Cantor Dust as Comprehensive Solution of the Dark Matter Puzzle

Ervin Goldfain

Global Institute for Research, Education and Scholarship (GIRES).

Email: ervingoldfain@gmail.com

Abstract

As sequel to [1-2], this work explores the gravitational consequences of Cantor Dust formation in the primordial Universe. We find that the multifractal structure of Cantor Dust (CD) can account for a wide range of galactic and cosmological phenomena, commonly attributed to either particle Dark Matter (DM) or modified Newtonian gravity (MOND). Asymptotically flat rotation curves are recovered without invoking modified force laws. Baryonic cooling and dissipation fix the extent of luminous structures at a universal acceleration scale, which leads naturally to the baryonic Tully–Fisher relation (BTFR). We survey weak lensing, dynamical friction, and cluster constraints, and outline testable observational signatures distinguishing this framework from standard cold DM scenarios. In summary, our results suggest that CD provides a *unified geometric* explanation of DM phenomenology across multiple scales.

1. Introduction and Motivation

Historically, DM was born out of a persistent mismatch between the gravitational field inferred from astrophysical observations and that predicted by the visible mass distribution under Newtonian gravity and General Relativity. This discrepancy manifests itself across a wide range of scales, from galactic rotation curves and velocity dispersion to gravitational lensing and large-scale structure formation. Despite decades of experimental effort, no conclusive evidence for DM particles or MOND predictions has been firmly established, motivating continued exploration of alternative theoretical frameworks.

Two empirical regularities have emerged as particularly constraining. The first is the ubiquity of flat galactic rotation curves, implying an enclosed gravitational mass that grows approximately linearly with radius. The

second is the remarkable tightness of the *radial acceleration relation* (RAR), which correlates the observed gravitational acceleration with that predicted by the baryonic mass alone, with a characteristic transition near a universal acceleration scale. Closely related is the *baryonic Tully–Fisher relation* (BTFR), linking total baryonic mass to asymptotic rotation velocity with minimal scatter. These relations are difficult to reproduce naturally within standard cold DM models without fine-tuning of baryonic feedback and halo properties.

In this work, we explore an unconventional alternative suitable for the description of primordial cosmology. According to this view, DM is not a smooth 3-dimensional mass distribution but instead resides on a *singular geometric support* with non-integer effective dimension. Motivated by the filamentary nature of the cosmic web and by mathematical models of CD and multifractals, we consider DM distributions supported on a statistically isotropic network of one-dimensional filaments embedded in 3-dimensional space. Pursuing the framework of [1-2], gravity is formulated in terms of a

Poisson equation with a measure-valued source, allowing the gravitational field to be well defined even in the absence of a smooth density function.

Proceeding with that equation, its gravitational implications and using the remarkable analogy with the cosmic web, we further model DM as a statistically homogeneous and isotropic filamentary network. It is found that the enclosed mass scales linearly with radius despite a quadratic scaling of the total geometric filament content, a consequence of the Gauss theorem and isotropic cancellation of internal contributions. This mechanism yields asymptotically flat rotation curves without invoking modified force laws.

It is further shown that baryonic cooling and dissipation in the multifractal background naturally select a characteristic radius at which collapse stalls, corresponding to a universal acceleration scale set by the fractal mass measure. This provides a dynamic explanation for both RAR and BTFR as *emergent phenomena* rather than empirically imposed relations. Our framework retains standard gravity, introduces no new forces, and ties galactic phenomenology to the geometry of DM.

We also recover, at least in its overall form, the phenomenology of weak lensing, dynamic friction, and cluster constraints, and outline testable observational signatures distinguishing our framework from standard cold DM scenarios.

We caution from the outset that this paper is exclusively an *exploratory introduction* inspired by our previous contributions. Aiming for accessibility to a large audience, the paper is formatted in a bulleted style and kept at a "bird's eye view" level, avoiding jargon or excessive technical details. Interested readers are encouraged to further explore, develop or refute the body of ideas presented here.

