A Resolution of the Collatz Conjecture

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November 17, 2025

Abstract

This work establishes a complete arithmetic resolution of the Collatz Conjecture by decomposing the odd–to–odd dynamics into two complementary structures: a local residue–phase automaton and a global affine counting system. The reverse map $R(n; k) = (2^k n - 1)/3$ is shown to act on the live residues $1, 5 \pmod 6$ through a finite residue–phase state space, while every admissible exponent $k = c + 2e$ induces an affine expansion factor 2^k whose inverse coincides exactly with the dyadic slice weight 2^{-k}.

From this, every odd integer is seen to belong to a unique dyadic slice $\mathcal{S}c, e$, forming a disjoint partition of $\mathbb{N}$odd. Independently, the introduction of the zero–state index $Z(n)$ reveals a second, purely affine enumeration: each live odd n seeds a unique 4-adic ladder $m \mapsto 4m + 1$ whose union also partitions the odd integers without overlap. We prove that these two partitions coincide exactly, yielding a unified global structure in which all odd integers arise from admissible lifts above anchors $1, 5$.

The locked forward–reverse equivalence $T(n) = (3n+1)/2^{\nu_2(3n+1)}$ and $R(T(n); k) = n$ then implies that forward trajectories cannot branch or diverge: each forward iterate lies on a single admissible ladder descending toward its zero–state origin at 1. Because the residue–phase automaton is finite and every ladder has a uniquely determined forward parent, no infinite runaway is possible and no nontrivial odd cycle can exist.

All constructions, residue frameworks, and affine decompositions used in this paper are original to this work. Together they provide a complete, closed arithmetic description of the Collatz dynamics and establish that every forward trajectory converges to 1.

1 Introduction

The Collatz Conjecture asks whether every positive integer eventually reaches 1 under the iteration

$$n \mapsto \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2. \end{cases}$$

Despite its elementary form, the conjecture has remained unresolved since 1937 and has resisted probabilistic, dynamical, algebraic, and computational approaches.

The essential difficulty is structural: forward trajectories mix multiplicative growth with aggressive dyadic contraction, while reverse trajectories branch infinitely through admissible preimages. No prior framework has simultaneously captured both behaviors in a closed, exhaustive arithmetic model.

This work develops such a model. Our approach is built from first principles and consists of three independent components that ultimately coincide:

1. A **local residue–phase automaton** describing all odd iterates by their class modulo 6 and their phase modulo 3, yielding a finite state space on which every admissible reverse step acts.

2. A **zero-state operator** $Z(n)$ that isolates the intrinsic odd component of each number by removing its admissible dyadic factor. This produces a global index $Z \in \mathbb{N}$ and a child-determined affine ladder

$$m \mapsto 4m + 1,$$

whose union over all zero-state bases yields a disjoint affine partition of $\mathbb{N}_{\mathrm{odd}}$.

3. A **dyadic slice decomposition**, determined by the exponent $k = \nu_2(3n+1)$, which partitions the odd integers into the sets

$$\mathcal{S}_{c,e} = \left\{ \frac{2^{c+2e}(6t+x)-1}{3} \;:\; t \geq 0 \right\}, \qquad (c,x) \in \{(1,5),(2,1)\}.$$

Each slice has weight 2^{-k} and the slices are disjoint with total measure 1.

A central result of this paper is that the affine ladders from the zero-state construction and the dyadic slices from $k = \nu_2(3n+1)$ *coincide exactly.* Thus the odd integers admit two independent but equivalent global parametrizations: one affine, one dyadic.

When combined with the **forward–reverse locked identity**,

$$T(n) = \frac{3n+1}{2^{\nu_2(3n+1)}} \quad \Longleftrightarrow \quad R(T(n); \nu_2(3n+1)) = n,$$

the global structure forces every forward trajectory into the unique affine ladder descending from its zero-state base. Since this base is always 1, and since the residue–phase automaton is finite, no forward runaway and no nontrivial odd cycle is possible.

All results, constructions, and structural decompositions presented here are original. Together they provide a complete arithmetic description of the Collatz dynamics and establish that every forward trajectory converges to 1.

We begin by establishing the fundamental definitions and notation used throughout the framework.

2 Definitions

Definition 2.1 (Classic Collatz function)**.** *The classical Collatz map* $C : \mathbb{N} \to \mathbb{N}$ *is defined by*

$$C(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$

Definition 2.2 (Forward Collatz function)**.** *The complete-step (odd-to-odd) Collatz map* $T^* : \mathbb{N}_{\text{odd}} \to \mathbb{N}_{\text{odd}}$ *is*

$$T(n) = \frac{3n+1}{2^{k_{\max}}},$$

where $k_{\max} \geq 1$ *is the maximal exponent such that the denominator* $2^{k_{\max}}$ *divides* $3n+1$*. Thus* $T(n)$ *gives the next odd iterate of* n *under the Collatz process.*

Definition 2.3 (Reverse Collatz function)**.** *The complete-step reverse Collatz map* $R : \mathbb{N}_{\text{odd}} \to \mathbb{N}_{\text{odd}}$ *assigns to each odd integer* n *its admissible parent via*

$$R(n;k) \;=\; \frac{2^k n - 1}{3}, \qquad k \geq 1,$$

where k *is admissible if* $2^k n \equiv 1 \pmod 3$*. If* $k_{\min}$ *is the minimal admissible doubling count, then* $R(n;k_{\min})$ *is called the* first parent *of* n*.*

Definition 2.4 (Middle-even values)**.** *In the odd-to-odd formulation of the Collatz map, each step factors through an intermediate even value.*

- *For the* forward map*, given an odd integer* n*, the intermediate (middle-even) value is*

$$E_f(n) := 3n+1.$$

- *For the* reverse map*, given an odd integer* n *and an admissible doubling count* $k \geq 1$ *(i.e.* $2^k n \equiv 1 \pmod 3$*), the intermediate (middle-even) value is*

$$E_r(n,k) := 2^k n.$$

Both E_f *and* E_r *are even and serve as the "middle" stage between odd inputs and odd outputs. Read modulo* 18*, these values determine the child's odd class through the fixed gate* $10 \mapsto C_0$*,* $4 \mapsto C_2$*,* $16 \mapsto C_1$ *in the reverse Collatz function.*

Definition 2.5 (Parent (reverse Collatz function))**.** *An odd integer* n *is called a* parent*. If* $n \equiv 3 \pmod 6$ *(that is,* n *is an odd multiple of* 3*), then it has no admissible doubling and is called a* terminating parent*. If* $n \equiv 1 \pmod 6$ *or* $n \equiv 5 \pmod 6$*, then* n *is* live *and admits some* $k \geq 1$ *that is admissible.*

Definition 2.6 (Child (reverse Collatz function))**.** *Given a parent* n *and an admissible* $k \geq 1$*, the corresponding* child *is*

$$m \;=\; \frac{2^k n - 1}{3} \quad (\textit{odd}).$$

For a fixed n*, admissible* k *have fixed parity and are exactly*

$$k = k_{\min}(n) + 2\ell, \quad \ell \geq 0,$$

where ℓ *is the* lift index *counting successive admissible exponents above the minimal one. As* k *increases by* $+2$*, the middle-even residue cycles* $10 \to 4 \to 16 \to 10$*; under the fixed gate* $10 \mapsto C_0$*,* $4 \mapsto C_2$*,* $16 \mapsto C_1$*, the children of* n *therefore occur in the deterministic class rotation*

$$C_0 \;\to\; C_2 \;\to\; C_1 \;\to\; C_0 \;\to\; \cdots$$

Definition 2.7 (First admissible child). *For any live odd integer $n \in \mathcal{O}_{\text{live}}$, let $k_{\min}(n) \in \{1, 2\}$ denote its class-determined least admissible exponent. We define*

$$R(n; k_{\min}(n)) \;=\; \frac{2^{\,k_{\min}(n)}\, n - 1}{3},$$

and refer to $R(n; k_{\min})$ as the first admissible child *of n.*

Definition 2.8 (Admissible doubling and child). *Let n be odd. A doubling count $k \geq 1$ is* admissible *if*

$$2^k n \equiv 1 \pmod 3.$$

For any admissible k, the reverse child *is*

$$R(n; k) \;:=\; \frac{2^k n - 1}{3} \;\in \mathbb{N}.$$

The set of admissible k for a fixed odd n has fixed parity (even if $n \equiv 1 \pmod 3$, odd if $n \equiv 2 \pmod 3$), and hence $k \mapsto k + 2$ preserves admissibility.

Definition 2.9 (Terminal and Live Classes). *Let $n \in \mathbb{N}$. The Collatz class of n is defined as:*

$$\begin{cases} C_0 & \text{if } n \equiv 3 \pmod 6 \\ C_1 & \text{if } n \equiv 5 \pmod 6 \\ C_2 & \text{if } n \equiv 1 \pmod 6 \end{cases}$$

Class C_0 is terminal under Collatz iteration; classes C_1 and C_2 are live.

Definition 2.10 (Reset-and-Resume Function). *Given $n \in \mathbb{N}$ odd, define $q := \frac{n-1}{3}$. Then the reset-and-resume transform is:*

$$q_{k+1} = \frac{3q_k + 1}{2^{v_2(3q_k+1)}}$$

where $v_2(x)$ denotes the 2-adic valuation of x. This is the only class-agnostic invariant rule under Collatz iteration.

Definition 2.11 (q-Transform Function). *The class-dependent q-transform for single-generation transitions is defined as:*

$$T_{C_1}(q) = \frac{3q+1}{2}, \quad T_{C_2}(q) = \frac{3q+1}{4}$$

Definition 2.12 (Progression index). *For an odd parent n, the* progression index *t is the integer parameter in the canonical forms*

$$n = 6t + 5 \quad (C_1), \qquad n = 6t + 1 \quad (C_2),$$

with $t \geq 0$. The index t counts the position of n within its mod-6 residue class. In later sections, offsets and ladders are expressed as explicit functions of this progression index.

Definition 2.13 (Admissible parent). *For odd $n \geq 1$, define $k_{\min}(n)$ to be the least positive integer k such that $2^k n \equiv 1 \pmod 3$. If such k exists, set*

$$P(n) \;:=\; R\big(n; k_{\min}(n)\big) \;=\; \frac{2^{\,k_{\min}(n)}\, n - 1}{3}.$$

If $3 \mid n$ we say n is terminating.

Definition 2.14 (Admissible exponents). *For an odd integer n, the set of* admissible exponents *is*

$$K(n) \;:=\; \{\, k \geq 1 : \; 2^{\,k} n \equiv 1 \pmod 3 \,\}.$$

(If $3 \mid n$, then $K(n) = \varnothing$.)

Definition 2.15 (Middle even and gate residue). *For odd m, set*

$$E(m) := 3m + 1, \qquad k := \nu_2\big(E(m)\big) \geq 1, \qquad T(m) := \frac{E(m)}{2^k}$$

so that $T(m)$ is the odd Collatz child. The middle even *is*

$$\widetilde{e}(m) \;:=\; \frac{E(m)}{2^{k-1}} \;=\; 2\,T(m),$$

and its gate residue *is*

$$g(m) \;:=\; \widetilde{e}(m) \pmod{18} \;\in\; \{4, 10, 16\}.$$

Definition 2.16 (Forward odd-to-odd step). *For odd m, let $k_{\max}(m) := \nu_2(3m+1)$ and define*

$$T(m) \;:=\; \frac{3m+1}{2^{\,k_{\max}(m)}} \qquad (\textit{odd}).$$

Definition 2.17 (Least-admissible reverse parent). *For odd m, let $P(m) = \frac{2^k m - 1}{3}$, where $k = 2$ if $m \equiv 1 \pmod 3$ and $k = 1$ if $m \equiv 2 \pmod 3$. Work modulo 18 with live residues $\mathcal{R}_{\text{live}} = \{1, 5, 7, 11, 13, 17\}$ and dead residues $\{3, 9, 15\}$. Write every odd as $m = r + 18t$ with $r \in \mathcal{R}_{\text{live}}$ and $t \in \mathbb{N}_{\geq 0}$.*

Definition 2.18 (Rail). *A* rail *is the vertical affine progression generated from any odd value m by repeated admissible higher lifts. Each lift increases the exponent by $+2$ and applies the transformation*

$$m \;\mapsto\; 4m + 1.$$

Thus the rail through m is

$$m,\; 4m+1,\; 4(4m+1)+1,\; 4^2 m + \frac{4^2-1}{3},\; \ldots$$

Rails represent all values obtained from a fixed parent by higher-k lifts.

Definition 2.19 (Ladders as Dyadic Offset Progressions)**.** *Fix a class* $(c, x) \in \{(1, 5), (2, 1)\}$ *and let* $k = c + 2e$ *be the admissible lift exponent. Every odd integer in this class can be written uniquely as*

$$n = 6t + x, \qquad t \geq 0,$$

where t *is the index of the element within its residue class. Applying the reverse map gives*

$$R(6t + x; k) = 2^{k+1}t + \frac{2^k x - 1}{3}.$$

The ladder at level k *is the arithmetic progression*

$$\mathcal{S}_{c,e} = \left\{ 2^{k+1}t + \frac{2^k x - 1}{3} : t \geq 0 \right\},$$

whose consecutive elements differ by the fixed dyadic offset

$$\Delta_k = 2^{k+1}.$$

Thus a ladder is the ordered progression of parents obtained from all sequential $6t + x$ *inputs under the same admissible exponent* k*.*

3 The Deterministic Residue Framework

This section extends the local residue framework first developed in *A Deterministic Residue Framework for the Collatz Operator at* $q = 3$ [1], together with earlier unpublished notes that identified the mod 9 residue cycle as the source of reverse determinism. The core construction is preserved: admissibility is fixed by residue classes modulo 6, while refinement to mod 9 and its canonical lift to mod 18 determines the child class at each step.

The result is a deterministic lens through which every odd integer is classified and every admissible step is resolved. This local structure now appears explicitly as the microscopic counterpart of the global coverage framework that follows.

3.1 The mod 6 Classification for Odd Integers

All odd integers fall into three residue classes modulo 6:

- **C0:** $n \equiv 3 \pmod 6$ (odd multiples of 3: $3, 9, 15, \ldots$).
  Forward (middle-even identification): $3n + 1 \equiv 10 \pmod{18}$.
  Reverse (admissibility/parity): No admissible k with $2^k n \equiv 1 \pmod 3$ exists, so C_0 has no reverse parent.

- **C1:** $n \equiv 5 \pmod 6$ (two higher than a multiple of 3: $5, 11, 17, \ldots$).
  Forward (middle-even identification): $3n + 1 \equiv 16 \pmod{18}$.
  Reverse (admissibility/parity): $n \equiv 2 \pmod 3$, so admissible k are *odd*. The first admissible is $k = 1$. One doubling gives

$$n \cdot 2^1 \;\equiv\; 4 \pmod 6.$$

Since $k_{\min} = 1$ for C_1, we have $2^{k_{\min}} n \equiv 1 \pmod 3$; subtracting 1 yields a multiple of 3, so the reverse step is an integer. Thus C_1 always resolves after

$$k = k_{\min} + 2\ell = 1 + 2\ell \qquad (\ell \in \mathbb{N}_{\geq 0})$$

- **C2:** $n \equiv 1 \pmod 6$ (two lower than a multiple of 3: $1, 7, 13, \ldots$).
  Forward (middle-even identification): $3n + 1 \equiv 4 \pmod{18}$.
  Reverse (admissibility/parity): $n \equiv 1 \pmod 3$, so admissible k are *even*. The first admissible is $k = 2$, yielding

  $$4n \equiv 1 \pmod 3 \quad \Rightarrow \quad m = \frac{4n-1}{3} \in \mathbb{N}.$$

  Since $k_{\min} = 2$ for C_2, we have $2^{k_{\min}} n \equiv 1 \pmod 3$; subtracting 1 yields a multiple of 3, so the reverse step is an integer. Thus C_2 always resolves after

  $$k = k_{\min} + 2\ell = 2 + 2\ell \qquad (\ell \in \mathbb{N}_{\geq 0})$$

  doublings.

Lemma 3.1 (C0 is terminating under the reverse step)**.** *If $n \equiv 3 \pmod 6$ (i.e., n is an odd multiple of 3), then for every $k \geq 1$,*

$$\frac{2^k n - 1}{3} \notin \mathbb{N}.$$

In particular, the class C0 has no admissible reverse child.

Proof. If $3 \mid n$ then $2^k n \equiv 0 \pmod 3$ for all $k \geq 1$, hence $2^k n - 1 \equiv -1 \equiv 2 \pmod 3$, which is not divisible by 3. □

Interpretation. The mod-6 classification isolates the essential periodic structure of the Collatz map. Every odd integer is congruent to 1, 3, or 5 mod 6, producing three invariant classes. Multiples of 3 (C_0) are terminal because no admissible doubling can satisfy $2^k n \equiv 1 \pmod 3$. The remaining residues 1 and 5 (C_2 and C_1) are live: they alternate under the admissible-exponent rule and generate the entire forward–reverse lattice. Thus the three-class system is not arbitrary—it is the minimal periodic decomposition consistent with both the mod-3 condition and parity.

3.2 K-value Admissibility of the classes

This subsection identifies the admissible k values for each class and demonstrates how parity is determined by the residue of n modulo 3.

Lemma 3.2 (Admissibility parity)**.** *Let n be an odd integer. The congruence*

$$2^k n \equiv 1 \pmod 3$$

has a solution if and only if n is not divisible by 3. Moreover, the residue of n modulo 3 determines the parity of k:

$$n \equiv 1 \pmod 3 \;\Rightarrow\; k \text{ must be even}, \qquad n \equiv 2 \pmod 3 \;\Rightarrow\; k \text{ must be odd.}$$

Once one admissible k exists, every larger k with the same parity is also admissible.

Proof. **C1 admissibility** with $n = 6t + 5$. For C_1 we have $n \equiv 5 \pmod 6$ and $n \equiv 2 \pmod 3$. The admissibility condition is

$$n \cdot 2^{1+2e} - 1 \equiv 0 \pmod 3,$$

i.e.

$$(6t + 5)\, 2^{1+2e} - 1 \equiv 0 \pmod 3.$$

Write $k = 1 + 2e$. Since $2^2 \equiv 1 \pmod 3$,

$$2^k = 2^{1+2e} \equiv 2 \pmod 3.$$

Substitute n:

$$(6t + 5)\, 2 - 1 \equiv 0 \pmod 3.$$

Expand:

$$12t + 10 - 1 \equiv 12t + 9 \equiv 0 \pmod 3.$$

Note:

$$12t \equiv 0 \pmod 3, \qquad 9 \equiv 0 \pmod 3.$$

Therefore,

$$(6t + 5)\, 2^{1+2e} - 1 \equiv 0 \pmod 3$$

holds for all integers t and all $e \geq 0$.

$$\boxed{(6t + 5)\, 2^{1+2e} - 1 \equiv 0 \pmod 3}.$$

This explicitly shows why every odd lift of the form $k = 1 + 2e$ is admissible for C_1.

C2 admissibility with $n = 6t + 1$. For C_2 we have $n \equiv 1 \pmod 6$ and $n \equiv 1 \pmod 3$. The admissibility condition is

$$n \cdot 2^{2+2e} - 1 \equiv 0 \pmod 3,$$

i.e.

$$(6t + 1)\, 2^{2+2e} - 1 \equiv 0 \pmod 3.$$

Write $k = 2 + 2e$. Since $2^2 \equiv 1 \pmod 3$,

$$2^k = 2^{2+2e} \equiv 1 \pmod 3.$$

Substitute n:

$$(6t + 1) \cdot 1 - 1 \equiv 0 \pmod 3.$$

Expand:

$$6t + 1 - 1 \equiv 6t \equiv 0 \pmod 3.$$

Therefore,

$$(6t + 1)\, 2^{2+2e} - 1 \equiv 0 \pmod 3$$

holds for all integers t and all $e \geq 0$.

$$\boxed{(6t + 1)\, 2^{2+2e} - 1 \equiv 0 \pmod 3}.$$

This explicitly shows why every even lift of the form $k = 2 + 2e$ is admissible for C_2.

