1)(6(m+x) \mp 1), n^2 < q < (n+1)^2\},$$

where $a, b, m, n \in \mathbb{Z}_{>0}$, $x \in \mathbb{Z}_{\geq 0}$, and $n \geq 4$. Then, for every $n \geq 4$, the inequality

$$|K| > |Q|$$

holds.

Verbal formulation

Among all the numbers of the form $6a \mp 1$ that lie between consecutive squares n^2 and $(n+1)^2$, the number of such integers always strictly exceeds the number of integers of the same form that can be represented simultaneously as products of the form $(6m \mp 1)(6(m+x) \mp 1)$.

Equivalently, in every interval between consecutive squares, the number of integers of the form $6a \mp 1$ strictly exceeds the number of integers of the same form with the specified special factorization.

Note. The proof of Theorem 1 is given in the Appendix.

3 Consequence for Legendre's Conjecture

Theorem 1 implies that in every interval $(n^2, (n+1)^2)$, there exists at least one integer of the form $6a \mp 1$ that is not representable as a product of two smaller numbers of the same form. Consequently, such an integer must be prime. Therefore, Legendre's conjecture follows.

References

- [1] A.-M. Legendre, *Essai sur la théorie des nombres*, Paris, 1808.
- [2] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig, 1909.