Optimal Symmetric Bounds on *P*

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Abstract: Various bounds on *p*, such as *Bertrand's Postulate* and *Legendre's Conjecture*, propose regions around *n* that have at at least one prime within them. Using *Prime Generator Theory*, I show more precise symmetric bounds on *p*, such that for *n* a prime exists symmetrically within a distance of $n^{1/2}$ below and above it. That is to say, a prime exists for: $n - n^{1/2} < p < n$ and $n < p < n + n^{1/2}$.

Introduction

In *Proof of Goldbach's Conjecture and Bertrand's Postulate Using Prime Generator Theory* (*PGT*) [1], I showed that *Bertrand's Postulate* (*BP*) is a structural property of modular groups $\mathbb{Z}_n$, and a necessary condition for *Goldbach's Conjecture* to be true. In fact, it's a weak statement on the distribution of primes, because as *n* increases the number of primes it bounds grows exponentially. A much tighter bound, derived from the properties of modular groups, and *PGT*, gives for even *n*: $n - n^{1/2} < p < n$.

Similarly for *Legendre's Conjecture*: for every *n* there's a prime *p* e.g. $n^2 < p < (n + 1)^2$, it diverges even faster, providing even less precise information on the distribution of primes. However, we can use the premise of the conjecture, i.e. characterizing the number of primes between two consecutive perfect squares, to formulate an optimal symmetric bound on *p* for *n* as it increases.

Also, Baker, Harman, and Pintz (*BHP*) [4] have shown there's always a prime *p* between $n - n^{\theta}$ and *n*, where originally $\theta = 23/42 = 0.5476$ (true for every $n > 3$ except for 11) which was lowered numerous times to now be $\theta = 0.525$. Thus using classical mathematical techniques and thinking, they derived a set of similar bounds to that from *PGT*.

Ideally, we'd like to find an ***optimal distance*** $d_{pmin} = n^{\theta}$ around *n* that contains primes as it increases. Here it will mean in the sense that this is the symmetric distance around *n* that contains all the primes between two consecutive perfect squares with no over or under prime count. We'll see this distance is for $\theta = 1/2 = 0.5$, i.e. for $n^{1/2} = \sqrt{n}$. Thus for $n \pm n^{1/2}$ there are a statistically equal number of primes on either side of any sufficiently large *n*, which precisely cover all the primes between two consecutive perfect squares n^2 and $(n + 1)^2$.

Finally, we compare and analyze these findings to estimates derived from the *Riemann Hypothesis* (*RH*). We will see the *RH* estimates have many of the same properties of the *PGT* symmetric bounds, and ultimately only differ by a factor of log *x*.

Thus in line with [1], we establish again using *PGT*, the deterministic (non-random) distribution of the primes. All the odd primes are the residues of modular groups $\mathbb{Z}_n$, and are evenly distributed along their residue tracks as their congruent modular values. Thus we should expect, and will find it to be true, we can derive tighter bounds on *p* than previously determined.

Bounds on *p*

Bertrand's Postulate (*BP*) states: ***for each n ≥ 2 there's at least one prime p such that n < p < 2n***, was conjectured in 1845, and is attributed to be first proven by Pafnuty Lvovich Chebyshev in 1852. Subsequent proofs followed, e.g. the widely cited elementary proof published by Paul Erdös in 1932. They used the classical analytical techniques of their time, primarily centered on the factorization of *n*.

When the *Prime Number Theorem* (*PNT*) was proven in 1896, the *BP* was seen as a weaker, though easier to formulate, statement on the distribution of primes, primarily for small *n*. But as *n* increases, its precision to characterize the distribution of primes greatly diminishes (as we'll see). That is to say, if you want a more finely tuned understanding of the distribution of primes as *n* grows, the *BP* is lacking.

Unknown at the time (or at least not widely understood, and appreciated) were the properties of the residues of modular groups $\mathbb{Z}_n$. For an even modulus *n*, the residues are the coprime integers to *n*, and all the odd primes are residues to some *n*. From [1] and [2] we see the *BP* is merely a fundamental property of residues symmetry, e.g. there's always at least one prime *p* in the ***high-half-residues* (*hhr*)** of a modular group, and thus equivalently for even $n > 2$, $\boldsymbol{n/2 < p < n}$.

Because residues exist as ***modular complement pairs* (*mcp*)**, and 1 is the first canonical residue value, for small even *n*, the largest *hhr* values $n - 1$, are mostly primes.

n	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$n - 1$	3	5	7	9	11	13	15	17	19	21	23	25	27	29

In [1] I derived a much more precise (tighter) bound on *p* I call the *Prime Generator Theory Postulate* (*PGTP*): ***for n ≥ 2 there is at least one prime p such that 2n – (2n)$^{1/2}$ < p < 2n***.

This is true for every lower bound except for $2n = 126$, but changing it to: $\boldsymbol{2n - (2n)^{1/2} - 2 < p < 2n}$ captures the prime 113, to make it true for all. But as we're most interested in the asymptotic behavior of the primes as *n* increases, the original bound is sufficient.

Unlike the *BP*, whose lower/upper *Bounds Ratio* (*BR*) is a constant value, the *PGTP*'s *BR* increases dynamically, asymptotically approaching 1. Therefore, as *n* (and thus $n^{1/2}$) increase, there exists a much smaller set of primes between their difference than identified from the *BP*. Thus we now know, the primes have a much tighter distribution that is non-random (and from *PGT* we know is deterministic).

The ramifications to the *PNT*, and prime number theory in general, is profound. The distribution of primes is governed by the $\sqrt{n}$. As *n* grows, there's always one prime in the interval $(n - \sqrt{n}, n)$.

We'll now see graphically, how profound the differences between the *BP* and *PGTP* look.

Bounds Lines

It is easiest to consider the profound differences between the *BP* and *PGTP* by graphing their bounds lines, and then mathematically understanding what they mean.