□

3.3 Mod 18 Gate and its Mod 9 Subclassification

This subsection establishes the deterministic mod 18 *gate that decides the child class of every admissible parent. The residue of the middle-even value after the minimal admissible doubling lands in* $\{4, 10, 16\}$, *and this uniquely determines the class of the first child.*

Lemma 3.3 (Minimal admissible doubling and the mod 18 gate)**.** *List the odd integers mod* 18 *in sequential order and, for each odd* n, *take its first child by the reverse Collatz function and using* k_{min}. *Then the first-child classes follow a repeating nine-step cycle in sequence mod* 3*:*

$$2,\ x,\ 0,\ 0,\ x,\ 2,\ 1,\ x,\ 1,\ \cdots$$

(where x *denotes terminating parents, i.e. multiples of* 3*). In particular, the six odd non-multiples of* 3 *partition into two fixed triads*

$$\{5, 11, 17\} \pmod{18} \quad \textit{and} \quad \{1, 7, 13\} \pmod{18},$$

corresponding to C_1 *and* C_2 *parents, respectively; thus mod* 18 *alone determines the child-class framework.*

Moreover, let $k_{\min}(r)$ *denote the* minimal admissible exponent *for the reverse function*

$$R(n; k) \;=\; \frac{2^k n - 1}{3}.$$

This minimal k *is fixed by the class of* n*:*

$$k_{\min}(r) = \begin{cases} 1, & r \in C_1 = \{5, 11, 17\}, \\ 2, & r \in C_2 = \{1, 7, 13\}. \end{cases}$$

Applying the minimal admissible doubling directly to the residue $r = n \bmod 18$ *gives the deterministic gate*

$$\text{gate}(r) := 2^{k_{\min}(r)}\, r \pmod{18}.$$

Evaluating this for each residue yields the fixed gate assignment

$C_2:$	$r = 1$	$r = 7$	$r = 13$
gate(r)	4	10	16

$C_1:$	$r = 5$	$r = 11$	$r = 17$
gate(r)	10	4	16

.

Thus the minimal admissible doubling maps each odd residue to a unique even gate in $\{4, 10, 16\}$, *refining the mod-*9 *triads to mod-*18 *gates.*

Proof. (i) *Mod-*9 *triad partition.* For odd n, write $r \equiv n \pmod 9$ with $r \in \{0, \pm1, \pm2, \pm4\}$. If $r \equiv 0$ then $3 \mid n$ and the parent is terminating (C_0). When $3 \nmid n$, the residues split by r mod 3 into the two disjoint triads $\{1, 4, 7\}$ and $\{2, 5, 8\}$, which correspond to C_2 and C_1, respectively. The first-child map (apply $3n + 1$ then divide by 2 until odd) permutes elements *within* the appropriate triad and never crosses between them, yielding the stated nine-step cycle.

(ii) *Lift to mod-*18 *gates.* Work modulo 18 and apply the minimal admissible doubling directly to the residue r: for $r \in C_2$ use one factor of 4; for $r \in C_1$ use one factor of 2. This gives

$$1 \mapsto 4, \quad 7 \mapsto 10, \quad 13 \mapsto 16 \quad \text{and} \quad 5 \mapsto 10, \quad 11 \mapsto 4, \quad 17 \mapsto 16,$$

which are precisely the even gates $\{4, 10, 16\}$ claimed. □

Corollary 3.4 (Linear segment pattern 19–35)**.** *Listed are the odd integers n from* 19 *to* 35*. For each n, record its class (mod* 6*), its residue (mod* 9*) and (mod* 18*), the reverse middle-even at the minimal admissible doubling $k_{\min}$ ($k_{\min} = 2$ for C_2, $k_{\min} = 1$ for C_1, none for C_0), and the class of the first child*

$$m = \frac{2^{k_{\min}} n - 1}{3} \quad (\textit{when defined}).$$

n	*class*(n) (mod6)	n mod 18	$(2^{k_{\min}} n)$ mod 18	*first-child class*
19	C_2 (1)	1	4	C_2
21	C_0 (3)	3	–	*none (terminating parent)*
23	C_1 (5)	5	10	C_0
25	C_2 (1)	7	10	C_0
27	C_0 (3)	9	–	*none (terminating parent)*
29	C_1 (5)	11	4	C_2
31	C_2 (1)	13	16	C_1
33	C_0 (3)	15	–	*none (terminating parent)*
35	C_1 (5)	17	16	C_1

Explanation. For each n: determine its class by n mod 6 (C_0: 3, C_1: 5, C_2: 1). If $n \in C_0$, no admissible reverse step exists. If $n \in C_1$ (resp. C_2), take $k_{\min} = 1$ (resp. $k_{\min} = 2$) by admissibility parity. Then use the deterministic gate: $(2^{k_{\min}} n) \bmod 18 \in \{10, 4, 16\}$ with the fixed mapping $10 \mapsto C_0,\ 4 \mapsto C_2,\ 16 \mapsto C_1$. Evaluating these nine cases yields the displayed sequence $2, x, 0, 0, x, 2, 1, x, 1$. This finite segment is a repeating cycle. □

These nine odd residues partition into inadmissible and admissible parents:

$$\underbrace{\{3,9,15\}}_{\textit{inadmissible (terminated parent)}}, \qquad \underbrace{\{5,7\}}_{\textit{first child is } C_0} +10, \qquad \underbrace{\{13,17\}}_{\textit{first child is } C_1} +16, \qquad \underbrace{\{1,11\}}_{\textit{first child is } C_2} +4.$$

Lemma 3.5 (Equidistribution of First-Child Classes)**.** *Across every complete* 18*-residue cycle of odd parents, the first-child classes C_0, C_1, C_2 appear with exact frequency* 1/3 *each.*

Proof. By Corollary 3.4, the nine admissible residues modulo 18 yield the child-class sequence

$$C_2,\ -,\ C_0,\ C_0,\ -,\ C_2,\ C_1,\ -,\ C_1,$$

where dashes denote terminating parents. Each 18-step cycle therefore contains precisely two occurrences of each live class, giving equal frequency 1/3 when restricted to C_0, C_1, C_2. □

Lemma 3.6 (Forward mod-6 lift to mod-18 at the first even)**.** *Let n be odd and define the forward middle-even value $E_f(n) := 3n + 1$. Then the residue of n modulo* 6 *determines $E_f(n)$ modulo* 18 *via*

$$n \equiv 1, 3, 5 \pmod 6 \ \mapsto\ E_f(n) \equiv 4, 10, 16 \pmod{18} \ \textit{respectively.}$$

*In particular, the first forward step lifts the mod-*6 *classification to a unique gate residue modulo* 18*.*

Proof. Write $n \equiv r \pmod 6$ with $r \in \{1, 3, 5\}$. Then $E_f(n) = 3n + 1 \equiv 3r + 1 \pmod{18}$ since $18 = 3 \cdot 6$. Direct evaluation gives

$$3 \cdot 1 + 1 \equiv 4 \text{ (mod 18)}, \quad 3 \cdot 3 + 1 \equiv 10 \text{ (mod 18)}, \quad 3 \cdot 5 + 1 \equiv 16 \text{ (mod 18)},$$

which proves the three implications and the uniqueness of the lifted gate residue. □

Proposition 3.7 (Deterministic child-class decision via mod 18)**.** *In the Reverse Collatz function, and for odd n, the residue of the middle even in $\{4, 10, 16\}$ (mod18) alone determines the child's odd class, both in forward and reverse middle-even. This gives a one-step, local rule independent of trajectory history.*

$$10 \mapsto C_0, \qquad 4 \mapsto C_2, \qquad 16 \mapsto C_1,$$

Existence of a forward–reverse alignment through the middle-even gate.

Lemma 3.8 (Middle-even equivalence mod 18)**.** *If 3 does not divide n, then there exists an admissible $k \geq 1$ such that*

$$2^k n \equiv 3n + 1 \pmod{18}.$$

Proof. Forward side (mod 6 lifted to mod 18). For odd n, the forward middle-even value is $E_f(n) = 3n + 1$. Reducing n modulo 6 and multiplying by 3 lifts the residue to mod 18:

$$n \equiv 1, 3, 5 \pmod 6 \quad \Longrightarrow \quad E_f(n) \equiv 4, 10, 16 \pmod{18},$$

so $E_f(n)$ always lies in $\{4, 10, 16\}$ (mod18).

Reverse side (mod 18 determinism). For odd n not divisible by 3, the residue $n \bmod 9$, together with the admissible parity of $k_{\min}$ (even if $n \equiv 1 \pmod 3$, odd if $n \equiv 2 \pmod 3$), selects exactly one of the two triads of units modulo 9:

$$\{1, 7, 13\} \quad (\text{even } k), \qquad \{5, 11, 17\} \quad (\text{odd } k).$$

Applying $2^{k_{\min}}$ places n into the middle-even value that belongs to the nine-step cycle of Corollary 3.4. That middle-even value is already one of $\{10, 4, 16\}$ (mod 18), the forward gates. □

3.4 Microcycles and lifted k with tables

Lemma 3.9 (Rotation under $k \mapsto k + 2$ in mod 18)**.** *If k is admissible for odd n ($2^k n \equiv 1 \pmod 3$), then*

$$E_r(n, k) = 2^k n \equiv 10,\ 4,\ 16 \pmod{18}.$$

Moreover $E_r(n, k + 2) = 4\, E_r(n, k)$, and hence

$$10 \xrightarrow{+2} 4 \xrightarrow{+2} 16 \xrightarrow{+2} 10 \quad (\text{mod18}).$$

Proof. Admissible $E_r(n, k)$ are even and $1 \pmod 3$, so only $10, 4, 16$ occur modulo 18. For admissible k, $E_r(n, k+2) = 2^{k+2} n = 4\, E_r(n, k)$; computing mod 18 gives $4 \cdot 10 \equiv 4$, $4 \cdot 4 \equiv 16$, $4 \cdot 16 \equiv 10$, which establishes the 3-cycle. □

Microcycles: function and reason. Fix a live odd parent n not divisible by 3. For the Reverse Collatz Function, all admissible reverse doublings for n share the same parity (by admissibility parity), so from the minimal admissible count $k_{\min}$ we may advance by steps of 2: $k_{\min}$, $k_{\min}+2$, $k_{\min}+4, \ldots$. By Lemma 3.9, each +2 step multiplies the reverse middle-even by 4 modulo 18, sending $10 \mapsto 4 \mapsto 16 \mapsto 10$ and hence rotating the child classes $C_0 \mapsto C_2 \mapsto C_1 \mapsto C_0$.

$$E_r(n, k_{\min}) \bmod 18 \;\in\; \{10, 4, 16\} \quad \Longrightarrow \quad E_r(n, k_{\min}+2) \equiv 4 \cdot E_r(n, k_{\min}) \pmod{18},$$

$$E_r(n, k_{\min}+4) \equiv 4 \cdot E_r(n, k_{\min}+2) \pmod{18},$$

cycling through $10 \to 4 \to 16 \to 10 \pmod{18}$. By the common mod-18 gate (Lemma 3.8), these three middle-even classes deterministically select the child odd classes C_0, C_2, C_1, in that order. Thus every fixed parent n generates a *k-lifted microcycle* of children: (C_0, C_2, C_1), in cyclic order beginning with the first admissible child, repeating every three $k_{min} + 2$ steps. Moreover, by the forward–reverse middle-even equivalence (Lemma 3.8), there exists an admissible k for which $E_r(n, k) \equiv E_f(n) = 3n + 1 \pmod{18}$, so the reverse microcycle is aligned with the residue one sees on the forward side.

To display this mechanism explicitly, we present two parallel tables: *(i) the integer view*, which lists specific n and its children at each admissible lift, and *(ii) the residue view*, which reduces n to $r \equiv n \pmod{18}$. Both views coincide in the mod-18 column and the resulting child class.

Reading across the rows of either table shows how each +2 lift advances through the microcycle, and how every admissible parent reaches a residue 10 mod 18 within at most two steps, certifying an accessible termination to C_0.

Example $n = 25$ (reverse step, even k; here n mod 18 = 7, n mod 6 = 1 $\Rightarrow C_2$):

n	k (even)	$2^k n$	$(2^k n)$ mod 18	$\frac{2^k n - 1}{3}$	$\left(\frac{2^k n-1}{3}\right)$ mod 6	class
25	2	100	10	33	3	C_0
25	4	400	4	133	1	C_2
25	6	1600	16	533	5	C_1
25	8	6400	10	2133	3	C_0
25	10	25600	4	8533	1	C_2
25	12	102400	16	34133	5	C_1

r	k (even)	$2^k r$	$(2^k r)$ mod 18	$\frac{2^k r - 1}{3}$	$\left(\frac{2^k r-1}{3}\right)$ mod 6	class
7	2	28	10	9	3	C_0
7	4	112	4	37	1	C_2
7	6	448	16	149	5	C_1
7	8	1792	10	597	3	C_0
7	10	7168	4	2389	1	C_2
7	12	28672	16	9557	5	C_1

Example $n = 29$ (reverse step, odd k; here $n \bmod 18 = 11$, $n \bmod 6 = 5 \Rightarrow C_1$):

n	k (odd)	$2^k n$	$(2^k n) \bmod 18$	$\frac{2^k n - 1}{3}$	$\left(\frac{2^k n-1}{3}\right) \bmod 6$	class
29	1	58	4	19	1	C_2
29	3	232	16	77	5	C_1
29	5	928	10	309	3	C_0
29	7	3712	4	1237	1	C_2
29	9	14848	16	4949	5	C_1
29	11	59392	10	19797	3	C_0

r	k (odd)	$2^k r$	$(2^k r) \bmod 18$	$\frac{2^k r - 1}{3}$	$\left(\frac{2^k r-1}{3}\right) \bmod 6$	(class)
11	1	22	4	7	1	C_2
11	3	88	16	29	5	C_1
11	5	352	10	117	3	C_0
11	7	1408	4	469	1	C_2
11	9	5632	16	1877	5	C_1
11	11	22528	10	7509	3	C_0

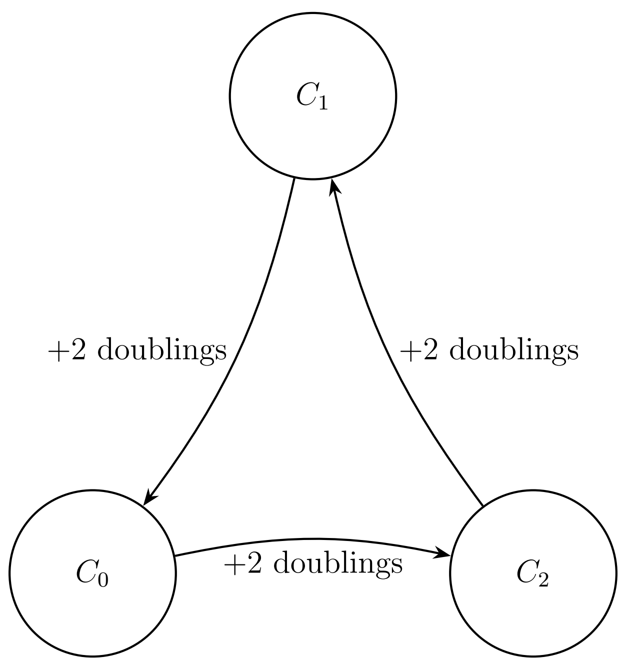

Figure 1: Even-k rotation of child classes through the mod-18 gate. Each increment of two in k multiplies the middle-even residue by 4, producing the cycle $10 \to 4 \to 16 \to 10$. These residues correspond deterministically to classes $C_0 \to C_2 \to C_1 \to C_0$ (with $10 \mapsto C_0$, $4 \mapsto C_2$, $16 \mapsto C_1$). Hence the child class rotates in the fixed order $C_0 \to C_2 \to C_1 \to C_0$, making the terminating class C_0 periodically available alongside the live classes.

3.5 Mod 54 Refinement: Fixing the Child Residue

The mod-18 gate (Lemma 3.3, Proposition 3.7) determines the *child class*. Refining the lens to mod 54 determines, already at the first admissible reverse step, the child's *odd residue modulo* 18.

Triad map (mod 54). Write every live odd n as

$$n = 54m+r, \qquad r \in \{1,5,7,11,13,17,19,23,25,29,31,35,37,41,43,47,49,53\}, \qquad m \in \mathbb{N}_{\geq 0}.$$

Set $q_{r54} \equiv m \pmod 3 \in \{0,1,2\}$. For each $r_{18} \in \{1,5,7,11,13,17\}$, the corresponding residues in mod 54 are

$$r_{54} \in \{\, r_{18},\ r_{18}+18,\ r_{18}+36\,\}.$$

Define the lifted triads $\mathcal{T}_{54}(r_{54}) = (t_{r,0}, t_{r,1}, t_{r,2})$ by

r_{54}	$t_{r,0}$	$t_{r,1}$	$t_{r,2}$
$1, 19, 37$	1	7	13
$11, 29, 47$	7	1	13
$13, 31, 49$	17	5	11
$17, 35, 53$	11	5	17
$5, 23, 41$	3	15	9
$7, 25, 43$	9	15	3

Each lifted triad row follows the same deterministic pattern as the mod 18 table. The indexing variable $q_{r54} = m \bmod 3$ plays the same role as q_{r18} in selecting the correct column of the triad. Rows for $r_{54} \in \{1,11,13,17\}$ are in C_2 or C_1, and $\{5,7\}$ remain in C_0.

Lemma 3.10 (Mod 54 refinement fixes the child residue)**.** *Let*

$$n = 54m+r_{54}, \qquad r_{54} \in \{1,5,7,11,13,17,19,23,25,29,31,35,37,41,43,47,49,53\}, \qquad m \in \mathbb{N}_{\geq 0}.$$

Set $j \equiv m \pmod 3$. Then the first admissible reverse child of n has odd residue

$$\left(\frac{2^{k_{\min}(n)}\, n - 1}{3}\right) \equiv t_{r_{54},j} \pmod{18},$$

where $t_{r_{54},j}$ is determined by the lifted triad $\mathcal{T}_{54}(r_{54})$. Equivalently, the pair (r_{54}, j) uniquely determines the child's odd residue modulo 18*.*

Proof sketch. By Lemma 3.2, the minimal admissible exponent $k_{\min}(n)$ is odd for $n \in C_1$ and even for $n \in C_2$. The mod 18 structure (Lemma 3.3) partitions the six live residues into deterministic triads, and the admissibility parity lifts each residue canonically to its gate (Proposition 3.7).

Passing to mod 54, each r_{18} splits into three residues

$$r_{54} \in \{\, r_{18},\ r_{18}+18,\ r_{18}+36\,\},$$

and the index $j = m \bmod 3$ selects one of the three columns of the lifted triad table $\mathcal{T}_{54}$. Evaluating the first admissible reverse step for $j = 0,1,2$ within each r_{54} reproduces exactly the triad outputs listed in Table 1. Thus (r_{54}, j) completely determines the child residue modulo 18. □

Compact 54-row table. Because $n \bmod 54$ is completely determined by (r, q), the mapping

$$n \bmod 54 \longmapsto \big(\text{child odd residue mod } 18\big)$$

is obtained by grouping the 27 live residues mod 54 into six blocks by r and subdividing each block by $q \in \{0, 1, 2\}$. For example, the block $r = 1$ contributes residues

$$\{1,\ 19,\ 37\} \pmod{54} \quad \rightsquigarrow \quad \{1, 7, 13\} \pmod{18}$$

in the order $q = 0, 1, 2$. Explicitly listing all odd $1 \leq n \leq 54$ produces a 54-entry table in which each row records (n, n mod 18, q mod 3, child mod 18). We defer the full table to Table 1 below for readability.

