ON THE RUN LENGTH-GAP THEOREM

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ABSTRACT. We prove that a certain class of infinite sequences whose finite truncation is an addition chain must have arbitrarily large gaps between their consecutive terms. This result is generic and can be applied to particular known infinite sequences with this property.

1. PRELIMINARIES AND SETUP

Let $l(n)$ be the length of an addition chain leading to n, denoted $E(n)$, of the form

$$E(n) : s_o = 1, s_1 = 2, \ldots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$ such that $l(n) := \beta(m)$. By adapting the ideas of the paper [1], we partition the steps in an addition chain into the following classes of steps

$$\mathcal{A} := \{i \ : \ s_i = 2s_{i-1}\} \quad (\textbf{doubling steps})$$

$$\mathcal{B} := \{i \ : \ \gamma s_{i-1} \leq s_i < 2s_{i-1}\} \quad (\textbf{large steps})$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden* ratio

$$\mathcal{C} := \{i \ : \ (1+\delta)s_{i-1} \leq s_i < \gamma s_{i-1}\} \quad (\textbf{medium} - \textbf{sized steps})$$

where $\delta := \delta(m) \longrightarrow 0$ as $m \longrightarrow \infty$. In particular

$$\delta := \delta(m) = \frac{1}{\log m}$$

$$\mathcal{D} := \{i \ : \ s_i < (1+\delta)s_{i-1}\} \quad (\textbf{small steps}).$$

We denote the cardinality of the sets to be

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

We call steps in $\mathcal{B}, \mathcal{C}, \mathcal{D}$ as *non-doubling steps*. We have therefore the relation

$$A + B + C + D = \beta(m).$$

Because the non-doubling steps in an addition chain cannot grow faster than a Fibonacci sequence, we have (by induction) the inequality

$$2^m \leq n \leq 2^A \gamma^{B+C+D} = 2^{\beta(m)} \left(\frac{\gamma}{2}\right)^{B+C+D}$$

and we deduce from this relation an upper control for the total number of non-doubling steps in an addition chain of length $\beta(m)$ to be

Lemma 1.1. *Put*

$$E(n) : s_o = 1, s_1 = 2, \ldots, s_{l(n)} = n$$

be an addition chain with $l(n) := \beta(m)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain of length $\beta(m)$ with cardinality A, B, C, D, respectively. Then we have

$$B + C + D \leq \frac{\beta(m) - m}{1 - \log_2 \gamma}.$$

It turns out that the non-doubling steps in an addition chain have certain structural pattern.

Lemma 1.2. *If $j \in \mathcal{B}$, then $j - 1 \in \mathcal{C} \cup \mathcal{D}$. In particular, each large step in an addition chain must be preceded by either a small step or a medium-sized step.*

Proof. Let $j \in \mathcal{B}$ (large step) then we have by definition

$$\gamma s_{j-1} \leq s_j < 2s_{j-1}$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden ratio*. Write $s_j = s_k + s_l$ with $k \geq l$. The inequality $s_j = s_k + s_l \leq s_{j-1} + s_l$ with $s_j < 2s_{j-1}$ implies that

$$\gamma s_{j-1} \leq s_j \leq s_{j-1} + s_{j-2}$$

which further implies

$$(\gamma - 1)s_{j-1} < s_{j-2} \iff s_{j-1} < \gamma s_{j-2}$$

since $\gamma = \frac{1}{\gamma - 1}$. This proves $j - 1 \in \mathcal{C} \cup \mathcal{D}$.   □

1.1. **The run and gap between terms in an addition chain.** We begin this section with the following definition.

Definition 1.3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain as in the setup. We call a **maximal** consecutive sequence of steps of a given type

$$j_1 < j_2 < \cdots < j_d$$

such that $j_1 - 1, j_d + 1$ cannot be a step of the given type a **run** of the step type. We call the number of terms in the *run* the **run length**.