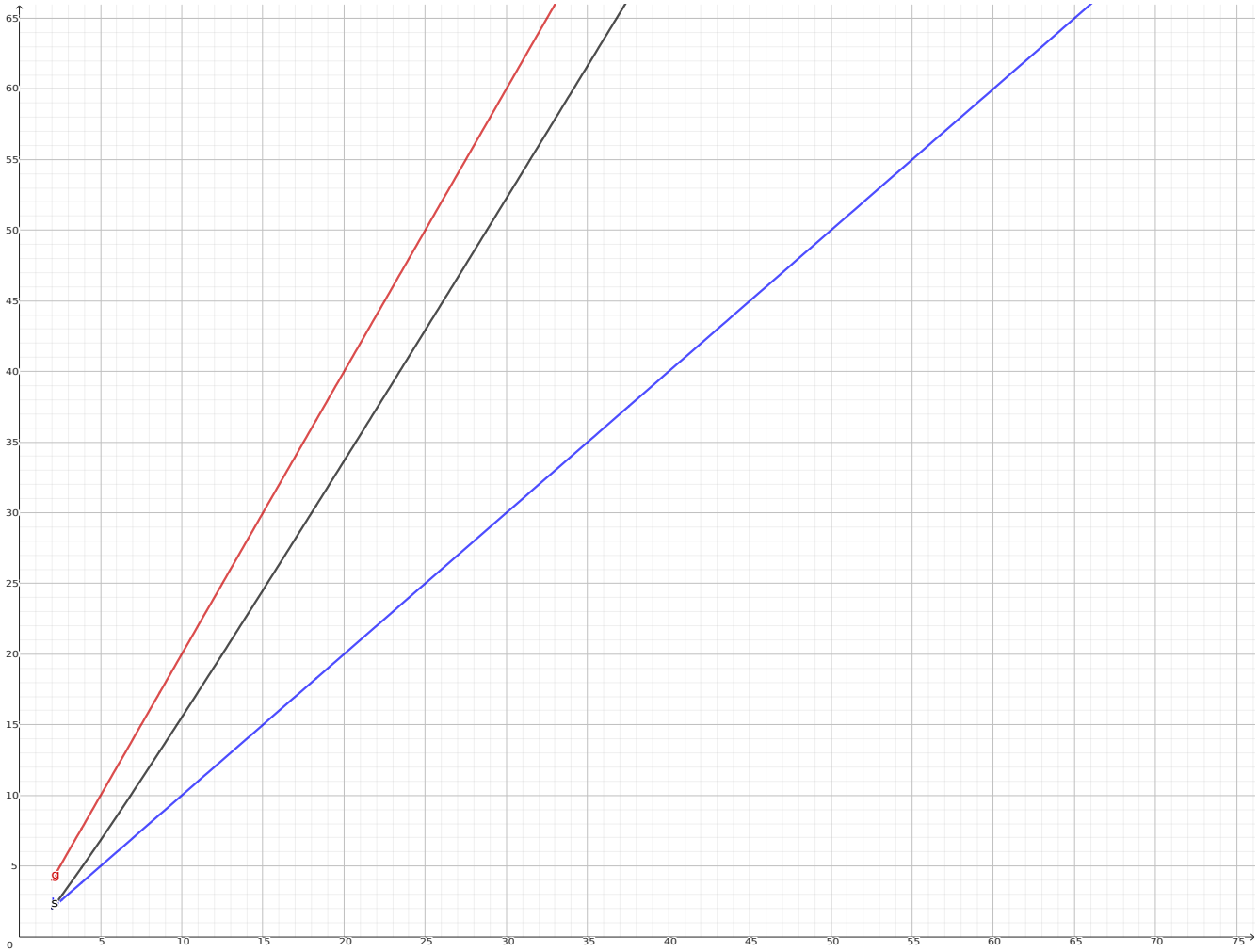

Fig. 1

This is a graph of both bounds shown as straight lines representing their upper and lower bounds.

For *Bertrand's Postulate* we have: $n < p < 2n$ (1)

While for *PGT's Postulate* it's: $2n - (2n)^{1/2} < p < 2n$ (2)

Upper bounds line for both: $y_1 = 2x$ (3)

BP lower bounds line: $y_2 = x$ (4)

PGTP lower bounds line: $y_3 = 2x - (2x)^{\frac{1}{2}}$ (5)

Each line starts at $x = 2$, to represent the integer domain starting values, and the integer x|y values correspond to the integer values of interest for each bound.

Here we see what the line for the floored integer lower bound of the *PGTP* looks like for comparison.