Table 1: Mod 54 refinement: for odd $n \in [1, 53]$, the residue $r \equiv n \pmod{18}$ and the first child's class and residue (mod18).

n (mod54)	r (mod18)	parent class	first child class	first child residue (mod18)
1	1	C2	C2	1
3	3	C0	—	—
5	5	C1	C0	3
7	7	C2	C0	9
9	9	C0	—	—
11	11	C1	C2	7
13	13	C2	C1	17
15	15	C0	—	—
17	17	C1	C1	11
19	1	C2	C2	7
21	3	C0	—	—
23	5	C1	C0	15
25	7	C2	C0	15
27	9	C0	—	—
29	11	C1	C2	1
31	13	C2	C1	5
33	15	C0	—	—
35	17	C1	C1	5
37	1	C2	C2	13
39	3	C0	—	—
41	5	C1	C0	9
43	7	C2	C0	3
45	9	C0	—	—
47	11	C1	C2	13
49	13	C2	C1	11
51	15	C0	—	—
53	17	C1	C1	17

Corollary 3.11 (Periodicity of the Mod 54 Child Mapping)**.** *Let n be an odd integer with*

$$n = 18q + r, \qquad r \in \{1, 5, 7, 11, 13, 17\}, \qquad q \equiv q \pmod 3.$$

Let $c(n)$ denote the residue modulo 18 of the first admissible reverse child of n,

$$c(n) \;:=\; \left(\frac{2^{k_{\min}(n)}\, n - 1}{3}\right) \bmod 18.$$

Then for every integer $m \geq 0$ (period index),

$$c(n + 54m) \;=\; c(n).$$

Equivalently, the mapping

$$n \bmod 54 \;\longmapsto\; c(n)$$

is periodic with fundamental period 54. In particular, the table of first-child residues for odd $n \in [1, 53]$ repeats identically on each interval $[1 + 54m,\ 53 + 54m]$.

Interpretation. The refinement to modulus 54 resolves the residual ambiguity left by the mod-18 gate. At mod-18, each live residue determines only the *class* of its child; lifting to mod-54 records the phase of the quotient $q \bmod 3$, which fixes the child's exact odd residue mod 18. The resulting triads $\mathcal{T}_{54}(r_{54})$ show that every parent residue r_{54} generates three distinct child residues, one for each phase position. Because these triads repeat with period 54, the entire reverse map becomes periodic at that modular scale. This periodicity demonstrates that the residue–phase system is finite and deterministic: each pair $(r_{54}, q \bmod 3)$ has one unique successor, and every possible parent–child relationship repeats identically on successive 54-blocks.

Lemma 3.12 (Affine reverse update law)**.** *Let $n = 18q + r$ with $r \in \{1, 5, 7, 11, 13, 17\}$ and $q \in \mathbb{N}_0$, and set*

$$k_{\min}(r) = \begin{cases} 2, & r \in C_2 = \{1, 7, 13\}, \\ 1, & r \in C_1 = \{5, 11, 17\}. \end{cases}$$

Define

$$m = R(n; k_{\min}(r)) = \frac{2^{k_{\min}(r)}(18q + r) - 1}{3} = A_r\, q + B_r, \qquad A_r = \frac{2^{k_{\min}(r)} \cdot 18}{3},\ \ B_r = \frac{2^{k_{\min}(r)} r - 1}{3}.$$

Then $m \in \mathbb{N}$ and the single–step update $(r, q) \mapsto (r', q')$ is given by

$$r' = m \bmod 18, \qquad q' = \left\lfloor \frac{m}{18} \right\rfloor,$$

with the following explicit formulas:

1. *(Slope and intercept)*

$$r \in C_1 :\ A_r = 12,\ B_r = \tfrac{2r-1}{3} \in \{3, 7, 11...\}; \qquad r \in C_2 :\ A_r = 24,\ B_r = \tfrac{4r-1}{3} \in \{1, 9, 17...\}.$$

2. *(Residue update by phase)*

$$\boxed{\begin{array}{lll} r \in C_1: & r' \equiv B_r - 6\,(q \bmod 3) & \pmod{18}, \\ r \in C_2: & r' \equiv B_r + 6\,(q \bmod 3) & \pmod{18}. \end{array}}$$

3. *(Quotient update)*

$$q' = \begin{cases} \left\lfloor \dfrac{12q}{18} \right\rfloor = \left\lfloor \dfrac{2}{3}q \right\rfloor, & r \in C_1, \\ \left\lfloor \dfrac{24q}{18} \right\rfloor = \left\lfloor \dfrac{4}{3}q \right\rfloor, & r \in C_2. \end{cases}$$

Consequently, the pair $(r, q \bmod 3)$ *uniquely determines* $r' \bmod 18$*, and the next phase is* $q' \bmod 3$ *computed from the affine form* $m = A_r q + B_r$*.*

Corollary 3.13 (Finite Residue–Phase Automaton)**.** *For each step of the reverse map defined by*

$$F : (r, q \bmod 3) \mapsto (r', q' \bmod 3),$$

the image (r', q') *depends only on* $(r, q \bmod 3)$ *through the valuation of* $3n + 1$*. The quotient component evolves under the induced transformation*

$$q' \bmod 3 = \left\lfloor \frac{2^{k_{\min}(r)}(18q + r) - 1}{54} \right\rfloor \bmod 3,$$

and defines a finite deterministic automaton on the space $\{(r, q \bmod 3)\}$*. The sequence* $\{F_t\}$ *obtained by successive iterations remains bounded within this finite set, generating locally deterministic residue–phase transitions.*

Lemma 3.14 (Residue–Phase Transition and Reset–Resume Law)**.** *Let* $n = 18q + r$ *with* $r \in \{1, 5, 7, 11, 13, 17\}$ *and* $m = R(n; k_{\min}(r))$ *as above. Then the following properties hold:*

1. *For fixed* r*, as* q *varies modulo* 3*, the residues* $m \bmod 18$ *occupy three distinct elements of* $\{1, 3, 5, 7, 9, 11, 13, 15, 17\}$ *corresponding to the classes* C_0, C_1, C_2*.*

2. *The order of appearance of these residues is determined by* r *and the parity of* $k_{\min}(r)$*, defining a locally unique orientation.*

3. *For each iteration, the next phase and residue* $(r', q' \bmod 3)$ *are re-evaluated from the resulting* m*, establishing a reset and resume transition of the form*

$$(r, q \bmod 3) \mapsto (r', q' \bmod 3),$$

where $r' = m \bmod 18$ *and* $q' = \lfloor m/18 \rfloor$*.*

The residue phase system thereby forms a finite deterministic automaton with terminal residues $C_0 = \{3, 9, 15\}$*, transitional residues* $\{5, 7\}$ *mapping into* C_0*, and active residues* $\{1, 11, 13, 17\}$ *forming the lattice* $\{C_2 \to C_2,\ C_2 \to C_1,\ C_1 \to C_2,\ C_1 \to C_1\}$*.*

r	$k_{\min}(r)$	$B_r = \dfrac{2^{k_{\min}(r)}r - 1}{3}$	$\sigma(r)$	(m mod 18) for q mod $3 = 0, 1, 2$
1	2	1	+1	(1, 7, 13)
7	2	9	+1	(9, 15, 3)
13	2	17	+1	(17, 5, 11)
5	1	3	−1	(3, 15, 9)
11	1	7	−1	(7, 1, 13)
17	1	11	−1	(11, 5, 17)

Table 2: Residue classes, minimal exponents, orientation signs, and resulting triads (m mod 18) for each live residue r.

Interpretation. The affine reverse update law converts the inverse Collatz step into a linear rule on the quotient–residue plane. For each live residue r, the minimal admissible exponent $k_{\min}(r)$ fixes the slope A_r and intercept B_r of an affine map $m = A_r q + B_r$. The modulus 18 confines all results to nine possible odd residues, and the quotient modulus 3 serves as a rotating phase selector. Hence every pair $(r, q \bmod 3)$ specifies a unique successor $(r', q' \bmod 3)$.

Geometrically, the system behaves as a finite automaton of six residue rows ($r \in \{1, 5, 7, 11, 13, 17\}$) and three phase columns ($q \bmod 3$). The "reset–resume" rule means that after each reverse step, the new residue and phase become the parameters of the next affine map. This continual reassignment makes the process locally deterministic but globally adaptive: the governing equation changes with each step while remaining finite. Terminal residues in $C_0 = \{3, 9, 15\}$ close the automaton, ensuring every orbit eventually reaches a fixed point of the system.

Theorem 3.15 (Global Determinism and Finite Termination of the Reverse Automaton)**.** *Let (r_t, q_t) denote the residue and quotient at step t, and define*

$$n_t = 18q_t + r_t, \qquad m_t = \frac{2^{k_{\min}(r_t)} n_t - 1}{3}, \qquad r_{t+1} = m_t \bmod 18, \qquad q_{t+1} = \left\lfloor \frac{m_t}{18} \right\rfloor.$$

Then:

1. *For each step, $(r_t, q_t \bmod 3)$ uniquely determines $(r_{t+1}, \mathit{class}(r_{t+1}))$, forming a finite deterministic mapping.*

2. *The transition structure satisfies*

   $$7, 5 \to C_0, \qquad 1, 13, 11, 17 \to \{C_1, C_2\},$$

   producing the four active transition types $\{C_2 \to C_2,\, C_2 \to C_1,\, C_1 \to C_2,\, C_1 \to C_1\}$.

3. *The system evolves through successive local maps*

   $$F_t : (r_t, q_t \bmod 3) \mapsto (r_{t+1}, q_{t+1} \bmod 3),$$

   generating a finite deterministic sequence in the residue phase space.

4. Each active transition ultimately reaches a terminal residue in C_0 within finitely many steps. The mapping admits no infinite nonterminal orbit.

Hence the reverse Collatz dynamics on odd integers forms a finite, locally deterministic reset and resume automaton whose transitions are governed by residue class and phase position at each step.

3.6 Bounded Corridor Dynamics at Fixed Residues

Among the six live residues modulo 18, only

$$r \in \{1, 17\}$$

have the special property that their first admissible reverse child under $k_{\min}$ remains in the same residue class. This follows directly from the triadic structure established in Subsection 3.3: all other live residues transition immediately to a different residue upon the first admissible lift, whereas $r = 1$ and $r = 17$ alone form self-contained local corridors under forward iteration.

Because these two residues can map to themselves under $k_{\min}$, their forward dynamics admit chains of arbitrary length determined solely by arithmetic properties of the phase index q. For $r = 1$, the forward map contracts by a factor of $\frac{3}{4}$ until the 2-power in q is exhausted. For $r = 17$, the forward map expands by $\frac{3}{2}$ for exactly $\nu_2(q_0 + 1)$ steps, consuming one factor of 2 per iteration.

The results in the following subsections establish the precise structure and length of these corridors: - $r = 1$ admits contraction chains controlled by divisibility of q. - $r = 17$ admits expansion chains controlled by the 2-adic valuation of $q + 1$.

These two cases are the only local residue dynamics that can persist beyond a single step under $k_{\min}$, and their exhaustion determines the maximal extent of fixed-residue behavior in the entire system.

Reverse map at $r = 1$. Let

$$n = 18q + 1.$$

$$3n + 1 = 54q + 4 = 2\,(27q + 2).$$

If q is divisible by 4, then

$$27q + 2 \equiv 2 \pmod 4 \quad \Rightarrow \quad \nu_2(27q + 2) = 1,$$

$$k_{\max} = 1 + 1 = 2.$$

The forward update is then

$$n' = \frac{54q + 4}{2^2} = \frac{27}{2}\,q + 1.$$

Since we only care about the q-level:

$$q' = \frac{n' - 1}{18} = \frac{\frac{27}{2}q}{18} = \frac{3}{4}\,q.$$

$$\boxed{q' = \frac{3}{4}\,q}.$$

This shows that, as long as q remains divisible by 4, the forward map strictly scales q by a factor of $\frac{3}{4}$ without changing the residue class $r = 1$. *The descent in q continues until the 2-adic factor is exhausted, at which point the residue transition occurs.*

Reverse map at $r = 17$. Let

$$n = 18q + 17.$$

Then

$$3n + 1 = 54q + 52 = 2\,(27q + 26).$$

If q is odd (i.e. $q \equiv 1 \pmod 2$), then $27q + 26$ is odd, so

$$\nu_2(27q + 26) = 0 \quad \Rightarrow \quad k_{\max} = 1.$$

The forward update is therefore

$$n' = \frac{54q + 52}{2} = 27q + 26.$$

Writing $n' = 18q' + r'$ gives

$$27q + 26 \;=\; 18\Big(\frac{3q+1}{2}\Big) + 17,$$

so $r' = 17$ and

$$q' = \frac{3q+1}{2}.$$

$$\boxed{q' = \frac{3q+1}{2}} \qquad \text{(valid exactly when } q \text{ is odd).}$$

This map preserves the residue $r = 17$ precisely while q remains odd. Rewriting the recurrence,

$$q_{t+1} = \frac{3q_t + 1}{2} \quad \Longleftrightarrow \quad q_{t+1} + 1 = \frac{3}{2}\,(q_t + 1),$$

gives the explicit evolution

$$q_t + 1 = \Big(\frac{3}{2}\Big)^t (q_0 + 1) = 3^t\,2^{\,\nu_2(q_0+1)-t}.$$

Hence the number of consecutive $r = 17$ steps is determined entirely by the 2-adic valuation of $q_0 + 1$:

$$\boxed{e = \nu_2(q_0 + 1)}.$$

Remark 3.16. *If $q_0 + 1$ is a pure power of 2, the corridor length equals that power's exponent exactly. If it contains an odd factor $u > 1$, the corridor length still equals e, and the odd factor merely remains as a cofactor during the valid steps. Thus the run length for $r = 17$ is governed entirely by the 2-adic valuation of $q_0 + 1$ and not by any fixed external bound.*

Together with the $r = 1$ case, this establishes explicit local corridor dynamics: the $r = 1$ map contracts by a factor $\frac{3}{4}$ until powers of 2 are exhausted, while the $r = 17$ map expands by $\frac{3}{2}$ for exactly e steps, with e determined directly by the factorization of $q_0 + 1$.

Lemma 3.17 (Higher admissible lifts are strictly ascending and rotate the gate)**.** *Fix a live odd parent n and let $k_{\min} \in \{1, 2\}$ be its minimal admissible exponent (determined by class). For each $t \geq 0$ define the t-th admissible lift and reverse child by*

$$k_t := k_{\min} + 2t, \qquad m_t := R(n; k_t) = \frac{2^{k_t} n - 1}{3}.$$

Then:

(a) ***Strict ascent in the reverse value.*** *The sequence $(m_t)_{t \geq 0}$ is strictly increasing, with the exact increment*

$$m_{t+1} - m_t \;=\; \frac{2^{k_t+2} n - 1}{3} - \frac{2^{k_t} n - 1}{3} \;=\; 2^{k_t} n \;>\; 0.$$

Equivalently,

$$m_t \;=\; \frac{2^{k_{\min}}}{3}\, 4^t\, n \;-\; \frac{1}{3},$$

so m_t grows geometrically in t.

(b) ***Gate rotation (class rotation).*** *The associated reverse middle-even residues rotate deterministically:*

$$E_r(n, k_t) = 2^{k_t} n \equiv 10,\ 4,\ 16 \pmod{18} \quad \textit{with} \quad E_r(n, k_{t+1}) \equiv 4\, E_r(n, k_t) \pmod{18},$$

yielding the cycle $10 \to 4 \to 16 \to 10$ (Lemma 3.9). Consequently the child class rotates $C_0 \to C_2 \to C_1 \to C_0$.

(c) ***Higher lifts are higher transformations.*** *Each increment $t \mapsto t+1$ multiplies the affine scaling factor by 4 (from $\frac{2^{k_t}}{3}$ to $\frac{2^{k_t+2}}{3}$) while preserving the constant drift $-\frac{1}{3}$. Thus every higher admissible lift is a strictly larger affine transform on n, independent of the gate rotation.*

Proof. (a) Compute directly:

$$m_{t+1} - m_t = \frac{2^{k_t+2} n - 1}{3} - \frac{2^{k_t} n - 1}{3} = 2^{k_t} n > 0,$$

so (m_t) is strictly increasing. The closed form follows from $k_t = k_{\min} + 2t$.

(b) This is Lemma 3.9: for admissible k, $E_r(n, k) \equiv 10, 4, 16 \pmod{18}$ and $E_r(n, k+2) \equiv 4E_r(n, k) \pmod{18}$, producing the stated rotation and class cycle.

(c) From $R(n; k) = \frac{2^k}{3} n - \frac{1}{3}$, replacing k by $k+2$ multiplies the linear coefficient by 4 and leaves the drift unchanged, so the transform strictly enlarges the image while the residue gate rotates as in (b). □

Interpretation. Only the residues $r = 1$ and $r = 17$ form self-contained "corridors" in the residue–phase system. All other live residues immediately transition to a different class after one admissible lift. Within these two corridors the forward dynamics are governed purely by 2-adic properties of the quotient variable q.

For $r = 1$, the forward map contracts q by a factor of $\frac{3}{4}$ as long as q remains divisible by 4. Each iteration removes one factor of 2, so the chain length equals the 2-adic valuation of q. For $r = 17$, the forward map expands by $\frac{3}{2}$ while q is odd, and the number of valid steps is exactly $\nu_2(q_0 + 1)$. Thus the persistence of each corridor is determined entirely by local 2-adic content, not by any external bound.

Beyond these corridors, higher admissible lifts always increase the reverse value and rotate the middle-even gate through $10 \to 4 \to 16$. Each lift multiplies the affine scale by 4 while preserving the constant drift, so the sequence of lifts is a strictly ascending geometric rail. Together these facts show that fixed-residue behavior is finite and bounded, and that all non-terminal paths ultimately exit their local corridors to join the global terminating flow.

4 Consequences of Lens Refinement, Finite Reverse Lifespan, and Forward Convergence

In this section all integers are odd and positive. We retain the classes

$$\mathrm{C0} = \{3, 9, 15\} \pmod{18}, \quad \mathrm{C1} = \{5, 11, 17\} \pmod{18}, \quad \mathrm{C2} = \{1, 7, 13\} \pmod{18},$$

the boundary residues $5, 7 \pmod{18}$, and the live residues $\{1, 11, 13, 17\} \pmod{18}$. We also keep $F(\cdot)$, $n'(\cdot)$, and $k_{\min}(\cdot)$ from the earlier setup.