Example 1.4. Consider the addition chain

$$E(n) : 1 = s_0, s_1 = 2, s_2 = 3, s_4 = 10, s_5 = 20, s_6 = 23.$$

Here, we observe that $2^4 < 23 < 2^5$ so that $m := 4$ and $\delta := \delta(m) = \frac{1}{\log m}$. Furthermore, we observe that $s_1 = 2s_0$, $s_2 < s_1(1 + \delta)$, $s_2 \cdot \gamma < s_3 < s_2 \cdot 2$, $s_4 = 2s_3$, $s_5 = 2s_4$, $s_6 < 23(1 + \delta)$. Thus, $\{1, 4, 5\}$ are steps of type $\mathcal{A}$, $\{2, 6\}$ are steps of type $\mathcal{D}$ and $\{3\}$ is the only step of type $\mathcal{B}$. We observe that $1 < 2$ is a consecutive sequence of steps of type $\mathcal{A} \cup \mathcal{D}$. Similarly, $4 < 5 < 6$ is a consecutive sequence of steps of type $\mathcal{A} \cup \mathcal{D}$. Since these are the only consecutive sequences of steps of type $\mathcal{A} \cup \mathcal{D}$ in the addition chain and the sequence $4 < 5 < 6$ contains more steps of type $\mathcal{A} \cup \mathcal{D}$ than $1 < 2$, it follows that the sequence of consecutive steps

$$4 < 5 < 6$$

is a run of type $\mathcal{A} \cup \mathcal{D}$. Also, $2 < 3$ is the only sequence of consecutive steps of type $\mathcal{B} \cup \mathcal{D}$. Thus, $2 < 3$ is a run of type $\mathcal{B} \cup \mathcal{D}$. Furthermore, we observe that $3 < 4 < 5$ is the only consecutive sequence of steps of type $\mathcal{A} \cup \mathcal{B}$ so that

$$3 < 4 < 5$$

is a run of type $\mathcal{A} \cup \mathcal{B}$.

Lemma 1.2 hints at the core idea that a **run** of step type $\mathcal{C} \cup \mathcal{B}$ or $\mathcal{D} \cup \mathcal{B}$ will always appear among the non-doubling steps in any addition chain, whether or not optimal. More likely it is for chains that are not optimal to have many **run** of types $\mathcal{C} \cup \mathcal{B}$. We begin with the following Lemmas

Lemma 1.5. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four step types as in the setup, with

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{B}$, then*

$$\frac{1}{2}\gamma^{d-2} < s_{j_d} - s_{j_d-1} < 2^{d-1+\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{B}$ of the form

$$j_1 < \cdots < j_d.$$

Then $\gamma s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1} \iff (\gamma-1)s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < s_{j_d-1}$. We obtain (by induction)

$$(\gamma-1)\gamma^{d-2} < s_{j_d} - s_{j_d-1} < 2^{d-1+\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. The inequality follows immediately since $\gamma - 1 > \frac{1}{2}$. □

Lemma 1.6. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four step types as in the setup, with

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{C}$, then*

$$\delta(1+\delta)^{d-2} < s_{j_d} - s_{j_d-1} < \gamma^{d-1}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

Then $(1+\delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1} \iff \delta s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < (\gamma - 1)s_{j_d-1}$. By induction, we deduce

$$\delta(1+\delta)^{d-2} \leq s_{j_d} - s_{j_d-1} < \gamma^{d-1}2^{\alpha_o}$$

for some $\alpha := \alpha_o(d) > 0$. □

Lemma 1.7. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. *Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type* $\mathcal{D}$, *then*

$$s_{j_d} - s_{j_d-1} < \delta(1+\delta)^{d-1} 2^{\alpha_o}$$

for some $\alpha_o = \alpha_o(d) > 0$.

Proof. Consider a **run** of a step of type $\mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

Then $s_{j_d} < (1+\delta)s_{j_d-1} \iff s_{j_d} - s_{j_d-1} < \delta s_{j_d-1}$. By induction, we deduce

$$s_{j_d} - s_{j_d-1} < \delta(1+\delta)^{d-1} 2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.8. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. *Denote by* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *the four step types as in the setup, with*

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type* $\mathcal{B} \cup \mathcal{C}$ *such that there are* d' *and* d'' *steps of type* $\mathcal{B}$ *and* $\mathcal{C}$, *respectively, in this **run** with* $d' + d'' = d$. *If* $j_d \in \mathcal{B}$, *then*

$$\frac{1}{2}(1+\delta)^{d''} \gamma^{d'-1} < s_{j_d} - s_{j_d-1} < 2^{d'-1+\alpha_o} \gamma^{d''}$$

for some $\alpha_o := \alpha_o(d) > 0$. *On the other hand, if* $j_d \in \mathcal{C}$, *then*

$$\delta(1+\delta)^{d''-1} \gamma^{d'} < s_{j_d} - s_{j_d-1} < \gamma^{d''} 2^{d'+\alpha_o}$$

for some $\alpha_o = \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{B} \cup \mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