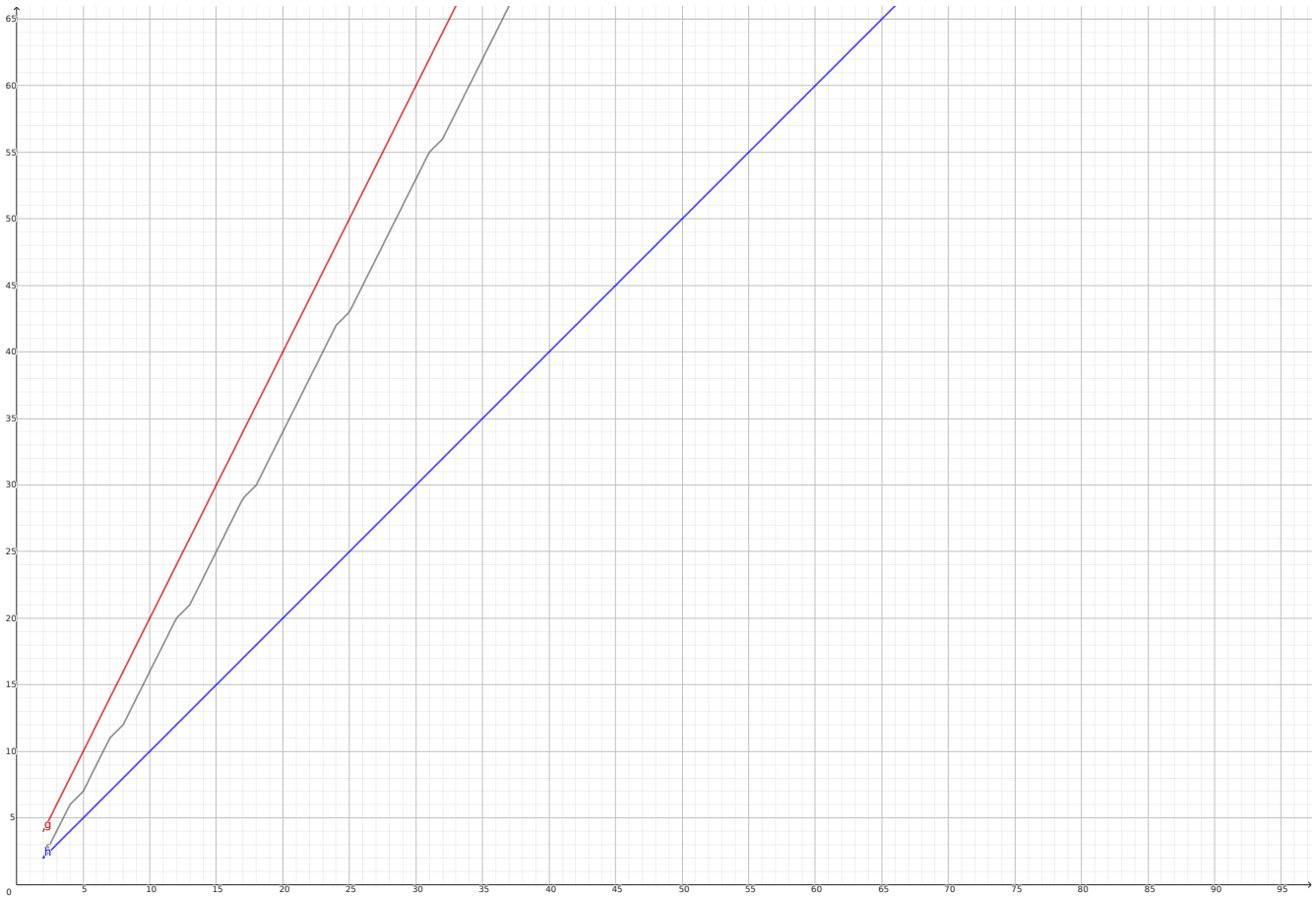

Fig. 2

The floored *PGT's Postulate*: $$2n - \left\lfloor (2n)^{\frac{1}{2}} \right\rfloor \leq p < 2n \tag{6}$$

PGTP floored lower bounds line: $$y_4 = 2x - \left\lfloor (2x)^{\frac{1}{2}} \right\rfloor \tag{7}$$

Here an x-integer value always correspond to a y-integer value, and the *PGTP* lower bound line looks like a stepped escalator (escalator staircase). Since the lower bound now produces strictly integer values, its lower bound condition is changed to be ≤ *p*.

Again for the *PGTP*, only for 2*n* = 126 is the lower bound not strictly true, but using 2 as the smallest integer value to make it true, lowers the line *y*4 by 2 to:

$$y_5 = 2x - \left\lfloor (2x)^{\frac{1}{2}} \right\rfloor - 2 \tag{8}$$

Now the lower bound catches the prime 113 for 2*n* = 126, and is true for all $n \geq 2$. We could also pick bigger constants than 2, and still have a more precise asymptotic bound than the *BP* as *n* grows larger. But the lower bound slope would still be the same, and it reveals one of the most significant differences between the two bounds.

This graph shows both *PGTP* lower bound lines compared to the *BP*.

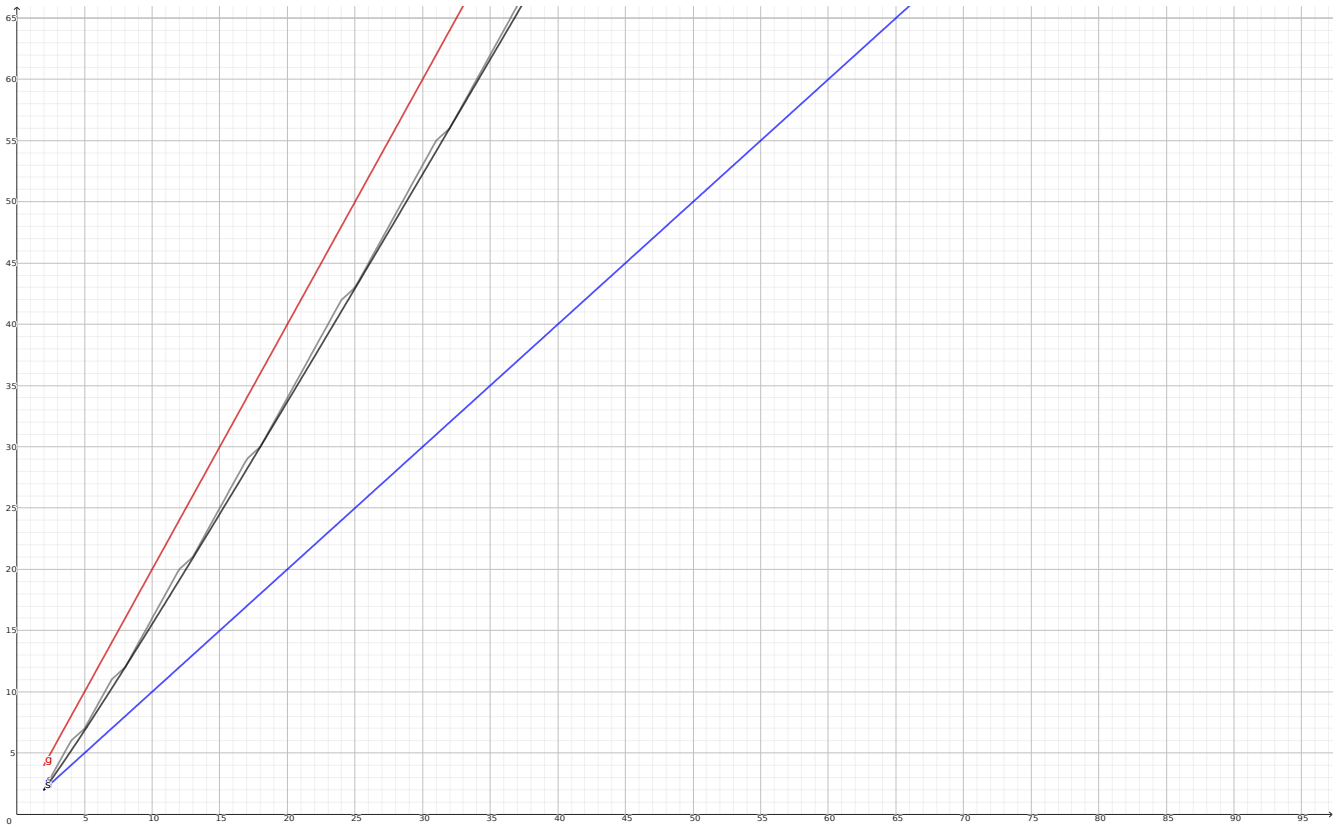

Fig. 3

Notice the floored *PGTP* line is always a more precise (tighter) bound. As $n^{1/2}$ grows larger, it has the same floored value (has the same escalator slope) for longer stretches of *n*, until it increases by 1 for the next perfect square $(n + 1)^2$, whose vertical distance from the upper line is smallest.

Graphically then, the vertical numerical distance between the upper and lower bounds lines represents the number space of the possible primes that exist within them.

The *PGTP* ensures primes exist within the smallest number space, at the optimal approaching slope.

The continuous lower bound line is: $$y = 2x - (2x)^{\frac{1}{2}} \tag{9}$$

For just integer values we can do: $$y = 2x - \left\lfloor (2x)^{\frac{1}{2}} \right\rfloor \tag{10}$$

Thus the lower bound slope is: $$y' = 2 - 1/(2x)^{\frac{1}{2}} \tag{11}$$

As $x \to \infty$ then $y' \to 2$, and the lower bound slope becomes parallel to the upper bound slope of 2.