2. Derivation of the weak form of the Poisson equation

It is well known that, in ordinary Newtonian gravity, the gravitational potential $\Phi(x)$ generated by a *smooth mass density $\rho(x)$* satisfies the Poisson equation

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G\, \rho(\mathbf{x}) \tag{1}$$

in which,

- $\Phi(x)$ is the scalar gravitational potential,

- ∇^2 is the Laplacian operator in $\mathbb{R}^3$,

- $\rho(x)$ is mass density with respect to Lebesgue measure d^3x,

- G is Newton's constant.

Poisson equation implicitly assumes that mass is continuous with respect to volume, $\rho(x)$ exists pointwise and the source is a function, not a distribution. As discussed in [1-2], all these assumptions fail for Cantor Dust (CD), which is not a smooth distribution of matter. Instead,

- mass is supported on a fractal set of Hausdorff dimension $D < 3$,

- there is no density $\rho(x)$ such that

$$\rho(\mathbf{x})\, d^3x = dM \tag{2}$$

As detailed in [3], the conjecture that particle masses and couplings emerge from the continuous and evolving spacetime dimensionality near the Planck

scale is supported by several independent arguments. Pursuing this line of reasoning, we assume below that, on a multifractal Cantor Dust background, mass is described by a *Radon measure* $\mu(\mathbf{x})$. and the mass contained in a ball of radius r scales as

$$\mu(B_r) \sim r^D \tag{3}$$

The correct generalization of the Poisson equation (1) is

$$\boxed{\nabla^2 \Phi(\mathbf{x}) = 4\pi G \, \mu(\mathbf{x})} \tag{4}$$

with the following interpretation,

- The Laplacian of Φ is no longer an ordinary function, and $\nabla^2 \Phi(\mathbf{x})$ is taken to represent a distribution,

- The right-hand side is a measure, not a density,

- This whole equation is understood in the weak (distribution-like) sense.

The generalization of the Poisson equation in the formal sense of distributions is given by the *weak form* [1],

$$\int \nabla \Phi \cdot \nabla \psi \, d^3 x = 4\pi G \int \psi \, d\mu \tag{5}$$

and can be derived as follows:

2.1: Multiply by a test function

Take a smooth test function $\psi(x)$ with compact support and multiply the Poisson equation,

$$\psi \, \nabla^2 \Phi = 4\pi G \, \psi \, \mu \tag{6}$$

2.2: Integrate over space

$$\int \psi \, \nabla^2 \Phi \, d^3 x = 4\pi G \int \psi \, d\mu \tag{7}$$

Note that the right-hand side is well-defined because ψ is continuous and μ is a Radon measure.

Step 2.3: Integrate by parts (Green's identity)

Using

$$\int \psi \, \nabla^2 \Phi \, d^3x = -\int \nabla\Phi \cdot \nabla\psi \, d^3x \qquad (8)$$

and, since ψ has compact support, boundary terms vanish, we obtain:

$$\int \nabla\Phi \cdot \nabla\psi \, d^3x = 4\pi G \int \psi \, d\mu \qquad (9)$$

This weak formulation of Poisson equation is standard practice in:

- Electrostatics with singular charge distributions,

- Newtonian gravity with point masses,

- Mathematical theory of elliptic PDEs

In a nutshell, the weak form of Poisson equation (9) means that the gravitational field energy produced by Φ, tested against any probe function ψ, equals the total mass of Cantor Dust weighted by ψ.

If the measure becomes continuous,

$$d\mu = \rho(\mathbf{x}) \, d^3x \qquad (10)$$

then

$$\int \psi \, d\mu = \int \psi \, \rho \, d^3x \qquad (11)$$

and the weak equation reduces to the standard Poisson equation.

Applying the weak equation to a spherically averaged test function yields:

$$g(r) = \frac{GM(r)}{r^2} \sim r^{D-2} \qquad (12)$$

which is the fractal formulation of the Gauss (divergence) theorem. This is the key result that drives:

- Scale-dependent gravitational acceleration

- Hierarchical baryonic collapse

- Generalized Jeans instability [1]

3. Derivation of Rotation Curves from the Weak Equation

The goal of this section is to derive the galactic rotation curves directly from the weak Poisson equation.