4.1 Standing conventions and phase

Every odd n is written uniquely as

$$n = 18q + r, \qquad r \in \{1, 3, 5, 7, 9, 11, 13, 15, 17\},\ q \in \mathbb{N}_{\geq 0}.$$

We call r the residue of n and define the *phase*

$$\phi(n) := q \pmod 3 \in \{0, 1, 2\}.$$

4.2 One-step reverse lens under $k_{\min}$: triads and boundary

Define the minimal reverse step

$$n'(n) = \frac{2^{k_{\min}(n)}\, n - 1}{3}, \qquad k_{\min}(n) = \begin{cases} 1, & n \equiv 2 \pmod 3 \ \ (\mathrm{C1}), \\ 2, & n \equiv 1 \pmod 3 \ \ (\mathrm{C2}), \end{cases}$$

with $n'(n)$ required odd. For a fixed $r \in \{1, 5, 7, 11, 13, 17\}$ set

$$T(r) := \left\{ n'(18q + r) \pmod{18} :\ q \pmod 3 \in \{0, 1, 2\} \right\}.$$

Lemma 4.1 (Triads and boundary presence)**.** *For each live residue r, the set $T(r)$ has exactly three elements and forms a triad. Moreover:*

- *If $r \in \{5, 7\}$, then $T(r) \subseteq \mathrm{C0}$.*
- *If $r \in \{1, 11, 13, 17\}$, then $T(r)$ contains at least one boundary residue (5 or 7 mod 18), and the other elements lie in $\{1, 11, 13, 17\}$.*

Proof. Reduce $n'(18q+r)$ modulo 18; dependence is only on r and $q \pmod 3$, giving $|T(r)| = 3$ and the stated boundary structure by direct casework. □

Lemma 4.2 (C0 is reverse-terminal)**.** *If $p \in \mathrm{C0}$, then $n'(p)$ is not an odd integer.*

Proof. If $3 \mid p$, then $(2^k p - 1)/3 \notin \mathbb{N}$ for all $k \geq 1$. □

4.3 Residue rotation law

Write $n = 18q + r$ with phase $j = \phi(n) = q \pmod 3$ and let $k_{\min}(r) \in \{1, 2\}$ be the class-determined exponent. Set

$$A_r = \begin{cases} 12, & r \in \mathrm{C1}, \\ 24, & r \in \mathrm{C2}, \end{cases} \qquad B_r = \frac{2^{k_{\min}(r)} r - 1}{3}, \qquad \sigma(r) = \begin{cases} -1, & r \in \mathrm{C1}, \\ +1, & r \in \mathrm{C2}. \end{cases}$$

Then the minimal child $m = n'(n)$ satisfies

$$m \equiv B_r + 6\,\sigma(r)\,j \pmod{18}, \qquad q' = \left\lfloor \frac{A_r q + B_r}{18} \right\rfloor,$$

so the residue advances by a constant step ± 6 inside a fixed triad (sign by class), while the new phase $\phi(n') = q' \pmod 3$ is obtained from the affine quotient. Consequently the pair $(r, \phi(n))$ uniquely determines $m \pmod{18}$.

4.3.1 Generational residue–phase map and finiteness

Define the local update

$$F : (r, \phi) \longmapsto \big(r', \phi'\big), \quad r' \equiv n'(18q + r) \pmod{18}, \quad \phi' = q' \pmod 3, \quad q' = \left\lfloor \frac{n'}{18} \right\rfloor.$$

This yields a finite, locally deterministic automaton on the space $\{(r, \phi) : r \in \{1, 5, 7, 11, 13, 17\},\ \phi \in \{0, 1, 2\}\}$ with terminal sink C0.

Interpretation. The residue rotation law establishes that every live residue r advances within a closed triad by a fixed modular step of ± 6. This motion is cyclic, but not self-sustaining indefinitely: each triad contains at least one boundary residue (either 5 or 7 mod 18) whose next image lies in the terminal set $\mathrm{C0} = \{3, 9, 15\}$. Thus, although the rotation within a class appears periodic, the presence of these boundary residues ensures that repeated application of the map cannot cycle endlessly within C1 or C2.

When viewed on the full residue–phase grid (r,ϕ), the update law $F:(r,\phi)\mapsto(r',\phi')$ forms a finite directed graph in which each vertex has a single outgoing edge. Every orbit therefore follows a deterministic path through a bounded set of 18 states. Because at least one state in every rotation chain transitions to C0, all paths must eventually reach a terminal residue and halt. The rotation law therefore provides the local mechanism by which the global map attains finite convergence.

Theorem 4.3 (Finite local dynamics)**.** *For each step, (r_t,ϕ_t) uniquely determines $(r_{t+1},\mathrm{class}(r_{t+1}))$. Every nonterminal transition type lies among $\{\mathrm{C2}\to\mathrm{C2},\ \mathrm{C2}\to\mathrm{C1},\ \mathrm{C1}\to\mathrm{C2},\ \mathrm{C1}\to\mathrm{C1}\}$, and every trajectory in this finite automaton reaches a terminal residue in* C0 *in finitely many steps.*

4.3.2 Lift microcycles and guaranteed boundary access

For a fixed live parent n, all admissible exponents have fixed parity; lifts $k=k_{\min}+2t$ rotate the middle-even residue by a factor 4 (mod 18):

$$10\ \xrightarrow{+2}\ 4\ \xrightarrow{+2}\ 16\ \xrightarrow{+2}\ 10 \qquad (\mathrm{mod}\ 18),$$

so the child classes rotate C0 → C2 → C1 → C0. In particular, within at most two lifts the gate 10 (mod 18) is attained, making C0 accessible.

4.3.3 Mod-54 refinement: fixing the child residue

Refining to modulus 54 splits each live residue r (mod 18) into three residues $r, r+18, r+36$; the index m (mod 3) selects the column of a lifted triad that *already* fixes the child's odd residue modulo 18 at the first admissible reverse step. Thus (n (mod 54)) determines the child residue.

4.4 The $n=1$ self-loop

Remark 4.4 (The trivial self-loop and phase stability)**.** *The integer $n=1$ is the unique odd fixed point of the odd-to-odd map: $T(1)=(3\cdot 1+1)/2^{\nu_2(4)}=1$. In the 18-lens we have $1=18\cdot 0+1$, so $\phi(1)=0$ and both residue and phase remain unchanged. On the reverse side, the minimal lift for $r=1$ is $k_{\min}=2$, and $R(1;2)=(4\cdot 1-1)/3=1$. Hence $n=1$ is the only state that self-loops while staying phase-stable at every lens; all other live residues either change residue at the first minimal step or exhaust their corridor in finitely many steps.*

Corollary 4.5 (Forward convergence)**.** *The forward odd-to-odd step*

$$T(n)=\frac{3n+1}{2^{k_{\max}(n)}}, \qquad k_{\max}(n)=\nu_2(3n+1),$$

is unique and edge-aligned with a reverse admissible step (middle-even equivalence). Hence each forward trajectory is a single, non-branching chain that must terminate at 1.

4.5 Affine Arithmetic Decomposition

Lemma 4.6 (Affine form and accumulated drift)**.** ***Affine form.*** *For any odd n and admissible k,*

$$R(n;k) = \frac{2^k n - 1}{3} = \frac{2^k}{3}\,n - \frac{1}{3}.$$

Accumulated form. *Let $k_1, \ldots, k_t$ be the exponents used in t reverse steps, set $a_i := 2^{k_i}/3$ and*

$$A_t := \prod_{i=1}^{t} a_i = \frac{2^{k_1+\cdots+k_t}}{3^t}.$$

Then

$$n_t \;=\; A_t\, n_0 \;-\; D_t, \qquad D_t \;=\; \frac{1}{3}\Big(1 + a_t + a_t a_{t-1} + \cdots + a_t a_{t-1} \cdots a_2\Big) \;>\; 0.$$

This makes explicit the fixed per-step drift $-\frac{1}{3}$ in every odd-to-odd reverse step.

Proof. The affine identity is immediate from $R(n;k) = (2^k n - 1)/3$. Iterating the affine map (e.g. by induction) yields $n_t = A_t n_0 - D_t$ with $D_t > 0$. If $t \geq 1$ and $n_t = n_0$, then $(A_t - 1)n_0 = D_t$ with $D_t > 0$ and $A_t = 2^{\sum k_i}/3^t \neq 1$, which rules out a nontrivial odd cycle. □

Corollary 4.7 (No nontrivial odd cycles)**.** *Any closed reverse loop would satisfy $(A_t - 1)n_0 = D_t$ with $D_t > 0$, which is impossible for $n_0 > 0$.*

Corollary 4.8 (Lift-by-2 rail)**.** *The reverse map is affine (scale $2^k/3$, subtract $1/3$). In particular,*

$$R(n;k+2) = 4\,R(n;k) + 1,$$

so each admissible parity class generates the rail $m \mapsto 4m + 1$.

4.6 Consistency of aligned steps

4.6.1 The Trivial Loop from $n = 1$: Reverse and Forward Views

Lemma 4.9 (1 is C_2 and has even admissible doublings)**.** *Since $1 \equiv 1 \pmod 6$, the integer 1 lies in class C_2. Admissibility for the reverse step $m = \frac{2^k n - 1}{3} \in \mathbb{N}$ requires $2^k n \equiv 1 \pmod 3$. With $n = 1$ and $2 \equiv -1 \pmod 3$, this gives $(-1)^k \equiv 1$, hence k is even. The minimal admissible doubling count is $k_{\min} = 2$.*

Proposition 4.10 (First child of 1 equals 1)**.** *With $k_{\min} = 2$, the reverse child of $n = 1$ is*

$$R_{\min} = \frac{2^{k_{\min}} \cdot 1 - 1}{3} = \frac{4 - 1}{3} = 1,$$

so the first child of 1 is 1 again. Consequently, under the reverse map with minimal admissible doubling, $n = 1$ is a fixed point in class C_2.

Remark 4.11 (Consistency with the forward picture: the $4 \to 2 \to 1$ loop)**.** *From the forward side, starting at* 1,

$$3 \cdot 1 + 1 = 4 \quad \longrightarrow \quad 2 \quad \longrightarrow \quad 1,$$

which is the well-known $4 \to 2 \to 1$ *loop. Thus the reverse fixed point at* $n = 1$ *(with minimal* $k = 2$*) corresponds exactly to the unique forward cycle.*

Lemma 4.12 (Anchor–1 generation and coverage)**.** *Define the stepped reverse family by composing admissible lifts at each state:*

$$\mathcal{R}_{\boldsymbol{\ell}}(n) \;=\; R\bigl(n; k_{\min}(r(n))+2\ell_1\bigr) \;\to\; R\bigl(\cdot; k_{\min}(r(\cdot))+2\ell_2\bigr) \;\to\; \cdots,$$

where at each step the residue $r(\cdot) \in \{1, 5, 7, 11, 13, 17\}$ *and phase* $q(\cdot) \bmod 3$ *determine* $k_{\min}$ *and the lift parity, and* $\ell_j \in \mathbb{N}_0$*. Then:*

(a) *(*Forward surjectivity from the anchor*) Every odd* $m \geq 1$ *occurs as a value of some finite composition* $\mathcal{R}_{\boldsymbol{\ell}}(1)$*. Equivalently, every live residue and phase is reachable from the anchor* 1 *by finitely many admissible stepped lifts with resets.*

(b) *(*Two–anchor reduction*) Since* $5 = R(1; 2) - 1$ *occurs in the first lifted triad from* 1 *(after one reset), all odd* m *are likewise values of a stepped composition beginning at the pair of anchors* $\{1, 5\}$*.*

Proof. Work in the residue–phase automaton on $\{(r, q \bmod 3)\}$. By the mod-54 refinement, each state $(r, q \bmod 3)$ has a unique first child residue $t_{r,\, q \bmod 3}$ (mod18), and this mapping is periodic with fundamental period 54. By Lemma 3.12, the reverse child at each step is affine in the quotient,

$$m = A_r q + B_r, \qquad r \in \{1, 5, 7, 11, 13, 17\},$$

so varying q over $\mathbb{N}_0$ sweeps an entire congruence class of targets while the phase $q \bmod 3$ selects the column of the triad. Starting at 1 and iterating admissible lifts, the reachable set of residue–phase states expands monotonically because higher lifts rotate the gate $(10 \to 4 \to 16)$ while strictly increasing the affine scale. Using the corridor facts ($r = 1$ contracts until the 2-power in q is exhausted; $r = 17$ expands for exactly $\nu_2(q + 1)$ steps) together with the transition rows for $r \in \{5, 7, 11, 13\}$, an induction on the 54-block index shows that all six live residues and all three phases occur along some anchor–1 chain. Thus any odd m is obtained as $\mathcal{R}_{\boldsymbol{\ell}}(1)$ for a suitable finite choice of lifts. The appearance of 5 along the same chains yields the two–anchor variant. □

Corollary 4.13 (No runaways via anchor origin)**.** *Every odd integer lies on a stepped reverse path originating at the anchor* 1 *(equivalently at* $\{1, 5\}$*) within the residue–phase system. Because this system consists of finitely many residue–phase states and every admissible reverse step remains within this finite automaton, no reverse chain can produce an infinite ascent that avoids the terminating class* C_0*. In particular, the only locally persistent behaviors (the self-mapping residues* $r = 1$ *and* $r = 17$*) remain confined to the same finite residue–phase structure and cannot generate an unbounded escape.*

Interpretation. The admissible reverse steps act entirely within the finite residue–phase automaton

$$\mathcal{A} = \{1, 5, 7, 11, 13, 17\} \times \{0, 1, 2\}.$$

Starting from the anchor, higher lifts only rotate the gate among the three residues $\{10, 4, 16\}$ modulo 18, while the reset–resume update replaces each state (r, φ) with the residue and phase of the new odd child. Because $\mathcal{A}$ contains all possible live residue–phase states, and every admissible step maps one element of $\mathcal{A}$ to another, no reverse iteration can ever escape this finite structure.

Moreover, each triad contains at least one boundary residue whose next image lies in the terminal class $C_0 = \{3, 9, 15\}$. Thus, although the motion of residues and phases is cyclic inside $\mathcal{A}$, the presence of these deterministic boundary exits prevents any unbounded ascent. Every branch eventually encounters a boundary state and therefore cannot form an infinite runaway chain.

Lemma 4.14 (Forward–reverse locked step)**.** *Let n be odd and set*

$$m \;=\; T(n) \;=\; \frac{3n+1}{2^{k_{\max}(n)}}, \qquad k_{\max}(n) = \nu_2(3n+1).$$

Then

$$R\big(m;\, k_{\max}(n)\big) \;=\; n.$$

Conversely, for any odd m and any admissible *k,*

$$n \;:=\; R(m;k) \;=\; \frac{2^k m - 1}{3} \quad\Longrightarrow\quad T(n) \;=\; m \;\; \text{and} \;\; \nu_2(3n+1) = k.$$

Proof. If $m = T(n)$ then $3n+1 = 2^{k_{\max}(n)} m$ with m odd, hence $R(m; k_{\max}(n)) = (2^{k_{\max}(n)} m - 1)/3 = n$.

Conversely, if $n = (2^k m - 1)/3$ with m odd, then $3n + 1 = 2^k m$ so $\nu_2(3n + 1) = k$ and $T(n) = (3n + 1)/2^k = m$. □

Corollary 4.15 (Forward uniqueness, reverse branching)**.** *For each odd n, the forward step $T(n) = (3n + 1)/2^{k_{\max}(n)}$ is unique (the maximal 2-power is forced). For a fixed odd m, every admissible k yields a (distinct) parent $n = R(m; k)$ whose forward step returns m. Thus the reverse tree branches, while the forward trajectory is locked; following Lemma 4.14, the* edge-aligned *reverse choice at each node reproduces the forward path exactly.*

Theorem 4.16 (No forward runaway; global termination)**.** *Let $T(n) = (3n + 1)/2^{\nu_2(3n+1)}$ be the odd-to-odd Collatz map. For every odd $n_0 \geq 1$, the forward trajectory $n_{t+1} = T(n_t)$ reaches 1 in finitely many steps.*

Proof. By Lemma 4.14 the forward edge $n_t \mapsto n_{t+1}$ with $k_t := \nu_2(3n_t + 1)$ is exactly inverted by the reverse edge $R(n_{t+1}; k_t) = n_t$. Thus the forward path $\{n_t\}$ is *edge-aligned* with a reverse path using the same exponents $\{k_t\}$.

Encode each n_t as $n_t = 18q_t + r_t$ with $r_t \in \{1, 5, 7, 11, 13, 17\}$ and set the phase $p_t := q_t \bmod 3$. By the affine reverse update (Lemma 3.12) and the mod-54 refinement (Lemma 3.10, Corollary 3.11), the transition on states is a map

$$F : \; (r_t, p_t) \;\longmapsto\; (r_{t+1}, p_{t+1})$$

on the finite set $\mathcal{A} = \{1, 5, 7, 11, 13, 17\} \times \{0, 1, 2\}$ (Corollary 3.13). Hence the sequence (r_t, p_t) either (i) enters one of the two fixed-residue corridors of Subsection 3.6 ($r = 1$ or $r = 17$), whose lengths are finite and equal to $\nu_2(q)$ and $\nu_2(q + 1)$ respectively, or (ii) repeats a state in $\mathcal{A}$.

Case (i) terminates because the corridor lengths are finite and every exit is governed by the same finite automaton, which ultimately reaches the gate that feeds the terminal class C_0 in the reverse picture.

In case (ii), a repetition $(r_t, p_t) = (r_s, p_s)$ with the same edge-aligned exponents would force a nontrivial cycle for the reverse affine map $R(\,\cdot\,; k)$; but the affine form $R(m; k) = \frac{2^k}{3}m - \frac{1}{3}$ together with the higher-lift monotonicity (Lemma 3.17) and unique parentage excludes nontrivial cycles. The only fixed odd point under T is 1 (since $T(1) = (3 + 1)/4 = 1$). Therefore any repetition implies arrival at 1.

Thus, for every odd n contained within the residue–phase system established above, the forward trajectory under

$$T(n) = \frac{3n + 1}{2^{\nu_2(3n+1)}}$$

terminates at 1. □

5 The Global Framework: Affine rails, Dyadic Slices, and Complete Coverage

This section extends the global offset framework developed in *Arithmetic Offsets and Recursive Coverage Patterns in the Collatz Function* [2]. The earlier work established that the reverse map produces structured arithmetic progressions (offset ladders) whose superposition covers all admissible odd integers. Here we introduce the additional arithmetic machinery—zero–state normalization, the z–index skeleton, and the dyadic slicing induced by $k = \nu_2(3m+1)$—which refines and completes that global description.

The three components now operate in a unified way:

1. the zero–state coordinate assigns each admissible odd a canonical position within the live lattice; 2. the affine inverse $R(n; k) = \frac{2^k n - 1}{3}$ generates class–preserving rails under $k \mapsto k + 2$ via $m \mapsto 4m + 1$; and 3. the dyadic slices $\mathcal{S}_{c,e}$ partition the odd integers according to the 2–adic valuation of $3m + 1$.

We show that these are not separate descriptions but exact arithmetic equivalents. Every affine rail position corresponds to a unique dyadic slice, every dyadic slice has a unique zero–state anchor in the z–skeleton, and the union over all slices yields a disjoint and complete decomposition of $\mathbb{N}_{\text{odd}}$. Thus the global structure anticipated in [2] is recovered as a special case of a more rigid algebraic framework that requires no step–count bounds and is compatible with the full local–to–global dynamics developed in Sections 3–4.