Thus as *n* increases, the *PGPT* identifies a tighter, and optimal, bounds on the distribution of primes. It shows for every increasing *n* that's a perfect square the floored ramp increases in value by 1,

Large *n* Bounds Behavior

We continue here to explore the significant differences between the bounds for large *n*.

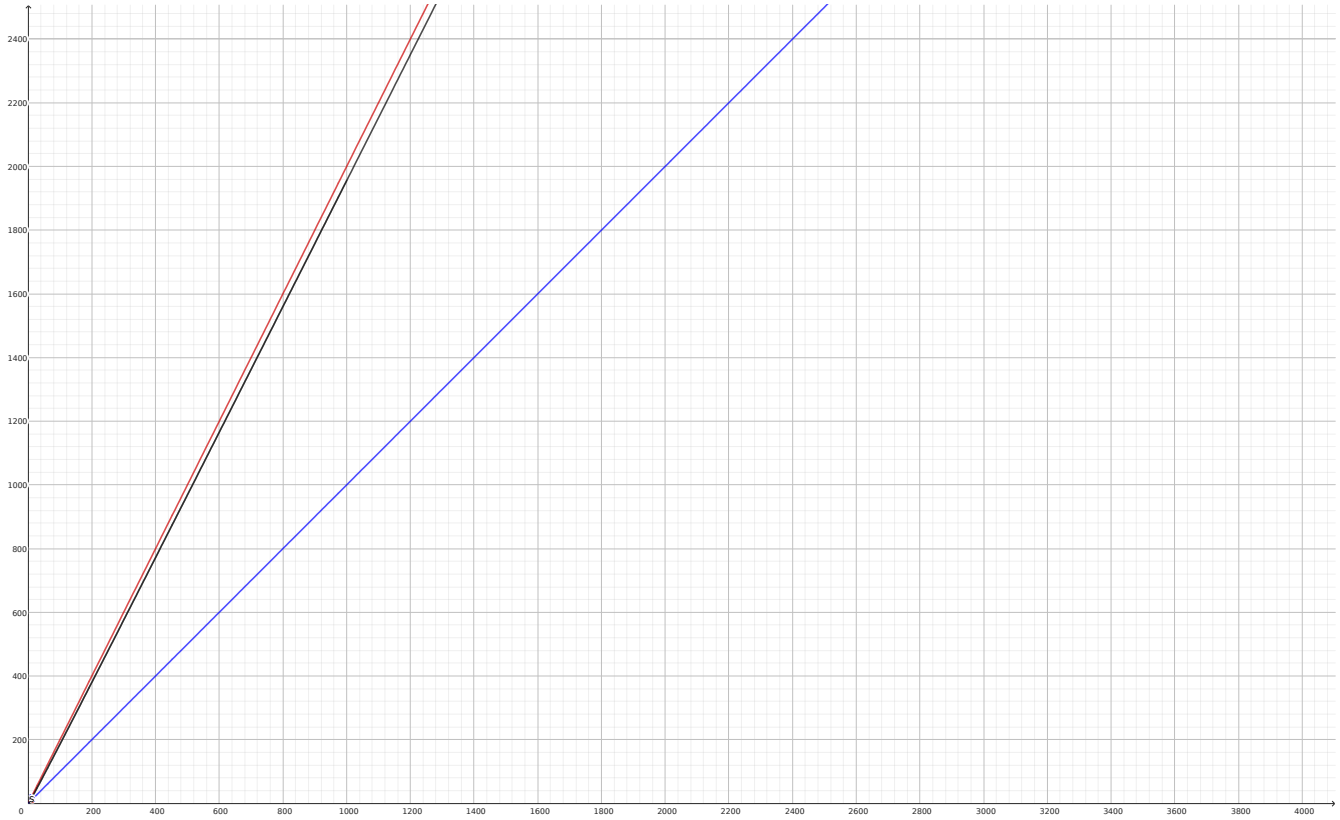

Fig. 4

We can clearly see here the behavior of the two bounds as *n* increases. For the *BP*, we see its bounds lines diverge (grow farther apart) as $x \to \infty$, However for the *PGTP*, we see its bounds lines tightly converge as $x \to \infty$. Thus their separation ratios have opposite characteristics.

For *Bertrand's Postulate* the lower/upper *Bounds Ratio* (*BR*) is a constant:

$$BR = \frac{n}{2n} = \frac{1}{2} \tag{12}$$

While *PGT's Postulate* ratio dynamically changes as:

$$BR = \frac{2n - (2n)^{\frac{1}{2}}}{2n} = 1 - \frac{1}{(2n)^{\frac{1}{2}}} \tag{13}$$

The *PGTP BR* starts at 1/2 for *n* = 2, and quickly approaches 1 as $n \to \infty$, as $1/(2n)^{\frac{1}{2}} \to 0$. Thus for it, as its lower bound line approaches becoming parallel to its upper bound line as *n* becomes larger, its bounds lines separation ratio approaches the closest possible, an optimum value of 1.

Numerical Comparisons

While the graphs show viscerally the profound differences between the two bounds, here I'll provide a numerical basis to understand, and appreciate, the ramifications of their differences.

From the *PNT*, the number of primes to some value *x* is given by $\pi(x)$. Let's first quantify the primes counts difference within their bounded regions as *n* grows. As an example for *n* = 500, the *BP* gives $\pi(500) - \pi(250) = 42$ primes, while the *PGTP* gives, $\pi(500) - \pi(477.64) = 4$ primes. Thus the *PGPT* provides a much more precise understanding of the primes distribution.

To provide a more expansive numerical sense to Fig. 4, the table below shows the orders of magnitude differences in bounded prime counts as *n* grows by powers of 10, and their growth ratios $n(10^e/10^{e-1})$.

Primes counts within BP|PGTP bounds				
n	$n - n^{1/2} < p < n$	PGTP Ratio Increase	$n/2 < p < n$	BP Ratio Increase
10^1	1		1	
10^2	1	1	10	10
10^3	5	5	73	7.3
10^4	9	1.8	560	7.671
10^5	29	3.222	4,459	7.963
10^6	65	2.241	36,960	8.289
10^7	191	2.938	316,066	8.552
10^8	533	2.791	2,760,321	8.733
10^9	1,518	2.848	24,491,667	8.873
10^{10}	4,306	2.837	220,098,288	8.987
10^{11}	12,514	2.906	1,998,400,235	9.079
10^{12}	36,400	2.909	18,299,775,876	9.157
10^{13}	105,283	2.892	168,773,875,190	9.223
10^{14}	310,457	2.949	1,566,017,986,235	9.279
10^{15}	916,294	2.951	14,606,736,768,049	9.327
10^{16}	2,713,819	2.962	136,860,923,837,561	9.369

Table 1.