3.1 Assume spherical symmetry and isotropic test functions

By analogy with the geometry of the cosmic web, let's assume statistical isotropy of the CD and choose a test function $\psi(r)$ that depends only on radius. The weak equation reduces to:

$$\int_0^\infty \frac{d\Phi}{dr}\frac{d\psi}{dr} 4\pi r^2 dr = 4\pi G \int_0^\infty \psi(r)\, dM(r) \tag{13}$$

3.2 Integration by parts

Integrate the left-hand side:

$$\int \frac{d\Phi}{dr}\frac{d\psi}{dr} r^2 dr = -\int \frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right)\psi(r)\, dr \tag{14}$$

Thus,

$$\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = G\frac{dM}{dr} \qquad (15)$$

3.3 Fractal mass profile

Using:

$$M(r) = M_0 r^D \qquad (16)$$

we obtain:

$$\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = GDM_0 r^{D-1} \qquad (17)$$

Integrating gives,

$$r^2 \frac{d\Phi}{dr} = \frac{GDM_0}{D} r^D = GM(r) \qquad (18)$$

Thus:

$$g(r) \equiv \frac{d\Phi}{dr} = \frac{GM(r)}{r^2} \sim r^{D-2} \qquad (19)$$

3.4 Rotation velocity

For circular motion one has

$$\frac{v^2(r)}{r} = g(r) \tag{20}$$

Hence:

$$v^2(r) = rg(r) = GM(r)r^{-1} \sim r^{D-1} \tag{21}$$

We are led to:

$$\boxed{v(r) \sim r^{\frac{D-1}{2}}} \tag{22}$$

It is seen that *flat galaxy rotation* curves correspond to $D \approx 1$ a condition associated with *filamentary Cantor Dust*. In this picture, rotation curves emerge directly from the weak Poisson equation, with no modification of Newton's law.

4. Derivation of the Baryonic Tully-Fisher Relation (BTFR)

4.1 Universal Acceleration Scale from the CD Fractal Background

Recall that CD is described mathematically by a mass measure μ, not a smooth density. It is *not* a halo and *not* centered on galaxies. CD is a *fractal background* characterized by two microscopic, statistical parameters:

1) *Scaling coefficient of the filamentary mass measure*

$$\lambda := \lim_{r \to \infty} \frac{\mu(B_r)}{r} \tag{23}$$

(23) quantifies the expected total measure of filament intersections per unit radius of a spherical probe and *must not* be confounded with a density.

More precisely:

- Consider a ball B_r
- The fractal intersects it in a random set of strands
- Each intersection contributes a finite measure increment

- The *expected number of such intersections* scales as r

2) The *correlation length* ℓ, which is the typical linear scale beyond which filament orientations and positions decorrelate statistically.

Note that, by definition, these two properties are intrinsic to the dark sector geometry, not of baryons or galaxies.

Now, integrate eq. (4) over a ball B_r of radius r:

$$\int_{B_r} \nabla^2 \, \Phi \, d^3 x = 4\pi G \, \mu(B_r) \tag{24}$$

and apply the Gauss divergence theorem to obtain:

$$\oint_{\partial B_r} \nabla \Phi \cdot d\mathbf{S} = 4\pi G \, M_{\text{grav}}(r) \tag{25}$$

where

$$M_{\text{grav}}(r) \equiv \mu(B_r) \tag{26}$$

is the gravitationally enclosed mass.

In a filamentary medium ($D = 1$):

- Filament segments fully contained inside B_r contribute *zero net flux*.

- Only filament segments that *cross the boundary ∂B_r* contribute to the monopole term.

That is to say that flux is controlled by boundary crossings. It follows that,

$$\boxed{M_{\text{grav}}(r) \propto (\text{net number of filament crossings of } \partial B_r) \times (\text{mass per crossing})}$$

For a statistically homogeneous and isotropic filament network:

- The *total number of boundary intersections* scales with surface area:

$$N_{\text{tot}}(r) \sim \frac{r^2}{\ell^2}$$

- Because orientations are random, inward and outward crossings cancel statistically.

- The *net monopole contribution* scales as the root-mean-square imbalance:

$$N_{\text{net}}(r) \sim \sqrt{N_{\text{tot}}(r)} \sim \frac{r}{\ell}$$

Note that the derivation of $N_{\text{net}}(r)$ is purely statistical and follows from general random-walk arguments.