In the case $j_d \in \mathcal{B}$, then $\gamma s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1} \iff (\gamma-1)s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < s_{j_d-1}$. By induction, we deduce

$$\frac{1}{2}(1+\delta)^{d''}\gamma^{d'-1} < s_{j_d} - s_{j_d-1} < 2^{d'-1+\alpha_o}\gamma^{d''}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{C}$, we have

$$(1+\delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1} \iff \delta s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < (\gamma-1)s_{j_d-1}.$$

By induction, we deduce

$$\delta(1+\delta)^{d''-1}\gamma^{d'} < s_{j_d} - s_{j_d-1} < \gamma^{d''}2^{d'+\alpha_o}$$

for some $\alpha_o = \alpha_o(d) > 0$. $\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.9. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four step types as in the setup, with

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{B} \cup \mathcal{D}$ such that there are d' and d'' steps of type $\mathcal{B}$ and $\mathcal{D}$, respectively, in this **run** with $d' + d'' = d$. If $j_d \in \mathcal{B}$, then*

$$\frac{1}{2}\gamma^{d'-1} < s_{j_d} - s_{j_d-1} < 2^{d'-1+\alpha_o}(1+\delta)^{d''}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{D}$, then

$$s_{j_d} - s_{j_d-1} < \delta(1+\delta)^{d''-1}\gamma^{d'}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{B} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

In the case $j_d \in \mathcal{B}$, then

$$\gamma s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1} \iff (\gamma-1)s_{j_d-1} \leq s_{j_d} < s_{j_d-1}.$$

By induction, we deduce

$$\frac{1}{2}\gamma^{d'-1} < s_{j_d} - s_{j_d-1} < 2^{d'-1+\alpha_o}(1+\delta)^{d''}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{D}$, then

$$s_{j_d} < (1 + \delta)s_{j_d-1} \iff s_{j_d} - s_{j_d-1} < \delta s_{j_d-1}.$$

By induction, we deduce

$$s_{j_d} - s_{j_d-1} < \delta(1 + \delta)^{d''-1}\gamma^{d'}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.   □

Lemma 1.10. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four step types as in the setup, with

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{C} \cup \mathcal{D}$ such that there are d' and d'' steps of type $\mathcal{C}$ and $\mathcal{D}$, respectively, in this **run** with $d' + d'' = d$. If $j_d \in \mathcal{C}$, then*

$$\delta(1 + \delta)^{d'-1} \leq s_{j_d} - s_{j_d-1} < \gamma^{d'}(1 + \delta)^{d''}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{D}$ then

$$s_{j_d} - s_{j_d-1} < \delta(1 + \delta)^{d''-1}\gamma^{d'}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a **run** of type $\mathcal{C} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

If $j_d \in \mathcal{C}$, then

$$(1 + \delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1} \iff \delta s_{j_d-1} \leq s_{j_d} - s_{j_d-1} < (\gamma - 1)s_{j_d-1}.$$

By induction, we deduce

$$\delta(1 + \delta)^{d'-1} \leq s_{j_d} - s_{j_d-1} < \gamma^{d'}(1 + \delta)^{d''}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$. On the other hand, if $j_d \in \mathcal{D}$, then

$$s_{j_d} < (1 + \delta)s_{j_d-1} \iff s_{j_d} - s_{j_d-1} < \delta s_{j_d-1}.$$

By induction, we deduce

$$s_{j_d} - s_{j_d-1} < \delta(1 + \delta)^{d''-1}\gamma^{d'}2^{\alpha_o}$$

for some $\alpha_o := \alpha_o(d) > 0$.   □

2. Main theorem

We now state and prove a criterion related to the natural density of infinite sequences whose finite truncation at any term in the sequence constitutes an addition chain leading to that term. This is a consequence of the intermediate results in the previous section.