The *BP* is less precise in the sense that each increasing *n* covers primes covered by more smaller *n*. Thus for *n* = 400 the *BP* bounds are: $200 < p < 400$, and for *n* = 500 its: $250 < p < 500$. This creates for 500 an overlap with the 25 primes from $250 < p < 400$ out of its overall total of 42 up to 500.

Thus as *n* increases, the *BP* provides no real understanding of the characteristics of how the primes are distributed around each integer. In other words, its field of view is very course, and has low resolution.

Uniform Distribution of Primes

The graphs|data provide a visceral and empirical understanding of the distribution of primes. As n increases by factors of 10, the primes count in both bounded regions grow by essentially constant factors. For example, for $n = 10^{16}$ we get 9.369/2.962 = 3.1631. Thus the graphs and data tells us, the distribution of primes follows a regular order, and ***is not random***! *PGT* shows and explains the primes deterministic existence within the integers number space is a property of modular groups $\mathbb{Z}_n$.

The odd primes exist as modular group residues, with the integers grouped as columns of size n. When $n = p_k\#$ (kth prime primorial) they have specific and unique properties. For $\mathbb{Z}_{30}$, $n = p_3\# = 5\# = 30$, we see its primes distribution. The first column coprime integers to n are its residues, and all subsequent primes are their congruent residue values, uniformly distributed along each residue row. Thus for $\mathbb{Z}_{30}$, its canonical residues are {1, 7, 11, 13, 17, 19, 23, 29}, with prime values: $p_m = 30k + r_i$, $k \geq 0$, i = 0..7.

0	30	60	90	120	150	180	210	240	270	300	330	360	390	420	450	480	510	540	570	
1	31	61	91	121	151	181	211	241	271	301	331	361	391	421	451	481	511	541	571	····
2	32	62	92	122	152	182	212	242	272	302	332	362	392	422	452	482	512	542	572	
3	33	63	93	123	153	183	213	243	273	303	333	363	393	423	453	483	513	543	573	
4	34	64	94	124	154	184	214	244	274	304	334	364	394	424	454	484	514	544	574	6
5	35	65	95	125	155	185	215	245	275	305	335	365	395	425	455	485	515	545	575	
6	36	66	96	126	156	186	216	246	276	306	336	366	396	426	456	486	516	546	576	
7	37	67	97	127	157	187	217	247	277	307	337	367	397	427	457	487	517	547	577	····
8	38	68	98	128	158	188	218	248	278	308	338	368	398	428	458	488	518	548	578	
9	39	69	99	129	159	189	219	249	279	309	339	369	399	429	459	489	519	549	579	4
10	40	70	100	130	160	190	220	250	280	310	340	370	400	430	460	490	520	550	580	
11	41	71	101	131	161	191	221	251	281	311	341	371	401	431	461	491	521	551	581	····
12	42	72	102	132	162	192	222	252	282	312	342	372	402	432	462	492	522	552	582	2
13	43	73	103	133	163	193	223	253	283	313	343	373	403	433	463	493	523	553	583	····
14	44	74	104	134	164	194	224	254	284	314	344	374	404	434	464	494	524	554	584	
15	45	75	105	135	165	195	225	255	285	315	345	375	405	435	465	495	525	555	585	**4**
16	46	76	106	136	166	196	226	256	286	316	346	376	406	436	466	496	526	556	586	
17	47	77	107	137	167	197	227	257	287	317	347	377	407	437	467	497	527	557	587	····
18	48	78	108	138	168	198	228	258	288	318	348	378	408	438	468	498	528	558	588	2
19	49	79	109	139	169	199	229	259	289	319	349	379	409	439	469	499	529	559	589	····
20	50	80	110	140	170	200	230	260	290	320	350	380	410	440	470	500	530	560	590	
21	51	81	111	141	171	201	231	261	291	321	351	381	411	441	471	501	531	561	591	4
22	52	82	112	142	172	202	232	262	292	322	352	382	412	442	472	502	532	562	592	
23	53	83	113	143	173	203	233	263	293	323	353	383	413	443	473	503	533	563	593	····
24	54	84	114	144	174	204	234	264	294	324	354	384	414	444	474	504	534	564	594	
25	55	85	115	145	175	205	235	265	295	325	355	385	415	445	475	505	535	565	595	
26	56	86	116	146	176	206	236	266	296	326	356	386	416	446	476	506	536	566	596	6
27	57	87	117	147	177	207	237	267	297	327	357	387	417	447	477	507	537	567	597	
28	58	88	118	148	178	208	238	268	298	328	358	388	418	448	478	508	538	568	598	
29	59	89	119	149	179	209	239	269	299	329	359	389	419	449	479	509	539	569	599	····

Fig. 5

As the residues increase for increasing $n = p_k\#$, the primes are more uniformly distributed along them.

Symmetric Bounds On *P*

We can generalize the bounds on *p* by restating what a bound represents. Here we will ask, what is the ***optimal symmetric distance*** around any integer *n* as it increases? This will be the precise distance from any *n* that will contain all the primes between two consecutive perfect squares n^2 and $(n + 1)^2$.

From *PGT,* the properties establishing the lower bounds on *p* for even *n* also establish them for odd *n*. Every odd prime is one less|more than an even *n*, and exists within $n - n^{1/2} < p < n$. Thus as *n* increases, we know primes exist between $(n + 1) - n^{1/2} < p < (n - 1)$, and also $n < p < (n - 1) + n^{1/2}$.

For $n = 30$, $\sqrt{30} = 5.477$ gives mathematical lower|upper bounds (24.523, 30) and (30, 35.477). Since primes are integers, we check for primes within the inclusive bounds [25, 29] and [31, 35].

24 **[**25 26 27 28 <u>29</u>**]** **30** **[**<u>31</u> 32 33 34 35**]** 36

For both odd and even perfect squares their bounded values are even. For $n = 100$, its bounded values are [90, 110], and for $n = 121$ they are [110, 132]. Here we see the bounded primes for $n = 100 = 10^2$.

89 **[**90 91 92 93 94 95 96 <u>97</u> 98 99**]** **100** **[**<u>101</u> 102 <u>103</u> 104 105 106 <u>107</u> 108 <u>109</u> 110**]** 111

Thus there's a sliding window whose value is the square root of the closest perfect square less than *n*. Since 25 is the closest perfect square less than 30, each *n* from 25 ↔ 35 has a window of $\sqrt{25} = 5$ on each side that contains primes. This increases to 6 for values 36 ↔ 48, etc, as shown in Figs 2 and 3.