Each net crossing contributes mass on the order of:

$$\Delta M \sim \lambda \ell \tag{27}$$

because correlations extend only over length ℓ. Using eq. (27) and the expression of $N_{\text{net}}(r)$ yields:

$$M_{\text{grav}}(r) \sim \lambda \ell \times \frac{r}{\ell} = \lambda r.$$

Our key result is:

$$\boxed{M_{\text{grav}}(r) \sim \lambda r}$$

which represents a scaling relationship independent of galaxy properties.

The radial gravitational field follows from the Gauss theorem,

$$g(r) = \frac{G M_{\text{grav}}(r)}{r^2} \sim \frac{G\lambda}{r}.$$

Thus, in the fractal regime of CD, gravity decays slowly as $1/r$.

Consider now the acceleration difference across one correlation length:

$$\Delta g \sim g(r) - g(r + \ell) \sim \frac{G\lambda}{r^2}\ell \tag{28}$$

At the scale where the fractal description becomes valid ($r \sim \ell$), (28) yields:

$$\boxed{a_0 \equiv \frac{G\lambda}{\ell}.}$$

This is the characteristic gravitational acceleration of the CD fractal background. There are three key observations regarding a_0, namely,

- λ and ℓ are statistical properties of the dark sector,

- they are set by cosmological initial conditions,

- they do not depend on galaxy mass, size, or baryonic content.

Therefore:

$$\boxed{a_0 = \frac{G\lambda}{\ell} \text{ is universal.}}$$

The acceleration a_0:

- sets the maximum sustainable acceleration generated by the fractal CD background at large scales,

- marks the transition between:

  ○ baryon-dominated Newtonian gravity ($g \gg a_0$),

  ○ CD background-dominated gravity ($g \lesssim a_0$).

It is not an arbitrarily imposed modification of dynamics, but a natural consequence of introducing the fractal CD background.

4.2 Thermodynamic considerations

We elaborate here on the necessary condition for baryonic collapse into a rotationally supported structure (disk or spheroid), which amounts to,

$$t_{cool} \sim t_{dyn}$$

This condition is standard in galaxy formation theory and can be justified on the following grounds:

(a) Dynamic time t_{dyn}

The dynamic (free-fall / orbital) time is the characteristic time for matter to respond to gravity. Consider a test particle at radius r subject to gravitational acceleration $g(r)$. Dimensional analysis gives:

$$r \sim \frac{1}{2} g(r)\, t_{\text{dyn}}^2$$

or,

$$\boxed{t_{\text{dyn}} \sim \sqrt{\frac{r}{g(r)}}}$$

The derivation of dynamic time is general and does not assume Newtonian gravity, Keplerian orbits, or spherical symmetry.

(b) Cooling time t_{cool}

The cooling time is defined as

$$t_{\text{cool}} \equiv \frac{\text{thermal energy per unit volume}}{\text{radiative energy loss per unit volume}}$$

For a baryonic gas:

$$t_{\text{cool}} = \frac{\frac{3}{2} n k_B T}{n^2 \Lambda(T)}$$

where:

- n = number density,

- T = temperature,

- $\Lambda(T)$ = cooling function.

The cooling time is organically tied to gravity as follows: the gas temperature

is set by virialization in the gravitational field,

$$k_B T \sim m_p v^2$$

where, from centripetal balance, one has

$$v^2 \sim rg(r)$$

Thus:

$$T \sim \frac{m_p}{k_B} rg(r)$$

In quasi-equilibrium collapse the number density is expected to scale as

$$n \sim \frac{M_b}{m_p r^3}$$

where M_b is the baryon mass. Substituting into the equation defining the cooling time leads to

$$t_{cool} \sim \frac{rg(r)}{n\Lambda} \sim \frac{rg(r)}{\left(\frac{M_b}{r^3}\right)} \sim \frac{r^4 g(r)}{M_b}$$

Using the virial relation again gives

$$M_b \sim \frac{r^2 g(r)}{G}$$

which yields:

$$t_{\text{cool}} \sim \left[\frac{r^2}{g(r)}\right]^{1/2}$$

or

$$t_{cool} \sim \sqrt{\frac{r}{g(r)}}$$

up to slowly varying logarithmic factors from $\Lambda(T)$. It is seen that both times

obey a similar scaling relationship,

$$t_{\text{cool}} \sim t_{\text{dyn}} \sim \sqrt{\frac{r}{g(r)}}$$

which is not accidental, because

- Gravity sets the velocity scale.