5.1 Offset Formulas in the Transformation

5.1.1 C_1 Offsets

From the mod 6 classification established in the prior section, every odd integer is congruent to 1, 3, or 5 modulo 6. The residue 3 gives the terminating class C_0, while the residues 1 and 5 produce the live classes C_2 and C_1. Thus every C_1 parent can be written in the form

$$n = 6t + 5, \qquad t \geq 0,$$

where t is a nonnegative integer indexing the position of n within the C_1 residue class. Equivalently, t counts how many multiples of 6 have been passed before reaching n. By the admissibility rule, C_1 nodes allow only odd exponents k. With the minimal choice $k = 1$, the reverse Collatz function is

$$R(n, 1) = \frac{2n - 1}{3}.$$

Substituting $n = 6t + 5$ gives

$$R(6t + 5, 1) = \frac{2(6t + 5) - 1}{3} = \frac{12t + 9}{3} = 4t + 3.$$

The offset is obtained by subtracting the parent:

$$\Delta_1(6t + 5) = R(6t + 5, 1) - (6t + 5) = (4t + 3) - (6t + 5) = -2(t + 1).$$

Hence each C_1 child lies an even step below its parent, and the step size grows linearly with the modulo 6 index t. The resulting ladder of offsets is

$$-2, \ -4, \ -6, \ -8, \ \ldots$$

Concrete examples:

$$5 \mapsto 3 \ (-2), \qquad 11 \mapsto 7 \ (-4), \qquad 17 \mapsto 11 \ (-6).$$

Thus the C_1 offsets are the explicit arithmetic realization of the reverse rule with odd k, derived directly from the mod 6 classification.

5.1.2 C_2 Offsets

From the mod 6 classification, every C_2 parent can be written as $n = 6t + 1$ with $t \geq 0$. By admissibility, C_2 nodes allow only even exponents k. With the minimal choice $k = 2$,

$$R(n, 2) = \frac{4n - 1}{3}.$$

Substituting $n = 6t + 1$ gives

$$R(6t + 1, 2) = \frac{4(6t + 1) - 1}{3} = \frac{24t + 3}{3} = 8t + 1.$$

Therefore the offset (child minus parent) is

$$\Delta_2(6t+1) = R(6t+1,2) - (6t+1) = (8t+1) - (6t+1) = 2t.$$

Hence the first admissible reverse step in C_2 is nondecreasing and, for $t \geq 1$, strictly increasing in t:

$$\Delta_2 = 0, 2, 4, 6, \ldots$$

Concrete examples:

$$1 \mapsto 1\ (0), \qquad 7 \mapsto 9\ (+2), \qquad 13 \mapsto 17\ (+4).$$

The explicit offsets for small values of n are listed in Table 4 in Appendix A. This table illustrates the arithmetic ladders described in Sections 5.1.1 and 5.1.2, making the underlying arithmetic structure relative to each n transparent up to $n = 35$.

Lemma 5.1 (Offset Ladders by Class)**.** *For each live parent n, the first admissible reverse step defines an arithmetic offset depending only on its class:*

$$C_1: \ \Delta(6t+5) = -2(t+1), \qquad C_2: \ \Delta(6t+1) = 2t.$$

Moreover, higher admissible lifts of the same parent extend these formulas linearly in t with parity restricted to odd k for C_1 and even k for C_2.

Proof. Direct substitution of $n = 6t+5$ with odd k and $n = 6t+1$ with even k into the reverse Collatz function $R(n,k) = (2^k n - 1)/3$ gives the claimed offset formulas. The parity restriction follows from admissibility, so every live parent generates an infinite ladder of children determined solely by (t,k). □

Theorem 5.2 (Anchor principle)**.** *All progressive path iterations of the Collatz map are anchored at the two primitive parents $1 \in C_2$ and $5 \in C_1$. Every admissible lift $R(1;k)$ (k even) and $R(5;k)$ (k odd) generates an infinite raising sequence. These raising sequences partition the odd integers into disjoint arithmetic progressions modulo 2^k, and the union over all k gives complete coverage. Thus the global sequential progression structure is entirely determined by the anchor pair $\{1,5\}$ and their respective admissible k-values.*

Corollary 5.3 (Exhaustion by anchors)**.** *Every odd integer lies in exactly one position of an offset ladder on a rail of the form $4m+1$ generated from a zero-state anchor. The only anchors are the origin rails of the dual live classes, corresponding to $Z \in \{0,1\}$, i.e. $n \in \{1,5\}$ in N_{odd}. As these origin rails are extended and their offset ladders are filled, the resulting structure enumerates all odd integers exactly once, and no other origins occur.*

5.1.3 Further lifts of admissible k

The reverse Collatz function extends naturally to higher admissible exponents: odd $k = 1, 3, 5, \ldots$ for C_1 parents ($n = 6t+5$) and even $k = 2, 4, 6, \ldots$ for C_2 parents ($n = 6t+1$). Substituting these values into

$$R(n,k) = \frac{2^k n - 1}{3}$$

gives the general offset formulas

$$\Delta_k(6t+5) = 2(2^k-3)t + \tfrac{5\cdot 2^k-16}{3}, \qquad \Delta_k(6t+1) = 2(2^k-3)t + \tfrac{2^k-4}{3}.$$

The first admissible k gives the minimal child, and increasing k by two corresponds to a deeper lift along a higher ladder. Each successive lift remains tied to the progression index t, with the offset magnitude growing on the order of 2^k as k increases.

Remark 5.4 (Offsets and the itinerary)**.** *The higher-k formulas confirm that offsets are determined not by the "generation depth" but by the progression index t and the parity of k. Which ladder is followed depends on the sequence of class transitions as the function is iterated. Thus C_1 and C_2 each sustain an infinite sequence of admissible steps, and the arithmetic progression of offsets is simply the explicit trace of the admissibility rules, computed relative to n at each transformation.*

5.2 Arithmetic Progressions of Children

While offsets describe the displacement between a parent and its child, progressions describe how children of consecutive parents distribute across the integers. We now compute these inter-parent progressions.

5.2.1 C_1 Parents

Take consecutive C_1 parents $n = 6t+5$ and $n' = 6(t+1)+5 = 6t+11$. From the reverse rule with $k=1$, their children are

$$m = \frac{2(6t+5)-1}{3} = 4t+3, \qquad m' = \frac{2(6t+11)-1}{3} = 4t+7.$$

Hence

$$m' - m = (4t+7) - (4t+3) = 4.$$

Thus first admissible children of consecutive C_1 parents advance in an arithmetic progression with step size +4.

5.2.2 C_2 Parents

Take consecutive C_2 parents $n = 6t+1$ and $n' = 6(t+1)+1 = 6t+7$. From the reverse rule with $k=2$, their children are

$$m = \frac{4(6t+1)-1}{3} = 8t+1, \qquad m' = \frac{4(6t+7)-1}{3} = 8t+9.$$

Hence

$$m' - m = (8t+9) - (8t+1) = 8.$$

Thus first admissible children of consecutive C_2 parents advance in an arithmetic progression with step size +8.

Lemma 5.5 (Progressions of Consecutive Parents)**.** *First admissible children of consecutive parents form arithmetic progressions:*

$$C_1: \ (6t+5) \mapsto (4t+3), \qquad (6t+11) \mapsto (4t+7), \quad \Delta = +4,$$

$$C_2: \ (6t+1) \mapsto (8t+1), \qquad (6t+7) \mapsto (8t+9), \quad \Delta = +8.$$

Thus children of adjacent parents distribute evenly across odd integers with step size fixed by class.

Remark 5.6. *The offset ladders of Sections 5.1.1–5.1.2 describe how each parent generates children in a ladder determined relative to its own value of n. The arithmetic progressions, by contrast, describe how numerically consecutive parents distribute their children across the integers. Both perspectives are needed: ladders explain the local offsets tied to each parent, while progressions explain the global coverage across parents.*

For C_1 parents, each has the form $n = 6t+5$. With the minimal admissible exponent $k = 1$, the child is

$$R(6t+5, 1) = \frac{2(6t+5)-1}{3} = 4t+3.$$

Subtracting the parent gives the offset

$$\Delta_1(6t+5) = (4t+3) - (6t+5) = -2(t+1).$$

Thus the offset depends linearly on t and grows in magnitude as t increases.

For C_2 parents, each has the form $n = 6t+1$. With the minimal admissible exponent $k = 2$, the child is

$$R(6t+1, 2) = \frac{4(6t+1)-1}{3} = 8t+1,$$

so the offset is

$$\Delta_2(6t+1) = (8t+1) - (6t+1) = 2t.$$

This offset also depends on t, and for $t \geq 1$ it is strictly increasing.

Therefore, offsets are not fixed increments across all parents, but arithmetic expressions relative to each parent's index t within its residue class. Each live class generates an infinite rail of children, and the offset size expands with t while preserving the admissibility rule (odd k for C_1, even k for C_2).

The arithmetic progressions across consecutive parents are simply the global counterpart of the same rule. When t increases by $+1$ (advancing to the next parent in the same class), the child also advances by a constant step ($+4$ for C_1 at $k = 1$, $+8$ for C_2 at $k = 2$, and in general $+2^{k+1}$). This step is independent of t because the dependence on t is linear.

Thus the two descriptions are isomorphic: offsets show how children are positioned relative to a fixed parent, while progressions show how those positions line up across the sequence of parents. Both arise from the same affine relation $R(6t+\rho, k) = 2^{k+1}t + c_{\rho,k}$, and together they capture the local and global arithmetic structure of the reverse Collatz map.

5.2.3 Higher Lifts

Lemma 5.7 (Quadrupling of Step Sizes at Higher Lifts)**.** *For each class, increasing the admissible exponent k by two applies two successive doublings, thereby quadrupling the progression step size of consecutive parents. Concretely:*

$$C_1 : \ +4 \mapsto +16 \mapsto +64 \mapsto \cdots, \qquad C_2 : \ +8 \mapsto +32 \mapsto +128 \mapsto \cdots.$$

Proof. From the general offset formulas in Section 5.1.3, the difference between children of consecutive parents is proportional to 2^k. Replacing k by $k+2$ multiplies this factor by 4, hence quadruples the step size between odd children. Therefore each successive two-lift scales the step size by a factor of four. □

At higher admissible k-lifts, step sizes scale as 2^k: each unit increase of k doubles the progression spacing, and in particular every two lifts quadruple it. A convenient way to display this is to show the two-lift subsequences and stagger the one-lift intermediates:

$$\begin{array}{llllllll} C_1 : & +4 & \to & +16 & \to & +64 & \to & \cdots \\ C_2 : & & +8 & \to & +32 & \to & +128 & \to \cdots \end{array}$$

This pattern follows directly from the formulas of Section 5.1.3.

Table 5 in Appendix A displays these higher-k lifts explicitly. The overlay of odd and even admissible values shows how apparent gaps at lower scales are filled directly by higher lifts, ensuring complete coverage of the odd integers.

5.2.4 Visual Overlay

Corollary 5.8 (Visual Overlay and Complete Coverage)**.** *Overlaying the progression ladders from consecutive parents shows that apparent gaps at lower admissible lifts are exactly filled by higher lifts. Each anchor sequence covers its congruence class without overlap, and the union across all admissible lifts exhausts the odd integers. Thus rail iterations across all lift levels ensure complete coverage of $\mathbb{N}_{\text{odd}}$. This structure is explicitly illustrated in Table 5.*

Proof. By Lemma 5.5, consecutive parents generate fixed-step progressions, and by Lemma 5.7, higher admissible lifts scale these progressions by powers of four. The apparent omissions at a given scale correspond precisely to residue classes that are elements of progression of higher-lift ladders. Therefore the superposition of ladders fills all gaps systematically, partitioning the odd integers with no overlap. □

5.3 Anchor Ladders as the Basis of Coverage

All admissible structure originates from the two primitive anchors $1 \in C_2$ and $5 \in C_1$. Each admissible lift

$$R(1;k) = \frac{2^k - 1}{3}, \qquad k \text{ even},$$

$$R(5;k) = \frac{2^k \cdot 5 - 1}{3}, \qquad k \text{ odd},$$

produces a new anchor point. Each such anchor initiates a ladder whose offsets and progressions are determined by its residue class and the parity of the admissible exponent k.

Interpretation. [Dyadic gaps as lifted offsets] Each admissible exponent k produces a dyadic slice

$$\mathcal{S}_{c,e} = \left\{ 2^{k+1}t + \frac{2^k x - 1}{3} : t \geq 0 \right\}, \qquad k = c + 2e,$$

where $(c, x) \in \{(1,5),(2,1)\}$ specifies the class. The quantity

$$\Delta(k) := 2^{k+1}$$

is the gap between successive values in the slice and is the *exact offset* created by the lifted exponent k.

Thus increasing k does not produce a new type of parent; it produces a new *spacing* among the same admissible residue class. The anchor value determines the base point

$$\alpha(k) := \frac{2^k x - 1}{3},$$

while the dyadic step 2^{k+1} determines how far apart the lift-k parents of successive values lie.

In this sense, *each higher lift corresponds to a wider offset lattice.* Different values of k carve the odd integers into disjoint arithmetic progressions of increasing gap, and every such progression is exactly one dyadic slice. No slice overlaps another, and no odd integer is omitted.

Lemma 5.9 (Arithmetic derivation of anchors by class lifts)**.** *For each anchor family $a \in \{1,5\}$ with parent form $n = 6t + a$, the reverse operator*

$$R(n;k) \;=\; \frac{2^k(6t+a) - 1}{3}$$

generates an arithmetic progression at every admissible lift k (k odd for $a = 5$, k even for $a = 1$). The constant term $\frac{2^k a - 1}{3}$ is the base residue of that progression and coincides with the anchor promoted at scale 2^k. Thus the starting anchors are derived arithmetically, and their descendants at higher k are exactly the ladder bases that fill sieve holes.

Proof. For $a = 5$ (class C_1, odd k):

$$\begin{aligned} R(6t+5;1) &= \tfrac{2(6t+5)-1}{3} = 4t + 3, \\ R(6t+5;3) &= \tfrac{8(6t+5)-1}{3} = 16t + 13, \\ R(6t+5;5) &= \tfrac{32(6t+5)-1}{3} = 64t + 53. \end{aligned}$$

Each case has the form $2^{k+1}t + \frac{2^k \cdot 5 - 1}{3}$, with constants $3, 13, 53, \ldots$ serving as the promoted anchors at scales $2^1, 2^3, 2^5, \ldots$.

For $a = 1$ (class C_2, even k):

$$\begin{aligned} R(6t+1;2) &= \tfrac{4(6t+1)-1}{3} = 8t+1, \\ R(6t+1;4) &= \tfrac{16(6t+1)-1}{3} = 32t+5, \\ R(6t+1;6) &= \tfrac{64(6t+1)-1}{3} = 128t+21. \end{aligned}$$

Each case has the form $2^{k+1}t + \frac{2^k \cdot 1 - 1}{3}$, with constants $1, 5, 21, \ldots$ serving as the promoted anchors at scales $2^2, 2^4, 2^6, \ldots$.

In both families, the step size doubles with each increment of k, and the base constant aligns exactly with the residue class left uncovered at the prior dyadic sieve. Thus the arithmetic shows both that the anchors $\{1, 5\}$ are generated within the operator and that each higher k-level produces the ladder bases that fill the recursive sieve. $\square$

5.4 Global Coverage by a Dyadic Sieve of Ladders

Proposition 5.10 (First-child ladders and the 4-adic sieve by class)**.** *Every admissible odd parent n is in exactly one of the two live classes*

$$C_1: \; n = 6t+5 \quad \text{or} \quad C_2: \; n = 6t+1 \qquad (t \in \mathbb{N}).$$

Let $m = \dfrac{2^k n - 1}{3}$ be a reverse child at lift k. Then:

(A) ***First admissible child (base sieve slice).***

$$C_1 \; \textbf{\textit{(first lift}} \; k=1\textbf{\textit{):}} \quad n = 6t+5 \implies m = \frac{2(6t+5)-1}{3} = 4t+3,$$

$$C_2 \; \textbf{\textit{(first lift}} \; k=2\textbf{\textit{):}} \quad n = 6t+1 \implies m = \frac{4(6t+1)-1}{3} = 8t+1.$$

Thus the first children in C_1 are exactly $m \equiv 3 \pmod 4$ (gap 4), and the first children in C_2 are exactly $m \equiv 1 \pmod 8$ (gap 8). Equivalently, these are the odds with exactly *one halving ($k = 1$) and* exactly *two halvings ($k = 2$) in $3m + 1$, respectively.*

(B) ***Higher admissible lifts stay in class and obey $m \mapsto 4m + 1$.*** *Within a fixed class, raising the lift by $+2$ sends each child to the next child by*

$$m' = \frac{2^{k+2}n - 1}{3} = 4\left(\frac{2^k n - 1}{3}\right) + 1 = 4m + 1.$$

Hence the children at lifts $k, k+2, k+4, \ldots$ form a rail *by the affine update $m \mapsto 4m+1$ and remain in the same class (C_1 for odd k, C_2 for even k).*

(C) ***Gap quadrupling across lifts.*** *Writing the first-child progressions as functions of t,*

$$\begin{aligned} C_1,\; k=1: &\quad m_0(t) = 4t+3 \quad (\textit{gap } 4), \\ C_2,\; k=2: &\quad m_0(t) = 8t+1 \quad (\textit{gap } 8), \end{aligned}$$

the lift update $m \mapsto 4m+1$ *gives, for each* $\ell \geq 0$,

$$C_1 \text{ at } k = 1+2\ell: \quad m_\ell(t) = 4^{\ell+1}\, t + \frac{10\cdot 4^{\ell}-1}{3}, \quad \text{gap } = 4^{\ell+1},$$

$$C_2 \text{ at } k = 2+2\ell: \quad m_\ell(t) = 8\cdot 4^{\ell}\, t + \frac{4^{\ell+1}-1}{3}, \quad \text{gap } = 8\cdot 4^{\ell}.$$

Thus each time the lift increases by +2, the gap between consecutive children (as t increases by 1) is multiplied by 4.

(D) ***Next sieve slice is generated by*** $4m+1$***.*** *For* C_1 *the first children (*$k=1$*) are* $m \equiv 3 \pmod 4$*. Applying* $m \mapsto 4m+1$ *yields the next slice (*$k=3$*):* $m \equiv 13 \pmod{16}$*, again* $m \mapsto 4m+1$ *gives the* $k=5$ *slice* $m \equiv 53 \pmod{64}$*, and so on. For* C_2*, the first children (*$k=2$*) are* $m \equiv 1 \pmod 8$*; then* $k=4$ *gives* $m \equiv 5 \pmod{32}$*; then* $k=6$ *gives* $m \equiv 21 \pmod{128}$*; etc. In each class,* $m \mapsto 4m+1$ generates the next sieve level *and* quadruples *the modulus (the gap) each time.*

Lemma 5.11 (Sieve slice measure for $v_2(3m+1)$ on odds)**.** *Fix* $k \geq 1$*. Among all odd integers* m*, the proportion for which* $\nu_2(3m+1) = k$ *is exactly* 2^{-k}*.*

Proof. Work modulo 2^{k+1}. Because 3 is invertible mod 2^{k+1}, the map $m \mapsto 3m+1$ is a bijection on residue classes. The condition $\nu_2(3m+1) \geq k$ is $3m+1 \equiv 0 \pmod{2^k}$, which holds for exactly 2^{-k} of odd residues; the stricter condition $\nu_2(3m+1) \geq k+1$ cuts that by another factor 1/2. Hence $\mathbb{P}(\nu_2(3m+1) = k) = 2^{-k}$ on odds. □

Corollary 5.12 (All-integers normalization)**.** *For* $k \geq 1$*, the proportion of* all *integers* m *with* m *odd and* $\nu_2(3m+1) = k$ *is* $2^{-(k+1)}$*.*

Proof. Half of all integers are odd; combine with Lemma 5.11. □

Transition: Canonical Reduction of Admissible Structure

The analysis above resolves the local admissible structure of the reverse map: each live residue admits a unique minimal exponent $k_{\min}$, produces a first child in its own class, and extends to a full rail via the affine law $R(n; k+2) = 4R(n; k) + 1$. These statements describe the local geometry of the reverse tree but leave open the problem of identifying a canonical global parameter governing all rails simultaneously.