Only a small number of values fall within one distinct square root band as *n* increases. For $10^2 = 100$, only <u>109</u> is contained solely within the window of 10 from 100, as the window for $(10 - 1)^2 = 9^2 = 81$ covers everything up to $99 + 9 = 108$, and $11^2 = 121$ covers everything down to $121 - 11 = 110$.

Thus as *n* increases, most values will be covered within two perfect square upper|lower windows.

Only the 5 small values 11, 29, 125, 126, 127 do not contain such a lower bound prime, while the 11 small values 3, 7, 8, 13, 23, 24, 31, 113, 114, 115, 116 do not contain an upper bound prime (no *n* fails both bounds). Thus, all values $x \geq 128$ satisfy both bounds, verified for all *n* up to 10^{22}.

Now let's observe the bounds behavior between two consecutive perfect squares $(n)^2$ and $(n + 1)^2$.

Their difference is: $(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 = (n) + (n + 1)$ (14)

Thus, the distance between any two perfect squares n^2 and $(n + 1)^2$ is the sum of their square roots. And the upper bound for n^2, with the lower bound for $(n + 1)^2$, contain all the primes between them. Below we see this for the consecutive perfect squares 10^2 and 11^2.

10^2 = **100** **[**<u>101</u> 102 <u>103</u> 104 105 106 <u>107</u> 108 <u>109</u> **]** 110 **[** 111 112 <u>113</u> 114 115 116 117 118 119 120**]** **121** = **11^2**

|---------- 10 ----------|---------- 11 ----------|

$10^2 < p < 10^2 + 10$ $\qquad$ $11^2 - 11 < p < 11^2$

From *BHP*, we can write the bounds generically in the form of: $n - cn^{\theta} < p < n \;||\; n < p < n + cn^{\theta}$
For our purposes we can always take c as 1, and will see in general we can always do so.

Using this nomenclature, the *Symmetric BP* becomes: $n - n^1 < p < n \;||\; n < p < n + n^1$
This says for any integer n there are primes a distance n on both sides of it, as n increases. But again, this is a very course statement on the distribution of primes.

Earlier we saw *BHP* showed there are always primes a distance n^{θ} from n for $\theta = 0.525$. We now know we can reduce that to exactly $\theta = 0.5 = 1/2$. Thus as n increases, any small deviation from $\theta = 0.5$ will produce over|under counts of the actual primes between consecutive perfect squares.

As an example, there are 152 primes between $1{,}000^2$ (1,000,000) and $1{,}001^2$ (1,002,001). When $\theta = 0.5$ the primes within the upper bound for $1{,}000^2$ and lower bound for $1{,}001^2$ sum to 152. For $\theta = 0.49$ it's 131 (an under count of 21), and for $\theta = 0.51$ its 175 (an over count of 23). We see this below.

n	$n^{0.49}$	$n^{0.50}$	$n^{0.51}$
$1{,}000^2$	870.964	1,000	1,148.154
Upper Bounds	$1{,}000^2 < p < 1{,}000^2 + 870.9$	$1{,}000^2 < p < 1{,}000^2 + 1{,}000$	$1{,}000^2 < p < 1{,}000^2 + 1{,}148.1$
Primes Counts	66	75	87
$1{,}001^2$	871.817	1,001	1,149.325
Lower Bounds	$1{,}001^2 - 871.8 < p < 1{,}001^2$	$1{,}001^2 - 1{,}001 < p < 1{,}001^2$	$1{,}001^2 - 1{,}149.3 < p < 1{,}001^2$
Primes Counts	65	77	88
Bounded Primes	131	152	175

Table 2.

In other words, ***the distribution of primes around n optimally follows a simple square root law***.

While *BHP* in [4] states for all $x > x_0$, the inclusive intervals $[x - x^{0.525}, x]$ contain primes, in [1] using *PGT*, we now know that the non-inclusive intervals are $(n - n^{0.5}, n)$, for all $x \geq 128$ (as 127 is a prime).

Thus *Legendre's Conjecture* is true: ***primes always exist between consecutive perfect squares, n ≥ 2***.

The data further establishes the deterministic (non-random) distribution of primes within the integers $\mathbb{Z}$. As the $\sqrt{n}$ bound windows slide up the number line, they increase by 1 at each larger perfect square, and we should expect each one to contain more primes. And this is (on average) exactly what we see. We can see in Fig. 5 this is true for all n (except for the few cited small values). And we can see their increasing windows sufficiently cover the gap structure between primes at the scale they exist within.

Thus, even though as established in [2], there are no bounds on the size of consecutive prime gaps, and there are an infinity of each, they increase in size and frequency in a mathematically structured manner. Therefore no matter their size, *PGT* establishes all the odd primes are modular group residues, and thus must be uniformly distributed along them, over sufficiently large ranges.

As before, we can graphically visualize the symmetric prime domains that are defined by these bounds.

Lower Bound	Upper Bound
$n - n^{1/2} < p < n$	$n < p < n + n^{1/2}$

Below is a graph of these bounds, with the middle line representing all integers *n*, as $y = x$.