Theorem 2.1 (The run length-gap theorem). *Let $L := \{T_n\}_{n=0}^{\infty}$ be an infinite sequence of positive integers with $T_0 := 1$, $T_1 := 2$ and $T_i := T_j + T_k$ for all $i \geq 2$. If for all sufficiently large r at least one of the following*

(i) *The sequence $L_r := \{T_n\}_{n=0}^{r}$ contains a run of at least one of the types of steps $\mathcal{B}$, $\mathcal{C}$, $\mathcal{B} \cup \mathcal{C}$ of length d that satisfies*

$$d \gg (\log\log T_r)^{1+\epsilon}$$

for a small $\epsilon > 0$

(ii) *The sequence $L_r := \{T_n\}_{n=0}^{r}$ contains a run of at least one of the types of steps $\mathcal{B} \cup \mathcal{D}$, $\mathcal{D} \cup \mathcal{C}$ such that the number of steps of type $\mathcal{B}$ and $\mathcal{C}$ in each run satisfies*

$$\gg (\log\log T_r)^{1+\epsilon}$$

for a small $\epsilon > 0$ and that the last step in any of these runs is not of type $\mathcal{D}$

holds, then

$$\lim_{n \longrightarrow \infty} \sup(T_n - T_{n-1}) = \infty.$$

Proof. Suppose $L := \{T_n\}_{n=0}^{\infty}$ is a sequence of positive integers with $T_0 := 1$, $T_1 := 2$ and $T_i := T_j + T_k$ for all $i \geq 2$. We observe that for each $r \geq 2$, the sequence $L_r := \{T_n\}_{n=0}^{r}$ is an addition chain that leads to T_r. If the sequence $L_r := \{T_n\}_{n=0}^{r}$ contains a run of a step of type $\mathcal{B}$ of the form

$$j_1 < \cdots < j_d$$

then Lemma 1.5 gives

$$T_{j_d} - T_{j_d-1} > \exp(d\log\gamma - \log 2\gamma).$$

Under the hypothesis for the run length $d \gg (\log\log T_r)^{1+\epsilon}$ for a small $\epsilon > 0$, the difference $T_{j_d} - T_{j_d-1}$ tends to infinity with r for those j_d. If the sequence $L_r := \{T_n\}_{n=0}^{r}$ contains a run of type $\mathcal{C}$ of the form

$$j_1 < \cdots < j_d$$

then Lemma 1.6 gives

$$T_{j_d} - T_{j_d-1} > \exp\left(\frac{d}{2}\delta - \delta + \log\delta\right)$$

with $\delta \asymp \frac{1}{\log\log T_r}$. Under the hypothesis $d \gg (\log\log T_r)^{1+\epsilon}$ for a small $\epsilon > 0$, the difference $T_{j_d} - T_{j_d-1}$ tends to infinity with r for those j_d. Furthermore, if the sequence $L_r := \{T_n\}_{n=0}^{r}$ contains a run of step type $\mathcal{B} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d$$

with $j_d \in \mathcal{B}$, then Lemma 1.9 gives

$$T_{j_d} - T_{j_d-1} > \exp(d'\log\gamma - \log 2\gamma)$$

where d' denotes the number of steps of type $\mathcal{B}$ in this run. Under the hypothesis $d' \gg (\log\log T_r)^{1+\epsilon}$ for a small $\epsilon > 0$, the difference $T_{j_d} - T_{j_d-1}$ tends to infinity with r for those j_d. Finally, if the sequence $L_r := \{T_n\}_{n=0}^{r}$ contains a run of step type $\mathcal{C} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d$$

with $j_d \in \mathcal{C}$, Lemma 1.10 gives

$$T_{j_d} - T_{j_d-1} > \exp\left(\frac{d'}{2}\delta + \log\delta - \frac{\delta}{2}\right)$$

where d' denotes the number of steps of type $\mathcal{C}$ in this run with $\delta \asymp \frac{1}{\log\log T_r}$. Under the hypothesis $d' \gg (\log\log T_r)^{1+\epsilon}$ for a small $\epsilon > 0$, the difference $T_{j_d} - T_{j_d-1}$ tends to infinity with r for those j_d. $\square$

References

1. J.M. De Koninck, N. Doyon and W. Verreault *On the minimal length of addition chains*, arXiv preprint arXiv:2504.07332, 2025.