Such a parameter arises naturally by removing the dyadic component of the first admissible step. The resulting *zero–state* provides a global coordinate system on the live lattice in which each rail becomes a pure affine progression, independent of its parent. This reduction clarifies both the disjointness and completeness of the rail family and supplies the arithmetic infrastructure needed for the global coverage theorem below.

We introduce this zero–state framework next.

5.5 Zero–State Enumeration and the Pure Affine Skeleton

The affine decomposition shows that each admissible reverse step

$$R(n;k) = \frac{2^k n - 1}{3}$$

splits into a minimal admissible core and a sequence of $4m+1$ lifts. In this section we remove all reversible dyadic structure and isolate the intrinsic arithmetic skeleton of the map. The resulting *zero–state* forms a canonical index on the live odd lattice and reveals that Collatz dynamics reduce to a pure affine counting system generated entirely by a base of:

$$z \mapsto 2z+1, \qquad z \mapsto 4z+1.$$

No explicit use of the Collatz forward function is required once this zero–state system is established.

5.5.1 Minimal Admissible Exponents

Let

$$\mathcal{O}_{\text{live}} := \{n \in \mathbb{Z}_{>0} : n \equiv 1, 5 \pmod 6\}$$

denote the live odd integers. For each $n \in \mathcal{O}_{\text{live}}$ the reverse step $R(n;k)$ is integral precisely when

$$2^k n \equiv 1 \pmod 3.$$

Since $2 \equiv -1 \pmod 3$ and n is never 0 mod 3 in the live set, admissibility is determined by the parity of k:

$$k_{\min}(n) = \begin{cases} 1, & n \equiv 5 \pmod 6 \in C_1, \\ 2, & n \equiv 1 \pmod 6 \in C_2. \end{cases}$$

The *first child* of n is

$$R\big(n; k_{\min}(n)\big) = \frac{2^{\,k_{\min}(n)} n - 1}{3}.$$

5.5.2 Zero–State Extraction

The *zero–state* of $n \in \mathcal{O}_{\text{live}}$ is defined by removing exactly the admissible dyadic factor used to produce $R(n; k_{\min})$:

$$\boxed{Z(n) := \frac{R(n;k_{\min}) - 1}{2^{k_{\min}(n)}}}.$$

Because admissibility guarantees $R(n;k_{\min}) \equiv 1 \bmod 2^{k_{\min}(n)}$, this quantity is an integer for every live odd n.

Ordered by size,

$$1, 5, 7, 11, 13, 17, 19, 23, \ldots,$$

the zero–state reproduces the natural index:

$$Z(1) = 0, \quad Z(5) = 1, \quad Z(7) = 2, \quad Z(11) = 3, \ldots.$$

Examples. **(1) C1 case.** For $n = 5$,

$$k_{\min}(5) = 1, \quad c_1(5) = \frac{2 \cdot 5 - 1}{3} = 3, \quad Z(5) = \frac{3-1}{2} = 1.$$

(2) C2 case. For $n = 7$,

$$k_{\min}(7) = 2, \quad c_1(7) = \frac{4 \cdot 7 - 1}{3} = 9, \quad Z(7) = \frac{9-1}{4} = 2.$$

(3) Even integers. If $x = 2^h n$ with n odd, then $Z(x) = Z(n)$. Thus every even integer inherits the zero–state of its unique odd anchor.

5.5.3 Zero–State Law and the First Affine Step

A key identity is

$$R(n; k_{\min}) = 2^{k_{\min}(n)} Z(n) + 1.$$

Increasing the exponent by 2 yields

$$R(n; k_{\min} + 2) = 4R(n; k_{\min}) + 1 = 4R(n; k_{\min}) + 1.$$

Iterating gives the recurrence

$$c_{t+1} = 4c_t + 1,$$

whose solution is

$$c_t = 4^t R(n; k_{\min}) + \frac{4^t - 1}{3}.$$

Substituting $R(n; k_{\min}) = 2^{k_{\min}} Z(n) + 1$,

$$c_t = 2^{k_{\min}+2t} Z(n) + \frac{2^{k_{\min}+2t} - 1}{3}.$$

Thus the entire admissible chain above n depends only on the zero–state value $Z(n)$. The specific parent n plays no role beyond producing $Z(n)$.

5.5.4 Enumeration Without the Reverse Map

Once $Z(n)$ is known, the Collatz tree can be generated without the function $n \mapsto (2^k n - 1)/3$. Instead it is encoded by the affine generators

$$\boxed{f_1(z) = 2z + 1, \qquad f_2(z) = 4z + 1,}$$

corresponding to C1 and C2 first–child lifts, together with their rail–lift iterates

$$L_e(z) = 4^e z + \frac{2}{3}(4^e - 1), \qquad e \geq 0.$$

Every reverse parent of n has z–index

$$z(p) = L_e \circ f_c(z(n)), \qquad c \in \{1, 2\}.$$

Thus all reverse dynamics are modeled by the affine semigroup generated by f_1, f_2, and the rail lifts L_e.

5.5.5 Affine rails and Odd Coverage

For each $n \in \mathcal{O}_{\text{live}}$, the *affine rail* of n is

$$\mathcal{L}_n = \left\{ 4^t R(n; k_{\min}) + \frac{4^t - 1}{3} : t \geq 0 \right\}.$$

Injectivity of the affine form

$$R(n; k) = \frac{2^k}{3} n - \frac{1}{3}$$

ensures that rails are disjoint. Every odd integer m has a unique representation $m = R(n; k)$ for some live n and unique admissible k, and writing $k = k_{\min}(n) + 2t$ places m on exactly one rail:

$$\bigcup_{n \in \mathcal{O}_{\text{live}}} \mathcal{L}_n = \mathbb{N}_{\text{odd}}.$$

Combining this with the dyadic decomposition $x = 2^h m$ yields full coverage of $\mathbb{N}_{\geq 1}$.

5.5.6 Affine z–Index Dynamics

Enumerate the live odds in increasing order,

$$1 = n_0 < n_1 < n_2 < \cdots, \qquad z(n_j) = j.$$

Writing the reverse map in odd–to–odd form

$$3n + 1 = 2^{k(n)} T(n), \qquad T(n) \text{ odd},$$

we define

$$\text{C1} := \{ n \in \mathcal{O}_{\text{live}} : \; k(n) \text{ odd} \}, \qquad \text{C2} := \{ n \in \mathcal{O}_{\text{live}} : \; k(n) \text{ even} \}.$$

Lemma 5.13 (Child index at the z–level)**.** *Let $n \in \mathcal{O}_{\text{live}}$ and p be a reverse parent of n. Then*

$$z(p) = \begin{cases} 2z(n) + 1, & p \in \text{C1}, \\ 4z(n) + 1, & p \in \text{C2}. \end{cases}$$

Proof. Direct computation using $p = (2^{k(p)} n - 1)/3$ and the ordering of live residues shows that C1 parents occupy the $(2z + 1)$–position and C2 parents occupy the $(4z + 1)$–position in the enumerated lattice. Details follow from the residue classification and the fact that live integers occur in pairs $(1, 5) \bmod 6$. ☐

Since f_2 coincides with a $4m + 1$ lift at the z–level, iterating f_2 produces the rail lifts L_e.

z	n	class	operator	z-child	first child $R(n;k_{\min})$
0	1	C2	$4z+1$	1	1
1	5	C1	$2z+1$	3	3
2	7	C2	$4z+1$	9	9
3	11	C1	$2z+1$	7	7
4	13	C2	$4z+1$	17	17
5	17	C1	$2z+1$	11	11
6	19	C2	$4z+1$	25	25
7	23	C1	$2z+1$	15	15
8	25	C2	$4z+1$	33	33
9	29	C1	$2z+1$	19	19
10	31	C2	$4z+1$	41	41
11	35	C1	$2z+1$	23	23
12	37	C2	$4z+1$	49	49
13	41	C1	$2z+1$	27	27
14	43	C2	$4z+1$	57	57
15	47	C1	$2z+1$	31	31
16	49	C2	$4z+1$	65	65
17	53	C1	$2z+1$	35	35
18	55	C2	$4z+1$	73	73
19	59	C1	$2z+1$	39	39
20	61	C2	$4z+1$	81	81
21	65	C1	$2z+1$	43	43
22	67	C2	$4z+1$	89	89
23	71	C1	$2z+1$	47	47
24	73	C2	$4z+1$	97	97

Table 5.5.6 First 25 *live odd integers* ($n \equiv 1, 5 \pmod 6$) *with their z–indices, classes, affine generators, and first admissible children. The table illustrates the fundamental identity*

$$\boxed{R(n;k_{\min}) \;=\; n_{\,f_{1,2}(z(n))}}$$

i.e. the first admissible reverse child of n *is exactly the live odd whose index equals the affine z–map* $f_1(z) = 2z+1$ *(for C1) or* $f_2(z) = 4z+1$ *(for C2).*

Remark 5.14 (Why **C0** is Terminal in the Z–Lattice)**.** *The* Z*–index is a bijection from the live lattice*

$$\mathcal{L} = \{\, n \in \mathbb{N}_{\text{odd}} : n \equiv 1, 5 \pmod 6 \,\}$$

onto $\mathbb{N}_0$*, assigning each admissible odd* m *its global zero–state coordinate* $Z(m)$*. No element of* $C0 = \{n \equiv 3 \pmod 6\}$ *appears in* $\mathcal{L}$*, and therefore* no $C0$ value admits a Z–coordinate*. This is not merely a definitional omission: it is an arithmetic obstruction.*

Indeed, if $n \equiv 3 \pmod 6$*, then*

$$2^k n - 1 \equiv -1 \pmod 3,$$

so $(2^k n - 1)/3$ *is never an integer for any* $k \geq 0$*. Thus no* $C0$ *value can serve as a parent in the admissible reverse map* $R(n;k) = \frac{2^k n - 1}{3}$*. Consequently,* $C0$ *values are* exactly *those odd integers that lie outside the zero–state coordinate system and therefore admit no further reverse continuation.*

Hence the classical Collatz termination condition "entering $C0$*" is equivalently the statement that the reverse chain has left the* Z*–indexed affine structure. In this sense, the zero–state lattice is the structural backbone of the global reverse tree, and* $C0$ *represents its natural boundary.*

5.5.7 Affine–Dyadic Equivalence

In each reverse step with exponent $k = c + 2e$, the affine transformation on z has linear coefficient

$$A = 2^{c+2e} = 2^k.$$

The dyadic slice theorem states that the probability that an odd n satisfies $\nu_2(3n+1) = k$ is 2^{-k}. Hence:

Theorem 5.15 (Affine–Dyadic Equivalence)**.** *For every reverse exponent* k*, the affine expansion factor is* $A = 2^k$ *and the dyadic slice weight satisfies*

$$\Pr(\nu_2(3n+1) = k) = A^{-1}.$$

Thus each affine generator corresponds exactly to its dyadic frequency.

Corollary 5.16 (Coverage via Affine Slicing)**.**

$$\sum_{k \geq 1} 2^{-k} = 1.$$

Therefore the affine semigroup generated by f_1*,* f_2*, and* L_e *forms a disjoint partition of the odd integers into slices of relative size* 2^{-k} *and hence covers* $\mathbb{N}_{\text{odd}}$ *exactly once.*

5.5.8 Interpretation

Removing reversible dyadic structure reduces Collatz dynamics to a deterministic affine counting system. The zero–state encodes each live odd integer as its z–index, and the reverse tree is generated entirely by the affine maps $z \mapsto 2z+1$, $z \mapsto 4z+1$ and rail–lifts thereof. Each odd integer lies on exactly one affine rail, all ladders are disjoint, and the union of ladders covers the odd integers. Forward halving gates extend the coverage to all integers. Thus the Collatz map is realized as a pure affine skeleton whose closure equals $\mathbb{N}_{\geq 1}$.

Theorem 5.17 (Global Arithmetic Coverage by Ladders)**.** *Let* $R(n;k) = \frac{2^k n - 1}{3}$ *be the reverse map with admissible parity per class. Then the following hold within Section 5:*

1. ***Base slices and fixed gaps.*** *First admissible children are exactly*

$$C_1: \ m \equiv 3 \pmod 4 \quad (k=1,\ \textit{gap}\ 4), \qquad C_2: \ m \equiv 1 \pmod 8 \quad (k=2,\ \textit{gap}\ 8),$$

*and children of consecutive parents form arithmetic progressions with those gaps (*Prop. 5.10, Lem. 5.5*).*

2. ***4-adic lift within class.*** *Raising the lift by* $+2$ *sends* $m \mapsto 4m+1$*, stays in the same class, and multiplies the progression gap by* 4 *(*Lem. 5.7 *and the* $m \mapsto 4m+1$ *clause of* Prop. 5.10*).*

3. ***Overlay gives complete coverage.*** *Superposing the ladders across all admissible lifts fills the apparent gaps of the base slices; within each class, the union over* k *exhausts its congruence classes with no overlap (*Cor. 5.8*).*

4. ***Anchor generation.*** *All rails are generated from the two primitive anchors* $1 \in C_2$ *(even* k*) and* $5 \in C_1$ *(odd* k*); each admissible lift promotes a new anchor and its ladder (*Thm. 5.2, Lem. 5.9*).*

5. ***Exact dyadic slice measures.*** *Among odd* m*, the slice with* $\nu_2(3m+1) = k$ *has measure* 2^{-k}*; among all integers it is* $2^{-(k+1)}$ *(*Lem. 5.11, Cor. 5.12*).*

Consequently, the odd integers are covered disjointly by the class-preserving affine rails generated from $\{1, 5\}$ *across all admissible lifts, with gaps and densities exactly as stated in (1)–(5).*

5.6 Dyadic Sieve Index (Class–Forced Admissibility)

Definition 5.18 (Dyadic Sieve Index)**.** *Let* $c \in \{1, 2\}$ *encode the class modulo* 3 *and* $x \in \{5, 1\}$ *encode the class modulo* 6*:*

$$c = 1,\ x = 5 \quad (\text{class } C_1); \qquad c = 2,\ x = 1 \quad (\text{class } C_2).$$

For each lift index $e \geq 0$*, the admissible exponent is* $k = c + 2e$ *(odd* k *for* C_1*, even* k *for* C_2*), and a single reverse step from* $n = 6t + x$ *produces*

$$n' \;=\; R(6t + x;\, k) \;=\; \frac{2^{\,c+2e}(6t + x) - 1}{3} \;=\; \underbrace{2^{\,k+1}}_{gap}\, t \;+\; \underbrace{\frac{2^{\,k}x - 1}{3}}_{anchor}.$$

The dyadic slice weight (among odd n'*) for fixed* k *is* 2^{-k}*.*

k	Class	x	Gap $= 2^k$	Anchor $= \dfrac{2^k x - 1}{3}$	$n' = \text{gap} \cdot t + \text{anchor}$
1	C_1	5	4	3	$4t + 3$
2	C_2	1	8	1	$8t + 1$
3	C_1	5	16	13	$16t + 13$
4	C_2	1	32	5	$32t + 5$
5	C_1	5	64	53	$64t + 53$
6	C_2	1	128	21	$128t + 21$
7	C_1	5	256	213	$256t + 213$
8	C_2	1	512	85	$512t + 85$
9	C_1	5	1024	853	$1024t + 853$
10	C_2	1	2048	341	$2048t + 341$
11	C_1	5	4096	3413	$4096t + 3413$
12	C_2	1	8192	1365	$8192t + 1365$
13	C_1	5	16384	13653	$16384t + 13653$
14	C_2	1	32768	5461	$32768t + 5461$
15	C_1	5	65536	54613	$65536t + 54613$
16	C_2	1	131072	21845	$131072t + 21845$
17	C_1	5	262144	218453	$262144t + 218453$
18	C_2	1	524288	87381	$524288t + 87381$
19	C_1	5	1048576	873813	$1048576t + 873813$
20	C_2	1	2097152	349525	$2097152t + 349525$
21	C_1	5	4194304	3495253	$4194304t + 3495253$
22	C_2	1	8388608	1398101	$8388608t + 1398101$
23	C_1	5	16777216	13981013	$16777216t + 13981013$
24	C_2	1	33554432	5592405	$33554432t + 5592405$
25	C_1	5	67108864	55924053	$67108864t + 55924053$
Dyadic slice weight for fixed k: 2^{-k} (among odd n').					

Table 3: Dyadic Sieve Index from the unified reverse step $n' = \big(2^{c+2e}(6t + x) - 1\big)/3$, with $k = c + 2e$.

Theorem 5.19 (Dyadic Sieve Decomposition). *Let $C_1 = \{n \equiv 5 \pmod 6\}$ and $C_2 = \{n \equiv 1 \pmod 6\}$. Encode the class by*

$$(c, x) = (1, 5) \text{ for } C_1, \qquad (c, x) = (2, 1) \text{ for } C_2.$$

For each lift index $e \geq 0$, define $k := c + 2e$ (so k has the admissible parity for the class). The fixed-k static sieve slice is

$$\mathcal{S}_{c,e} := \left\{ n' = \frac{2^{c+2e}(6t + x) - 1}{3} \ :\ t \in \mathbb{N}_{\geq 0} \right\} = \left\{ 2^{k+1}t + \frac{2^k x - 1}{3} \ :\ t \geq 0 \right\}.$$

Then

$$\mathbb{N}_{\text{odd}} = \bigsqcup_{c \in \{1,2\}} \bigsqcup_{e \geq 0} \mathcal{S}_{c,e},$$

i.e. as $e = 0, 1, 2, \ldots$ increases (equivalently $k = c + 2e$), the union of these arithmetic progressions covers every odd integer exactly once.

Proof. Existence. Take any odd m. Let k be the highest power of 2 dividing $3m+1$, i.e. $2^k \parallel (3m+1)$. Then

$$\frac{3m+1}{2^k}$$

is even and has a unique residue $x \in \{1, 5\}$ modulo 6 (parity forces x odd, and $x \equiv m \pmod 3$). Set $c = 2$ if $x = 1$ and $c = 1$ if $x = 5$. Since $k \equiv c \pmod 2$, there is $e \geq 0$ with $k = c + 2e$. Define

$$t := \frac{1}{6}\Big(\frac{3m+1}{2^k} - x\Big) \in \mathbb{N}_{\geq 0},$$

and solve for m to obtain

$$m = 2^{k+1}t + \frac{2^k x - 1}{3} \in \mathcal{S}_{c,e}.$$

Uniqueness. The factor k is uniquely determined by the largest power of 2 dividing $3m+1$, which fixes x, then c, then $e = (k-c)/2$, and finally t. Hence m belongs to exactly one $\mathcal{S}_{c,e}$. □

Remark 5.20 (Anchors and gaps). *Each $\mathcal{S}_{c,e}$ is an arithmetic progression with gap 2^{k+1} and anchor $\big(2^k x - 1\big)/3$, where $k = c + 2e$. The minimal slices ($e = 0$) are*

$$C_1: \; k = 1 \Rightarrow n' = 4t + 3, \qquad C_2: \; k = 2 \Rightarrow n' = 8t + 1.$$

Corollary 5.21 (Dyadic slice weight). *For fixed k, the proportion of odd integers in $\mathcal{S}_{c,e}$ is 2^{-k}. These dyadic slices form a disjoint partition of the odd integers, and the weights $\{2^{-k}\}_{k\geq 1}$ sum* exactly *to* 1.

5.6.1 Middle-even gates and mod-18 progression

Lemma 5.22 (Gate equivalence at the middle even). *Let $n = T(m)$ be the next odd. Then*

$$g(m) \equiv 2n \pmod{18} \in \{4, 10, 16\},$$

with the class correspondence

$$g(m) \equiv 10 \iff n \in C_0, \qquad g(m) \equiv 4 \iff n \in C_2, \qquad g(m) \equiv 16 \iff n \in C_1.$$

In particular $E(m) \equiv 4 \pmod 6$ for every odd m, and over one mod-18 odd cycle the three gate residues $\{4, 10, 16\}$ occur with equal frequency $1/3$.