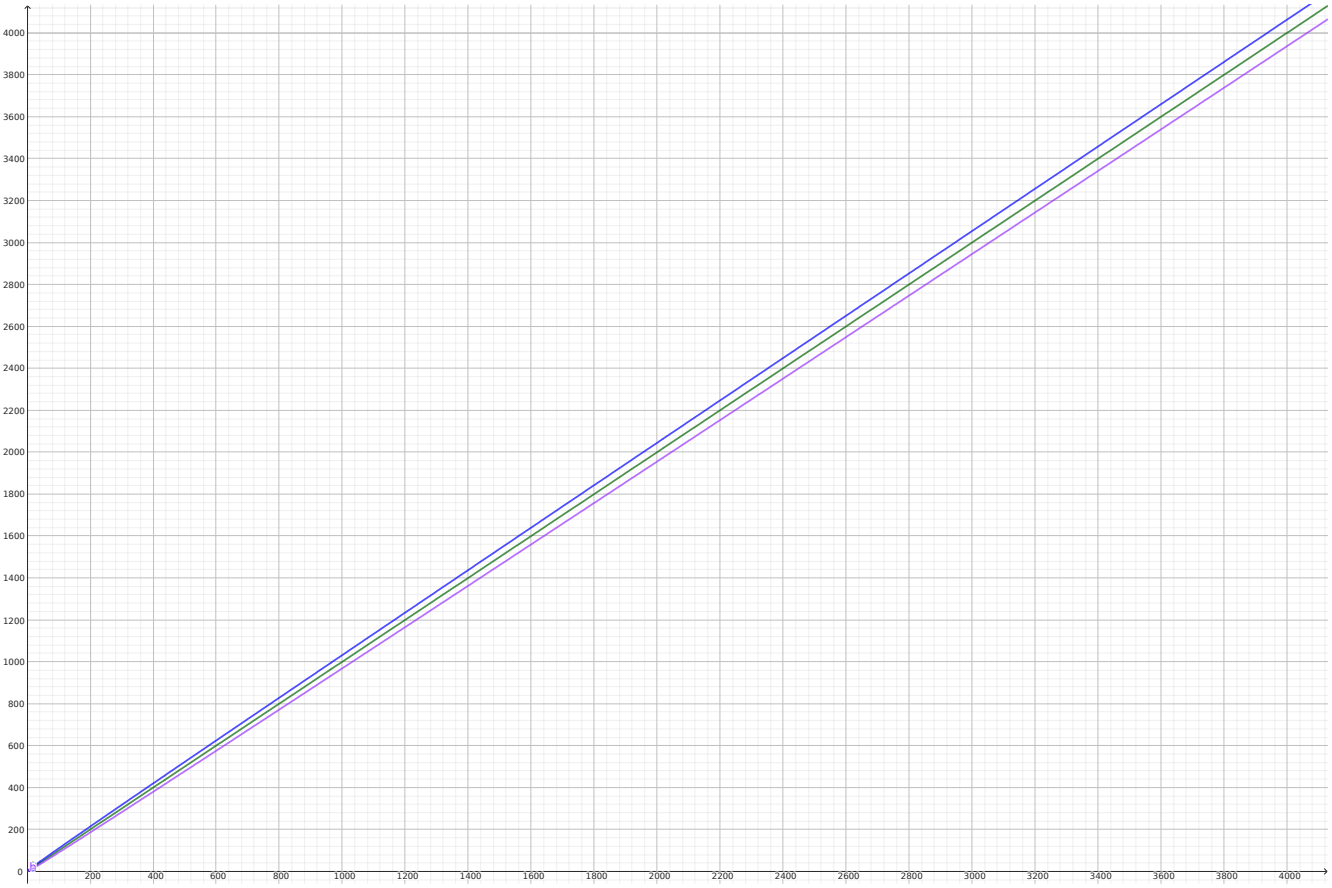

Fig. 6

Here the upper|lower bound lines are: $y = x \pm x^{\frac{1}{2}}$ (15)

For just integer values we can do: $y = x \pm \left\lfloor x^{\frac{1}{2}} \right\rfloor$ (16)

And the upper|lower bound slopes are: $y' = 1 \pm 1/2x^{\frac{1}{2}}$ (17)

As $x \to \infty$ their slopes approach $y' \to 1$, with the upper bound becoming parallel to $y = x$ from above and the lower bounds from below, separated vertically from the line $y = x$ by $\sqrt{x}$, i.e. $n^{1/2}$.

Thus as *x* increases, the graph shows there is a narrow vertical strip of values that encapsulates the distribution of primes around every value.

The table below provides empirical data that shows as *n* increases, the number of primes within the lower and upper bounds for each *n* increases uniformly, i.e. statistically in a 1:1 ratio. We also see as *n* increases by a factor of 10, the number of primes within the bounds increase by about a factor of 3.

Primes Counts Within Symmetric Lower|Upper Bounds					
n	$n - n^{1/2} < p < n$	$pcnt_{n+1} / pcnt_n$	$n < p < n + n^{1/2}$	$pcnt_{n+1} / pcnt_n$	Upper/Lower
10^1	1		2		2
10^2	1	1	4	2	4
10^3	5	5	5	1.25	1
10^4	9	1.8	11	2.2	1.2222
10^5	29	3.2222	24	2.1818	0.8276
10^6	65	2.2414	75	3.125	1.1538
10^7	191	2.9385	197	2.6267	1.0314
10^8	533	2.7906	551	2.7970	1.0338
10^9	1,518	2.8480	1,510	2.7405	0.9947
10^{10}	4,306	2.8366	4,306	2.8517	1
10^{11}	12,514	2.9062	12,491	2.9008	0.9982
10^{12}	36,400	2.9087	36,249	2.9020	0.9959
10^{13}	105,283	2.8924	105,785	2.9183	1.0048
10^{14}	310,457	2.9488	310,582	2.9360	1.0004
10^{15}	916,294	2.9514	915,261	2.9469	0.9881
10^{16}	2,713,819	2.9617	2,714,904	2.9663	1.0004
10^{17}	8,076,035	2.9759	8,083,056	2.9773	1.0009
10^{18}	24,122,961	2.9870	24,127,085	2.9849	1.0002
10^{19}	72,281,028	2.9964	72,277,024	2.9957	0.9999
10^{20}	217,151,027	3.0043	217,134,492	3.0042	0.9999
10^{21}	653,970,346	3.0116	653,976,888	3.0119	1.0000
10^{22}	1,974,049,195	3.0186	1,974,057,086	3.0185	1.0000
10^{23}	5,971,163,290	3.0248	5,971,116,556	3.0248	0.9999
10^{24}	18,096,893,288	3.0307	18,095,715,858	3.0305	0.9999
10^{25}	54,934,470,763	3.0356	54,934,399,096	3.0358	0.9999

Table 3.

The data vividly shows the uniform deterministic distribution of the primes within the integer number space. As *n* grows larger, the number of primes a distance $\sqrt{n}$ on both sides are statistically the same.

Riemann Hypothesis

The *Riemann Hypothesis* (*RH*) is a statement on the distribution of primes. In [5] it proves the estimate that there exists a prime in the interval $(x - \frac{4}{\pi}\sqrt{x}\log x, x]$ for all $x \geq 2$. It further states $c = 4/\pi$ can be reduced to $(1 + \epsilon)$ for sufficiently large x. Here's its graph for $4/\pi$ against the symmetric *PGT* bounds.