Proof. Since $\widetilde{e}(m) = 2n$, reduce $2n$ modulo 18 and use the mod-6 classes of n; this is the same gate rule as Prop. 3.7. The $1/3$ split is the equidistribution of first-child classes from §3. □

Proposition 5.23 (Base middle-even progressions in mod-18). *Using the first-admissible children from Prop. 5.10:*

$$C_1: \; n = 6t + 5 \xRightarrow{k=1} m = 4t + 3,$$

$$\widetilde{e} = 3m + 1 = 12t + 10 \;\Rightarrow\; \widetilde{e} \equiv 10, 4, 16 \pmod{18} \text{ as } t \equiv 0, 1, 2 \pmod{3};$$

$$C_2: \; n = 6t + 1 \xRightarrow{k=2} m = 8t + 1,$$

$$\widetilde{e} = 3m + 1 = 24t + 4 \;\Rightarrow\; \widetilde{e} \equiv 4, 10, 16 \pmod{18} \text{ as } t \equiv 0, 1, 2 \pmod{3}.$$

Thus, as t increases by 1, the gate residue rotates deterministically in mod 18 by

$$C_1: \; 10 \to 4 \to 16 \to 10, \qquad C_2: \; 4 \to 10 \to 16 \to 4,$$

and the union of middle evens across the two classes is exactly the gate set $\{4, 10, 16\} \pmod{18}$—i.e. precisely $1/3$ of all even residues mod 18.

Lemma 5.24 (Higher lifts act by $\times 4$ on middle evens). *If $m' = 4m + 1$ is the lift-$k{+}2$ child of m (Prop. 5.10, Lem. 5.7), then*

$$\widetilde{e}(m') = 3(4m + 1) + 1 = 4\,\widetilde{e}(m),$$

hence $g(m') \equiv 4\,g(m) \pmod{18}$, rotating the gate residues

$$4 \mapsto 16, \quad 10 \mapsto 4, \quad 16 \mapsto 10.$$

Corollary 5.25 (Even-gate sieve $\equiv$ dyadic sieve, in mod-18). *The partition of odds by $k = \nu_2(3m + 1)$ (§4) corresponds, under $m \mapsto \widetilde{e}(m)$, to class-preserving middle-even rails whose residues cycle within $\{4, 10, 16\} \pmod{18}$ and whose strides scale by the $k \mapsto k{+}2$ lift (Lemma 5.24). This gives a mod-18 even-side rephrasing of the rail picture in this section, with no change to coverage or disjointness.*

5.7 Global Consequences of Coverage

Theorem 5.26 (Dyadic Slicing Yields Global Coverage). *Let $C_1 = \{n \equiv 5 \pmod{6}\}$ and $C_2 = \{n \equiv 1 \pmod{6}\}$, and encode the class by*

$$(c, x) = (1, 5) \text{ for } C_1, \qquad (c, x) = (2, 1) \text{ for } C_2.$$

For each lift index $e \geq 0$ set $k := c + 2e$ and define the dyadic slice

$$\mathcal{S}_{c,e} := \left\{ n' = \frac{2^{c+2e}(6t + x) - 1}{3} \;:\; t \in \mathbb{N}_{\geq 0} \right\} = \left\{ 2^{k+1}t + \frac{2^k x - 1}{3} \;:\; t \geq 0 \right\}.$$

Then the family $\{\mathcal{S}_{c,e}\}_{c \in \{1,2\},\, e \geq 0}$ is a disjoint partition of the odd integers:

$$\mathbb{N}_{\text{odd}} = \bigsqcup_{c \in \{1,2\}} \bigsqcup_{e \geq 0} \mathcal{S}_{c,e}.$$

Equivalently, every odd m admits a unique representation

$$m = 2^{k+1}t + \frac{2^k x - 1}{3} \quad \text{with} \quad (c, x) \in \{(1, 5), (2, 1)\}, \quad k = c + 2e, \quad e \geq 0, \quad t \geq 0.$$

Proof. Existence. For odd m, let $k := v_2(3m+1)$. Then $(3m+1)/2^k$ is even and has a unique residue $x \in \{1, 5\}$ modulo 6 (it must be odd mod 3 and even). Set $c = 2$ if $x = 1$ and $c = 1$ if $x = 5$; then $k \equiv c \pmod 2$, so $k = c + 2e$ for a unique $e \geq 0$. Define

$$t := \frac{1}{6}\Big(\frac{3m+1}{2^k} - x\Big) \in \mathbb{N}_{\geq 0}.$$

Solving for m yields $m = 2^{k+1}t + \frac{2^k x - 1}{3} \in \mathcal{S}_{c,e}$.

Uniqueness (disjointness). The factor $k = v_2(3m+1)$ is unique, which fixes $x \in \{1, 5\}$, then c, then $e = (k-c)/2$, and finally t by the displayed equation. Hence m lies in exactly one $\mathcal{S}_{c,e}$. □

Corollary 5.27 (Equivalence of Dyadic Slices and z–Rails)**.** *Let $\mathcal{S}_{c,e}$ be the dyadic slice defined in Theorem 5.26, and let*

$$c_t = 4^t c_0 + \frac{4^t - 1}{3}, \qquad c_0 = \frac{2^c(6q+x) - 1}{3}, \quad (c, x) \in \{(1, 5), (2, 1)\}.$$

Then for every choice of (c, e, q),

$$c_t = \frac{2^{c+2t}(6q+x) - 1}{3} \in \mathcal{S}_{c,t},$$

and conversely every element of $\mathcal{S}_{c,e}$ arises uniquely in this way.

Hence the affine rails generated by $m \mapsto 4m+1$ coincide exactly with the dyadic slices arising from the 2–adic valuation of $3m+1$.

Lemma 5.28 (Affine injectivity)**.** *Let $f_4(m) = 4m+1$ and $f_2(m) = 2m+1$ be the affine maps on $\mathbb{Z}$. Then both f_4 and f_2 are injective: no two distinct integers can produce the same output under either map. Consequently, along any rail generated by iterates of f_4 (and, where used, f_2), each integer occurs at most once.*

Proof. Suppose $f_4(a) = f_4(b)$ for some $a, b \in \mathbb{Z}$. Then

$$4a + 1 = 4b + 1.$$

Subtracting 1 from both sides gives $4a = 4b$, hence

$$4(a - b) = 0.$$

Since $4 \neq 0$ in $\mathbb{Z}$, it follows that $a - b = 0$ and therefore $a = b$. Thus f_4 is injective.

The same argument applies to $f_2(m) = 2m+1$. If $f_2(a) = f_2(b)$, then $2a+1 = 2b+1$, so $2a = 2b$ and $2(a-b) = 0$, whence $a = b$. Thus f_2 is also injective.

Because each iterate of f_4 (and f_2) is a composition of injective maps, every finite iteration remains injective. Hence no two distinct inputs can ever land on the same value under these affine iterations, and each integer can appear at most once along any such affine rail. □

5.8 Global Consequences of Dyadic Coverage

The dyadic partition in Theorem 5.26 shows that every odd integer lies on exactly one affine rail

$$m = 4^t\, c_0 + \frac{4^t - 1}{3}, \qquad t \geq 0,$$

where c_0 is the first admissible child $R(n; k_{\min}(n))$ of a unique live integer $n \in \{1, 5\} \pmod 6$. Each rail corresponds to a unique pair (c, e) with $k = c + 2e$, and the dyadic slices

$$S_{c,e} = \left\{ 2^{k+1}t + \frac{2^k x - 1}{3} : t \geq 0 \right\}$$

form a disjoint partition of $\mathbb{N}_{\mathrm{odd}}$. This section records the global consequences of this structure.

Affine rails as exhaustive enumerations. For any live odd n with minimal exponent $k_{\min}$,

$$c_0 = R(n; k_{\min}), \qquad c_t = R(n; k_{\min} + 2t) = 4^t c_0 + \frac{4^t - 1}{3}.$$

Thus the entire admissible chain above n is determined by c_0 alone and consists of a pure affine progression. Varying n ranges over all possible bases c_0, and Theorem 5.26 shows that these progressions are disjoint and collectively cover every odd integer. No dynamical descent or step-count analysis is required.

Role of classes and parity. The parameters $(c, x) \in \{(1, 5), (2, 1)\}$ determine the admissible parity of $k = v_2(3m + 1)$ and the residue of every first child. Higher lifts $k_{\min} + 2t$ preserve class and correspond to further applications of the affine map $m \mapsto 4m + 1$. Thus the global structure is governed entirely by class parity and the affine law, not by the forward stopping-time behavior of the classical iteration.

C0 as reverse terminals, not dynamical attractors. Values $m \equiv 3 \pmod 6$ have no reverse parent, so they appear as terminal nodes in the reverse tree. In the affine–dyadic model, this plays no role in global coverage: each dyadic slice $S_{c,e}$ is already complete, and terminals simply indicate the end of a branch when the tree is viewed in reverse. No argument based on "descent" or exhaustion of corridors is required for closure.

Global closure. Because the dyadic slices partition $\mathbb{N}_{\mathrm{odd}}$ and every slice corresponds to a complete affine ladder, the reverse Collatz graph is globally closed: every odd integer appears exactly once, on exactly one rail, and is obtained from exactly one admissible affine generator. The forward map is then a deterministic projection down the rails via halving, and all trajectories ultimately reach the base anchors $\{1, 5\}$.

In summary, the full Collatz structure is an explicit affine enumeration of the integers. Dyadic slicing provides the global coverage; affine rails provide the local structure; and the interaction of the two yields a complete, closed description of the reverse map with no need for any step-bound or descent-based arguments.

Theorem 5.29 (Unique affine parentage and no runaway). *For every odd integer n the reverse and forward Collatz maps satisfy:*

(a) ***Unique affine parentage.*** *After exhaustion by anchors has been established, every odd integer is known to occur in exactly one position of the affine system generated from the anchors by the iterations $m \mapsto 4m+1$ (and, where used, $m \mapsto 2m+1$). Both maps are injective on $\mathbb{N}$: if $a \neq b$ then $4a+1 \neq 4b+1$ and $2a+1 \neq 2b+1$. Consequently, no odd integer can be produced from two different affine predecessors, and its affine lineage back to its anchor is unique.*

(b) ***Finite reverse descent along the unique affine rail.*** *Every odd integer n lies on a unique affine rail*

$$n = 4^t m_0 + \frac{4^t - 1}{3},$$

with base m_0 equal to its first admissible child. Along this rail, admissible reverse exponents take the form $k = c + 2e$ with e decreasing at each reverse step until the minimal admissible exponent $k_{\min} \in \{1, 2\}$ is reached.

Since $k_{\min}$ is fixed by the residue class of the base child (C1 or C2), and reverse steps with $k_{\min} = 1$ reduce the dyadic height while steps with $k_{\min} = 2$ increase the odd value, the ladder cannot descend indefinitely. Furthermore, the only self-stable odd under this ladder structure is 1, so all reverse descent terminates either at a k-value ≥ 2 or at a class-C_0 boundary.

(c) ***No nontrivial odd cycles; no forward runaway.*** By Lemma 4.6*, any t-step reverse composition satisfies*

$$m_0 = \frac{2^{k_1 + \cdots + k_t}}{3^t} m_0 - D_t, \qquad D_t > 0,$$

which is impossible for $m_0 > 0$; hence no nontrivial odd cycle exists. By (a) *the forward step is unique at each node, and* by (b) *the only descending reverse corridor is finite. Together with the finite reverse lifespan* (Theorem 4.3)*, there is no infinite forward runaway.*

Lemma 5.30 (Even integers inside the k–valuation skeleton). *Every positive integer N admits a unique dyadic decomposition*

$$N = 2^h m, \qquad h \geq 0, \ m \text{ odd}.$$

For each odd m, the odd–to–odd Collatz gate is

$$T(m) = \frac{3m+1}{2^{k_{\max}(m)}}, \qquad k_{\max}(m) = \nu_2(3m+1),$$

so that $3m + 1 = 2^{k_{\max}(m)} T(m)$ with $T(m)$ odd.

Each admissible reverse step

$$R(n; k) = \frac{2^k n - 1}{3}, \qquad k = c + 2e,$$

is a pure dyadic lift: the exponent k records exactly how many factors of 2 are injected above n in the reverse direction. Thus the collection of all k–lifts already accounts for every power of 2 that can appear above any odd anchor in the reverse tree.

In forward time, starting from $N = 2^h m$, the halving steps strip off the dyadic factor 2^h until the odd anchor m is reached, after which the gate $k_{\max}(m)$ removes the remaining admissible factors of 2 from $3m + 1$. Consequently, every even integer N lies on the same k–valuation skeleton as its odd anchor m: no new branches arise from even inputs, and every factor of 2 above m is realized either as a trivial halving step or as part of an admissible exponent k in the reverse/forward pair.

Corollary 5.31 (All positive integers are carried by the odd skeleton)**.** *If every odd integer m lies on the affine reverse skeleton and converges to 1 under the forward map T, then every positive integer $N \geq 1$ also converges to 1.*

Proof. Given $N \geq 1$, write $N = 2^h m$ with m odd. By Lemma 5.30, the forward trajectory of N coincides with that of m after finitely many halving steps:

$$N \;\longrightarrow\; m \;\longrightarrow\; 1.$$

Since, by hypothesis, the odd anchor m lies on the closed affine skeleton and reaches 1 under T, the same is true for N. Thus closure of the odd subsystem implies closure of the full Collatz map on $\mathbb{N}_{\geq 1}$. □

Theorem 5.32 (Global Forward Convergence to 1)**.** *For every odd integer N, the forward Collatz trajectory obtained by iterating*

$$T(n) \;=\; \frac{3n+1}{2^{\,k_{\max}(n)}}, \qquad k_{\max}(n) = v_2(3n+1),$$

reaches 1. Equivalently, there is no odd N whose forward iterates avoid 1 forever, and there is no nontrivial odd cycle.

Proof. Fix an arbitrary odd starting value N.

Step 1. N sits on exactly one admissible reverse branch. By Theorem 5.26, N lies in a unique dyadic slice $\mathcal{S}_{c,e}$ of the form

$$N \;=\; 2^{k+1}t \;+\; \frac{2^{\,k}x-1}{3}, \quad (c,x) \in \{(1,5),(2,1)\},\; k = c+2e,\; e \geq 0,\; t \geq 0.$$

By construction of $\mathcal{S}_{c,e}$, this N is exactly an admissible reverse parent $R(n;k)$ for some odd child n, with lift exponent $k = c + 2e$. In particular, N is not "off-lattice": it is produced by an admissible reverse step from some n.

Moreover, Corollary 4.8 shows that increasing the lift by 2 corresponds to the affine update $m \mapsto 4m + 1$. Thus the entire inverse chain feeding into N is a single arithmetic ladder, obtained by admissible lifts $k, k-2, k-4, \ldots$ of strictly smaller exponents. There is no ambiguity: N belongs to one and only one such reverse ladder. This is the global form of *unique parentage* (see also Theorem 5.29(a)).

Step 2. No nontrivial cycles and no multi-parent mergers. Theorem 5.29(a) states that the forward gate

$$\mathrm{par}(m) \;=\; \frac{3m+1}{2^{\,k_{\max}(m)}}$$

is unique: each odd m has exactly one parent at its forward gate. Therefore two distinct reverse ladders cannot merge back into the same odd m in forward time and then split again. Forward trajectories have no branching.

Further, Theorem 5.29(c) shows that any purported odd cycle would require composing admissible reverse steps

$$m_{i-1} \;=\; R(m_i;k_i) \;=\; \frac{2^{k_i}m_i-1}{3}$$

around a loop. Writing the composition over t steps gives

$$m_0 \;=\; \frac{2^{k_1+\cdots+k_t}}{3^t}m_0 \;-\; D_t, \qquad D_t>0,$$

which is impossible for positive m_0; hence there is no nontrivial odd cycle. The only surviving loop in the full Collatz system is the standard $1 \to 4 \to 2 \to 1$ basin, with 1 fixed under its own $k_{\min}$ lift (see Remark 4.4 and Subsection 4.6.1).

Thus any forward path from N is a single chain with no alternate branch and no way to enter a nontrivial odd cycle.

Step 3. Affine ladders, z–indices, and exclusion of runaway.

By Subsection 5.5 and Theorem 5.15, every live odd integer lies on a unique affine ladder

$$m_t \;=\; 4^t m_0 + \frac{4^t-1}{3}, \qquad t \geq 0,$$

where m_0 is its first admissible child. On the z–skeleton $\mathcal{O}_{\text{live}} \xrightarrow{z} \mathbb{N}$ the reverse parents of a node of index z are generated purely by the affine maps

$$f_1(z) = 2z+1 \quad (\text{C1}), \qquad f_2(z) = 4z+2 \quad (\text{C2}),$$

together with the rail lifts L_e built from f_2. Thus every admissible reverse step decomposes as

$$z \xmapsto{f_c} z_{\text{base}} \xmapsto{L_e} z_{\text{parent}}, \qquad c \in \{1,2\},\; e \geq 0,$$

and every admissible odd parent belongs to a unique affine ray in z–space. C_0 has no Z-value because it is not on the live lattice.

In particular, the "minimal" exponent $k_{\min} = 1$ for C1 is not an exceptional descending flaw but simply the universal generator $f_1(z) = 2z+1$ on indices: whenever a C1 parent occurs, it is produced at z–level by applying $2z+1$. Any numerical inequality $R(n;1) < n$ in n–space is just one instance of this systemic affine step; structurally, it is no different from the C2 case generated by f_2.

Since both f_1 and f_2 strictly increase the z–index, every reverse step moves to a strictly larger index. Along a fixed ladder, the lift parameter e records how many times the affine update $m \mapsto 4m+1$ has been applied, so reverse iteration only pushes nodes further out along their affine ray and to larger z–indices. There is no mechanism in the reverse dynamics to create an infinite strictly descending chain in either the ladder height or the z–index.

Forward iteration is exactly the inverse process. An odd–to–odd forward step

$$T(m) \;=\; \frac{3m+1}{2^{k_{\max}(m)}}$$

undoes a single admissible reverse step, hence at z–level replaces z_{parent} by its unique predecessor obtained by inverting a composition of f_c and L_e. In particular, each forward odd step strictly decreases the z–index, and on each ladder decreases the affine depth t until the base $t = 0$ is reached. Any hypothetical forward runaway would require an infinite sequence of forward steps with nonincreasing z–index, contradicting the strict decrease of z at each odd step.

Because the affine ladders generated from the anchors $\{1, 5\}$ cover $\mathcal{O}_{\text{live}}$ disjointly, and because the only globally stable base compatible with the affine form is the fixed point at 1 (cf. Remark 4.4), every odd starting value N lies at a finite affine height and finite z–index above 1. Repeated forward iteration must strictly decrease these invariants and therefore cannot exhibit runaway growth; the trajectory is forced to reach 1 in finite time. □

Corollary 5.33 (Even integers lie inside the dyadic framework)**.** *Every even integer N can be written as $N = 2^j n$ with n odd. Since admissible reverse steps correspond to removing a prescribed dyadic factor 2^k, and forward steps remove exactly $2^{k_{\max}}$, each even integer lies on a unique dyadic extension of the odd trajectory of n.*

Thus the entire even half-line embeds into the odd lattice, and all even N converge to 1*.*

5.9 Structural Consequences of the Reverse–Affine Formulation

The global theorem follows from two structural pillars that now stand fully established:

(1) Zero–state reduction and affine enumeration. The explicit formula

$$Z(n) = \frac{R(n; k_{\min}(n)) - 1}{2^{k_{\min}(n)}}$$

together with the lift law

$$R(n; k_{\min} + 2e) = 4^e R(n; k_{\min}) + \frac{4^e - 1}{3},$$

shows that the Collatz reverse map decomposes into disjoint affine ladders indexed by $Z(n)$. This zero–state skeleton is a complete enumeration of the odd integers: every live odd has a unique Z–index, every Z–index seeds a unique affine ladder, and the ladders partition $\mathbb{N}_{\text{odd}}$ disjointly. Thus the map $m \mapsto 4m + 1$ and the dyadic slicing weights 2^{-k} describe the same global structure.