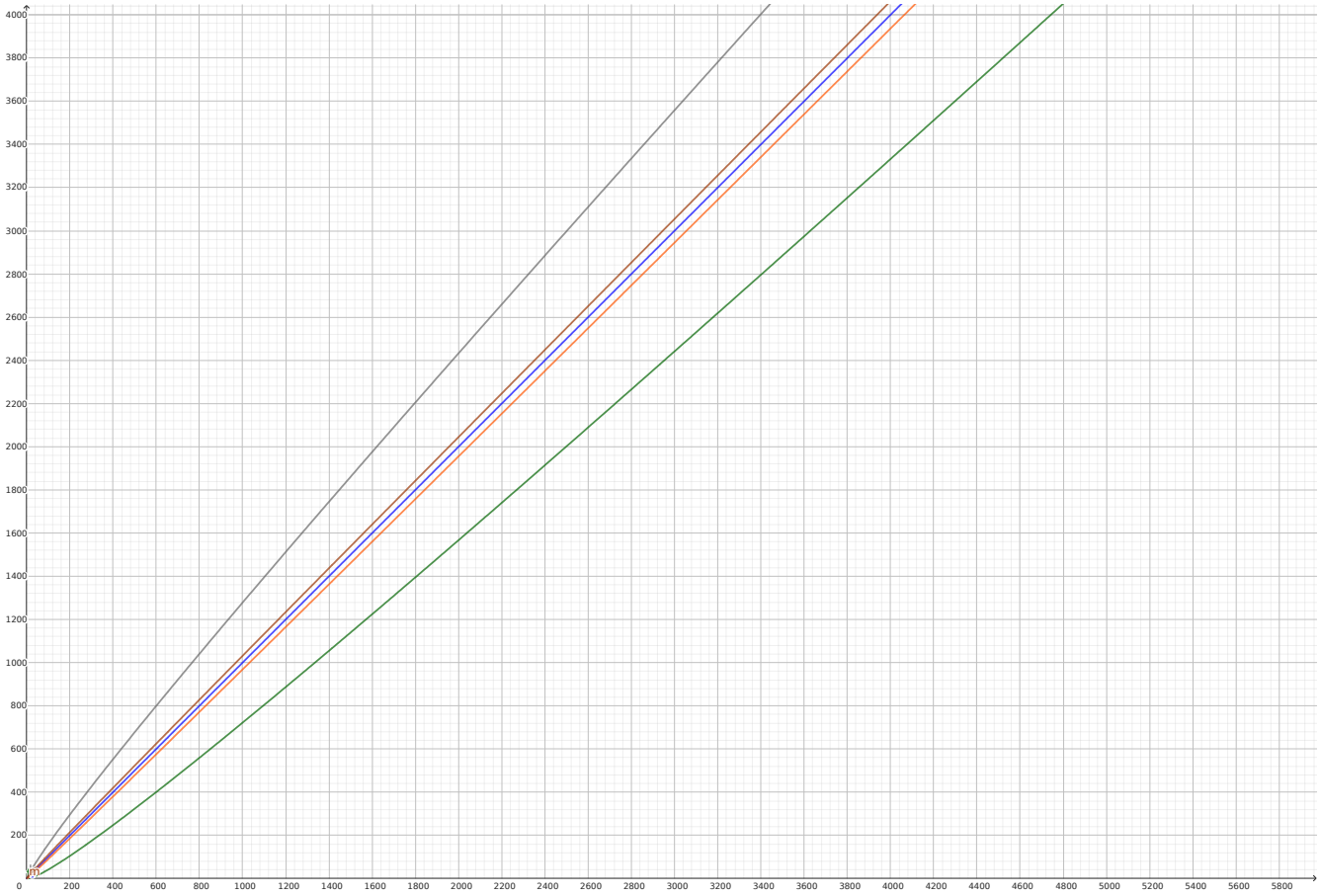

Fig. 7

The symmetric *RH* bounds are: $$y = x \pm \frac{4}{\pi} x^{\frac{1}{2}} \log x \tag{18}$$

And their slopes are: $$y' = 1 \pm \frac{4}{\pi}(\frac{1}{2} x^{\frac{-1}{2}} \log x + x^{\frac{1}{2}} \frac{1}{x}) = 1 \pm \frac{2}{\pi\sqrt{x}}(\log x + 2) \tag{19}$$

They also have the property that as $x \to \infty$ then $y' \to 1$, becoming parallel to $y = x$ from above|below, but at a growing greater vertical distance from x compared to the symmetric *PGT* bounds.

What does this mean for the *RH*? As in [5], we see we can improve this estimate by setting $\epsilon = 0$ so that $c = 1$, and get a tighter bound, that only differs from the *PGT* bounds by the log x factor.

Thus we see both bounds have similar arithmetic properties, with the *PGT* bounds providing a more precise picture of the nature of the distribution of primes, as x increases with no end.

Conclusion

PGT provides an understanding of the distribution of primes based on the structural framework of the residues of modular groups $\mathbb{Z}_n$. From it we can determine the minimum symmetric distance from *n* that contains a prime is $n^{1/2}$, i.e. $\sqrt{n}$.

The *PGTP* provides a higher resolution look into the primes distribution. It tells us the primes are distributed more uniformly than from the *BP*, and always exist within much smaller number spaces, which means there's a uniform gap structure between them, and the *BP* is merely a property of modular group symmetry, i.e. there's at least one prime in the *high-half-residues* of $\mathbb{Z}_n$, and thus $n/2 < p < n$. And as *n* grows, its bounds intervals diverge, become farther and farther apart, which contain more and more prime residues.

We also see *Legendre's Conjecture* statement on the primes between consecutive perfect squares, leads precisely to symmetric bounded regions of $n^{1/2}$. Thus as we traverse the increasing number space, each value is bounded below and above by the square root of the immediate smaller perfect square to it.

This provides for a much greater understanding of the deterministic distribution of the primes. And because we know the distribution of primes is not random in this manner, this implies the *Riemann Hypothesis* is true, as it's a statement on the distribution of primes, which are bounded simply by $n^{1/2}$.

References

[1] Jabari Zakiya – Proof of Goldbach's Conjecture and Bertrand's Postulate Using Prime Generator Theory (PGT). International Journal of Mathematics and Computer Research (IJMCR), Vol 13 No 3 (2025), 4923–4942
https://ijmcr.in/index.php/ijmcr/article/view/911/703 (pdf)

[2] Jabari Zakiya – On The Infinity of Twin Primes and other K-tuples. International Journal of Mathematics and Computer Research (IJMCR), Vol 13 No 1 (2025), 4739–4761.
https://ijmcr.in/index.php/ijmcr/article/view/867/678 (pdf)

[3] Jabari Zakiya – Twin Primes Segmented Sieve of Zakiya (SSoZ) Explained. J Curr Trends Comp Sci Res 2(2), (2023), 119–147.
https://www.opastpublishers.com/open-access-articles/twin-primes-segmented-sieve-of-zakiya-ssoz-explained.pdf

[4] R.C. Baker, G. Harman, and J. Pintz – The difference between consecutive primes, II. Proceedings of the London Mathematical Society, Vol. 83, Issue 3, (2001), 532–562.
https://www.cs.umd.edu/~gasarch/BLOGPAPERS/BakerHarmanPintz.pdf

[5] Adrian W. Dudek – On the Riemann Hypothesis and the Difference Between Primes.
arXiv:1402.6417v2 [math.NT]
https://arxiv.org/pdf/1402.6417 (pdf)

* All graphs produced using Linux desktop version of Classic GeoGebra 6 – www.geogebra.org.