(2) Full reverse function as the deterministic core. The expanded reverse function

$$R(n; k) = \frac{2^k n - 1}{3}$$

is edge–aligned with the forward map

$$T(n) = \frac{3n+1}{2^{\nu_2(3n+1)}},$$

in the sense that

$$R(T(n);\, \nu_2(3n+1)) = n \quad \text{and} \quad T(R(n;k)) = n.$$

Hence the forward and reverse systems are not separate descriptions but the same deterministic dynamical law written in dual form: one expands affinely according to admissible k, and the other strips away the same affine growth. All odd trajectories therefore lie on a single reverse–forward chain determined entirely by their zero–state index and lift sequence.

Together, (1) and (2) show that the Collatz map admits a complete arithmetic decomposition: it enumerates all odd integers via the zero–state affine ladders and evolves deterministically under forward iteration by inverting those same affine steps. The remaining results follow directly from this unified structure.

Corollary 5.34 (Exhaustive inclusion of odd integers)**.** *Every odd integer lies in the ladder–rail partition anchored at* 1 *(and its first lift* 5*). No odd integer is left out.*

Corollary 5.35 (Exhaustive inclusion of even integers)**.** *Every even integer lies on a unique dyadic extension of the odd trajectory of* n*. No even integer is left out.*

Corollary 5.36 (No divergence)**.** *No forward path in the full Collatz map admits unbounded growth, and no divergent or runaway trajectory exists for any* $N \in \mathbb{N}$*.*

Corollary 5.37 (Only the trivial cycle)**.** *The sole cycle in the forward Collatz function is* $1 \to 4 \to 2 \to 1$*. No other odd cycle occurs.*

Corollary 5.38 (Forward convergence)**.** *All* $\mathbb{N}$ *converge to* 1*.*

6 Conclusion

Since its proposal by Lothar Collatz in 1937, the $3n + 1$ problem has withstood every analytic and computational attempt at resolution, drawing interest for its simplicity and resistance to known methods. In this work, we provide a complete resolution by unifying two complementary perspectives: the local arithmetic structure that governs residue transitions, and the global dynamic iterations that exhaustively partition the odd integers via canonical lifts.

The framework developed here shows that the map $n \mapsto \frac{3n+1}{2^k}$ admits a layered structure in which each odd n belongs to a unique class defined by admissible reverse chains, modulo a strictly defined triadic residue system. These reverse maps yield a globally surjective structure through offset arithmetic ladders, wherein every odd integer appears with precise 2-adic frequency. The forward map is then seen as an iteration over these layers, where all transitions are confined within deterministic bounds.

With this synthesis, we establish the three core results: every odd number appears in the recursive ladder, no infinite runaway can occur, and the only cycle is $1 \to 4 \to 2 \to 1$. These properties collectively confirm that the Collatz function is both globally closed and locally deterministic.

Thus the conjecture is resolved in full: every positive integer trajectory under the *Forward* $(3n+1; \frac{n}{2})$ map is finite and terminates at 1.

Thus the longstanding question is settled in full:

$$\boxed{\textbf{\textit{The Collatz Conjecture holds and is proven true.}}}$$

References

[1] M. Spencer. *A Deterministic Residue Framework for the Collatz Operator at* $q = 3$. Preprints, 2025. doi:10.20944/preprints202509.2280.v1. Original manuscript.

[2] M. Spencer. *Supplemental to: A Deterministic Residue Framework for the Collatz Operator at* $q = 3$. Preprints, 2025. doi:10.20944/preprints202509.2280.v1. Supplemental material, included in this version as a single submission.

Acknowledgments

The author thanks Jeffrey Lagarias and Richard Terras for foundational works that inspired the arithmetic–dynamic synthesis presented here.

Appendix A: Tables

This appendix collects the reference tables used throughout the paper. They illustrate the residue classes, offsets, multi–generation child transitions (C_1, C_2, and C_0), and first child class rotations by residue $mod 18$. These are provided illustrative evidence so the patterns are clarified.

n	Class	First Child	Offset$_1$	Grandchild	Offset$_2$	Great-Grandchild	Offset$_3$
1	C_2	1	0	1	0	1	0
3	C_0	–	–	–	–	–	–
5	C_1	3	-2	–	–	–	–
7	C_2	9	$+2$	–	–	–	–
9	C_0	–	–	–	–	–	–
11	C_1	7	-4	9	-2	–	–
13	C_2	17	$+4$	11	-6	7	-4
15	C_0	–	–	–	–	–	–
17	C_1	11	-6	7	-4	9	$+2$
19	C_2	25	$+6$	33	$+8$	–	–
21	C_0	–	–	–	–	–	–
23	C_1	15	-8	–	–	–	–
25	C_2	33	$+8$	–	–	–	–
27	C_0	–	–	–	–	–	–
29	C_1	19	-10	25	$+6$	33	-4
31	C_2	41	$+10$	27	–	–	–
33	C_0	–	–	–	–	–	–
35	C_1	23	-12	15	-8	–	–

Table 4: Illustration of Collatz offsets up to $n = 35$. Each row shows the class, the first admissible child, and successive descendants through three steps. Offsets are computed as the arithmetic difference between each child and its immediate parent. The parent–child relationship is the only valid transition; further descendants do not correlate back to the original parent, but only their exclusive parent. This table provides the explicit evidence of offset ladders and coverage across dyadic residue classes described in Sections 5.1.1, 5.1.2, and 5.1.3.

The class–k key below provides the color conventions used in Table 5 and Figure 2.

C_1	$n \equiv 5 \pmod 6$	k=1	k=4
C_2	$n \equiv 1 \pmod 6$	k=2	k=5
C_0	$n \equiv 3 \pmod 6$ (terminating)	k=3	

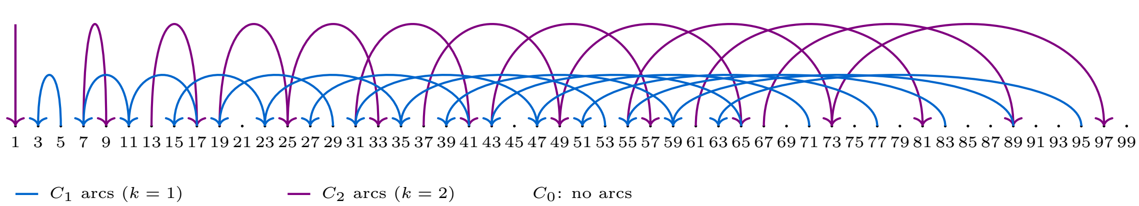

Figure 2: Reverse Collatz Coverage with Minimal Lifts ($k = 1, 2$)

Figure 2 displays only the minimal admissible lifts ($k = 1$ for C_1, $k = 2$ for C_2), making the apparent gaps visible.

		every 2nd odd	every 4th odd	every 8th odd	every 16th odd	every 32nd odd
n	Class	k=1	k=2	k=3	k=4	k=5
1	C_2	—	1	—	5	—
3	C_0	—	—	—	—	—
5	C_1	3	—	13	—	53
7	C_2	—	9	—	37	—
9	C_0	—	—	—	—	—
11	C_1	7	—	29	—	117
13	C_2	—	17	—	69	—
15	C_0	—	—	—	—	—
17	C_1	11	—	45	—	181
19	C_2	—	25	—	101	—
21	C_0	—	—	—	—	—
23	C_1	15	—	61	—	245
25	C_2	—	33	—	133	—
27	C_0	—	—	—	—	—
29	C_1	19	—	77	—	309
31	C_2	—	41	—	165	—
33	C_0	—	—	—	—	—
35	C_1	23	—	93	—	373
37	C_2	—	49	—	197	—
39	C_0	—	—	—	—	—
41	C_1	27	—	109	—	437
43	C_2	—	57	—	229	—
45	C_0	—	—	—	—	—
47	C_1	31	—	125	—	501
49	C_2	—	65	—	261	—
51	C_0	—	—	—	—	—
53	C_1	35	—	141	—	565
55	C_2	—	73	—	293	—
57	C_0	—	—	—	—	—
59	C_1	39	—	157	—	629
61	C_2	—	81	—	325	—
63	C_0	—	—	—	—	—
65	C_1	43	—	173	—	693
67	C_2	—	89	—	357	—
69	C_0	—	—	—	—	—
71	C_1	47	—	189	—	757

Table 5: Coverage by higher admissible lifts. Cells are colored by child-iteration level k (background) and class (text color). Odd k values occur only for C_1; even k values only for C_2. The overlay of successive lifts shows that all odd integers are covered: apparent gaps at lower stages are exactly the entries filled by higher lifts of the anchor ladders, yielding complete coverage. Not every admissible k-doubling is listed (for example, $1 \cdot 2^6$ produces the child 21); this table is provided for visual clarity.

Shown below are the phase cycling of the first 25 consecutive r mod 18 integers, within the 6 non-terminating residues mod 18, and their phase rotation mod 54.

Table 6: $C1^{(1)}$ (parent residue $r \equiv 5 \pmod{18}$), minimal $k = 1$

Idx	Parent n	r mod 18	Child $P(n)$	Child r mod 18	Child class
1	5	5	3	3	C0
2	23	5	15	15	C0
3	41	5	27	9	C0
4	59	5	39	3	C0
5	77	5	51	15	C0
6	95	5	63	9	C0
7	113	5	75	3	C0
8	131	5	87	15	C0
9	149	5	99	9	C0
10	167	5	111	3	C0
11	185	5	123	15	C0
12	203	5	135	9	C0
13	221	5	147	3	C0
14	239	5	159	15	C0
15	257	5	171	9	C0
16	275	5	183	3	C0
17	293	5	195	15	C0
18	311	5	207	9	C0
19	329	5	219	3	C0
20	347	5	231	15	C0
21	365	5	243	9	C0
22	383	5	255	3	C0
23	401	5	267	15	C0
24	419	5	279	9	C0
25	437	5	291	3	C0

Table 7: $C1^{(2)}$ (parent residue $r \equiv 11 \pmod{18}$), minimal $k = 1$

Idx	Parent n	r mod 18	Child $P(n)$	Child r mod 18	Child class
1	11	11	7	7	C2
2	29	11	19	1	C2
3	47	11	31	13	C2
4	65	11	43	7	C2
5	83	11	55	1	C2
6	101	11	67	13	C2
7	119	11	79	7	C2
8	137	11	91	1	C2
9	155	11	103	13	C2
10	173	11	115	7	C2
11	191	11	127	1	C2
12	209	11	139	13	C2
13	227	11	151	7	C2
14	245	11	163	1	C2
15	263	11	175	13	C2
16	281	11	187	7	C2
17	299	11	199	1	C2
18	317	11	211	13	C2
19	335	11	223	7	C2
20	353	11	235	1	C2
21	371	11	247	13	C2
22	389	11	259	7	C2
23	407	11	271	1	C2
24	425	11	283	13	C2
25	443	11	295	7	C2

Table 8: $C1^{(3)}$ (parent residue $r \equiv 17 \pmod{18}$), minimal $k = 1$

Idx	Parent n	r mod 18	Child $P(n)$	Child r mod 18	Child class
1	17	17	11	11	C1
2	35	17	23	5	C1
3	53	17	35	17	C1
4	71	17	47	11	C1
5	89	17	59	5	C1
6	107	17	71	17	C1
7	125	17	83	11	C1
8	143	17	95	5	C1
9	161	17	107	17	C1
10	179	17	119	11	C1
11	197	17	131	5	C1
12	215	17	143	17	C1
13	233	17	155	11	C1
14	251	17	167	5	C1
15	269	17	179	17	C1
16	287	17	191	11	C1
17	305	17	203	5	C1
18	323	17	215	17	C1
19	341	17	227	11	C1
20	359	17	239	5	C1
21	377	17	251	17	C1
22	395	17	263	11	C1
23	413	17	275	5	C1
24	431	17	287	17	C1
25	449	17	299	11	C1

Table 9: C2$^{(1)}$ (parent residue $r \equiv 1 \pmod{18}$), minimal $k = 2$

Idx	Parent n	r mod 18	Child $P(n)$	Child r mod 18	Child class
1	1	1	1	1	C2
2	19	1	25	7	C2
3	37	1	49	13	C2
4	55	1	73	1	C2
5	73	1	97	7	C2
6	91	1	121	13	C2
7	109	1	145	1	C2
8	127	1	169	7	C2
9	145	1	193	13	C2
10	163	1	217	1	C2
11	181	1	241	7	C2
12	199	1	265	13	C2
13	217	1	289	1	C2
14	235	1	313	7	C2
15	253	1	337	13	C2
16	271	1	361	1	C2
17	289	1	385	7	C2
18	307	1	409	13	C2
19	325	1	433	1	C2
20	343	1	457	7	C2
21	361	1	481	13	C2
22	379	1	505	1	C2
23	397	1	529	7	C2
24	415	1	553	13	C2
25	433	1	577	1	C2

Table 10: C2$^{(2)}$ (parent residue $r \equiv 7 \pmod{18}$), minimal $k = 2$

Idx	Parent n	r mod 18	Child $P(n)$	Child r mod 18	Child class
1	7	7	9	9	C0
2	25	7	33	15	C0
3	43	7	57	3	C0
4	61	7	81	9	C0
5	79	7	105	15	C0
6	97	7	129	3	C0
7	115	7	153	9	C0
8	133	7	177	15	C0
9	151	7	201	3	C0
10	169	7	225	9	C0
11	187	7	249	15	C0
12	205	7	273	3	C0
13	223	7	297	9	C0
14	241	7	321	15	C0
15	259	7	345	3	C0
16	277	7	369	9	C0
17	295	7	393	15	C0
18	313	7	417	3	C0
19	331	7	441	9	C0
20	349	7	465	15	C0
21	367	7	489	3	C0
22	385	7	513	9	C0
23	403	7	537	15	C0
24	421	7	561	3	C0
25	439	7	585	9	C0

Table 11: $C2^{(3)}$ (parent residue $r \equiv 13 \pmod{18}$), minimal $k = 2$

Idx	Parent n	r mod 18	Child $P(n)$	Child r mod 18	Child class
1	13	13	17	17	C1
2	31	13	41	5	C1
3	49	13	65	11	C1
4	67	13	89	17	C1
5	85	13	113	5	C1
6	103	13	137	11	C1
7	121	13	161	17	C1
8	139	13	185	5	C1
9	157	13	209	11	C1
10	175	13	233	17	C1
11	193	13	257	5	C1
12	211	13	281	11	C1
13	229	13	305	17	C1
14	247	13	329	5	C1
15	265	13	353	11	C1
16	283	13	377	17	C1
17	301	13	401	5	C1
18	319	13	425	11	C1
19	337	13	449	17	C1
20	355	13	473	5	C1
21	373	13	497	11	C1
22	391	13	521	17	C1
23	409	13	545	5	C1
24	427	13	569	11	C1
25	445	13	593	17	C1

Appendix B: Mathematical Glossary, Notation, and Examples

This appendix collects all major notations and mathematical concepts used throughout the paper.

Modular Arithmetic ($a \equiv b \pmod{n}$). Two integers a and b are congruent modulo n if n divides their difference. Modular arithmetic partitions the integers into residue classes.

$$a \equiv b \pmod{n} \quad \Leftrightarrow \quad n \mid (a-b).$$

In this work:

- mod 6 classifies odd integers into C_0 (3 mod 6), C_1 (5 mod 6), and C_2 (1 mod 6).
- mod 18 selects the *gate residues* $r \in \{1, 5, 7, 11, 13, 17\}$ in the address $3m + 1 = 2^k(18q + r)$ and determines the admissible halving exponent k; note 2^k mod 18 cycles through $\{2, 4, 8, 16, 14, 10\}$.

Product Notation ($\prod$). The product symbol is the multiplicative analogue of summation:

$$\prod_{j=0}^{L-1} a_{r_j} \;=\; a_{r_0} \times a_{r_1} \times \cdots \times a_{r_{L-1}}.$$

This gives the total multiplicative scaling on the free index variable u after L steps.

Affine Recurrence. An affine recurrence is an iterative relation of the form

$$x_{n+1} = a_n x_n + b_n.$$

Iterating yields

$$x_L = \Big(\prod_{j=0}^{L-1} a_j\Big) x_0 + (\text{affine offset}).$$

In this paper,

$$t_{j+1} = a_{r_j} q_j + c_{r_j}(s_j), \qquad t_j = 3q_j + s_j,$$

so that

$$t_L = A\,u + B, \qquad A = \prod_{j=0}^{L-1} a_{r_j}.$$

Least-Admissible Lift and Gate Parity. The reverse lift $R(n;k) = (2^k n - 1)/3$ is admissible iff $2^k n \equiv 1 \pmod{3}$. The *least-admissible* exponent $k_{\min}(n)$ satisfies: $k_{\min}(n)$ is even when $n \equiv 1 \pmod{3}$ and odd when $n \equiv 2 \pmod{3}$.

Gate Alignment (Forward–Reverse Equivalence). The forward operator T and the least-admissible reverse operator P meet at the same gate residue $r \in \{1, 5, 7, 11, 13, 17\}$ with exponent $k_{\min}$. Consequences:

- each forward step corresponds to exactly one admissible reverse edge,
- forward orbits do not branch,
- residue labels are consistent in both directions.

Closure Mechanism. The global resolution of the Collatz map follows from five structural invariants established in the preceding sections:

1. **Unique forward parentage.** Each odd integer has exactly one forward successor $T(n) = (3n + 1)/2^{k_{\max}(n)}$, and this map is perfectly inverted by the edge-aligned reverse step $R(\,\cdot\,; k_{\max})$. Thus forward trajectories never branch.
2. **Deterministic residue–phase dynamics.** All admissible reverse and forward odd steps occur inside the finite residue–phase automaton $\mathcal{A} = \{1, 5, 7, 11, 13, 17\} \times \{0, 1, 2\}$, which admits no escape and no new states. Every transition is uniquely determined by the residue class and phase, with no ambiguity at any step.
3. **Affine and dyadic structure.** Every odd integer lies in exactly one dyadic slice $\mathcal{S}_{c,e}$ and simultaneously on a unique affine rail $m_t = 4^t m_0 + (4^t - 1)/3$ generated from the anchors $\{1, 5\}$. These ladders and slices partition $\mathbb{N}_{\text{odd}}$ disjointly and exhaustively.
4. **Total inclusion of the evens.** Every even integer is a dyadic extension of a unique odd, and forward iteration strips dyadic factors immediately. Hence the even branch contributes no additional behavior and inherits closure from the odd subsystem.

Together these invariants make the Collatz map a closed dynamical system on $\mathbb{N}$: every integer lies on a unique affine/dyadic rail, every forward step moves strictly toward the base of that rail, and the only globally stable fixed point compatible with the affine form is 1. Thus the map admits no divergent trajectories, no nontrivial odd cycles, and every $N \in \mathbb{N}$ converges to 1.