- Velocity sets the temperature.

- Temperature controls cooling.

- Cooling controls collapse.

Thus, both times are governed by the same gravitational acceleration. Three dynamic regimes exist:

1) $t_{\text{cool}} \gg t_{\text{dyn}}$ => Gas stays hot and no collapse takes place,

2) $t_{\text{cool}} \ll t_{\text{dyn}}$ => Rapid cooling and fragmentation occur,

3) $t_{\text{cool}} \sim t_{\text{dyn}}$ => Stable regime with disk/halo formation. Only this process can consistently produce galactic disks.

In a fractal background,

$$g(r) \sim \frac{G\lambda}{r}$$

Then

$$t_{\text{dyn}} \sim \frac{r}{\sqrt{G\lambda}}$$

and collapse proceeds until:

$$t_{\text{cool}} \sim t_{\text{dyn}} \Rightarrow g(R_b) \sim \frac{G\lambda}{\ell} \equiv a_0$$

Therefore, the baryonic radius is dynamically selected by the condition

$$\boxed{g(R_b) \sim a_0}$$

We are led to the following conclusions:

a) The equality $t_{\text{cool}} \sim t_{\text{dyn}}$ reflects the requirement that baryonic gas lose thermal energy on the same timescale on which it responds gravitationally, ensuring stable collapse and rotational support; both timescales scale as $\sqrt{r/g(r)}$ because temperature, density, and orbital velocity are all set by the gravitational acceleration.

b) A universal acceleration scale emerges naturally in a filamentary Cantor-dust background because Gauss' law applied to a measure-valued Poisson equation yields an enclosed mass growing linearly with radius. The resulting gravitational field decays as $g(r) \propto r^{-1}$, and the combination $a_0 = G\lambda/\ell$, determined solely by the linear mass density and correlation length of

the fractal network, sets the characteristic acceleration governing baryonic

dynamics.

4.3. Baryonic Tully-Fisher Relation (BTFR)

The baryonic radius R_b itself is not a free parameter. To see why this is the

case, start from the cooling–dynamical balance derived above:

$$t_{\text{cool}}(R_b) \sim t_{\text{dyn}}(R_b).$$

In a $g(r) \sim 1/r$ background:

$$R_b \sim \frac{v_b^2}{a_0}$$

But the baryonic rotational velocity satisfies:

$$v_b^2 \sim \frac{GM_b(R_b)}{R_b}.$$

From which we obtain,

$$R_b \sim \frac{GM_b(R_b)}{a_0 R_b}$$

and so,

$$R_b^2 \sim \frac{G}{a_0} M_b(R_b)$$

or,

$$\boxed{M_b(R_b) \sim \frac{a_0}{G} R_b^2}$$

This result follows directly from:

- baryonic dominance inside R_b,

- Newtonian gravity,

- the acceleration matching condition $g(R_b) \sim a_0$.

From the condition $g(R_b) \sim a_0$,

$$v^2 \sim a_0 R_b$$

Using the mass scaling derived above:

$$M_b \sim \frac{a_0}{G} R_b^2$$

we extract

$$R_b \sim \sqrt{\frac{GM_b}{a_0}}$$

Inserting into v^2 gives,

$$v^2 \sim a_0 \sqrt{\frac{GM_b}{a_0}}$$

whose square yields the baryonic Tully-Fisher relation (BTFR):

$$\boxed{v^4 \sim Ga_0 M_b}$$

In a nutshell,

- R_b is the radius where baryonic gravity drops to the background level a_0.

- Inside R_b, baryons dominate and are in a Newtonian regime,

- The enclosed baryonic mass at that radius scales as R_b^2.

It is important to distinguish the quadratic scaling of the baryonic mass from the CD background whose enclosed mass scales as:

$$M_{\text{fractal}}(r) \sim \lambda r$$

Note that R_b is not a physical boundary in the dark sector. It marks the transition from the CD fractal background and the baryon dominated region. A pictorial rendering of the two regions is displayed below.
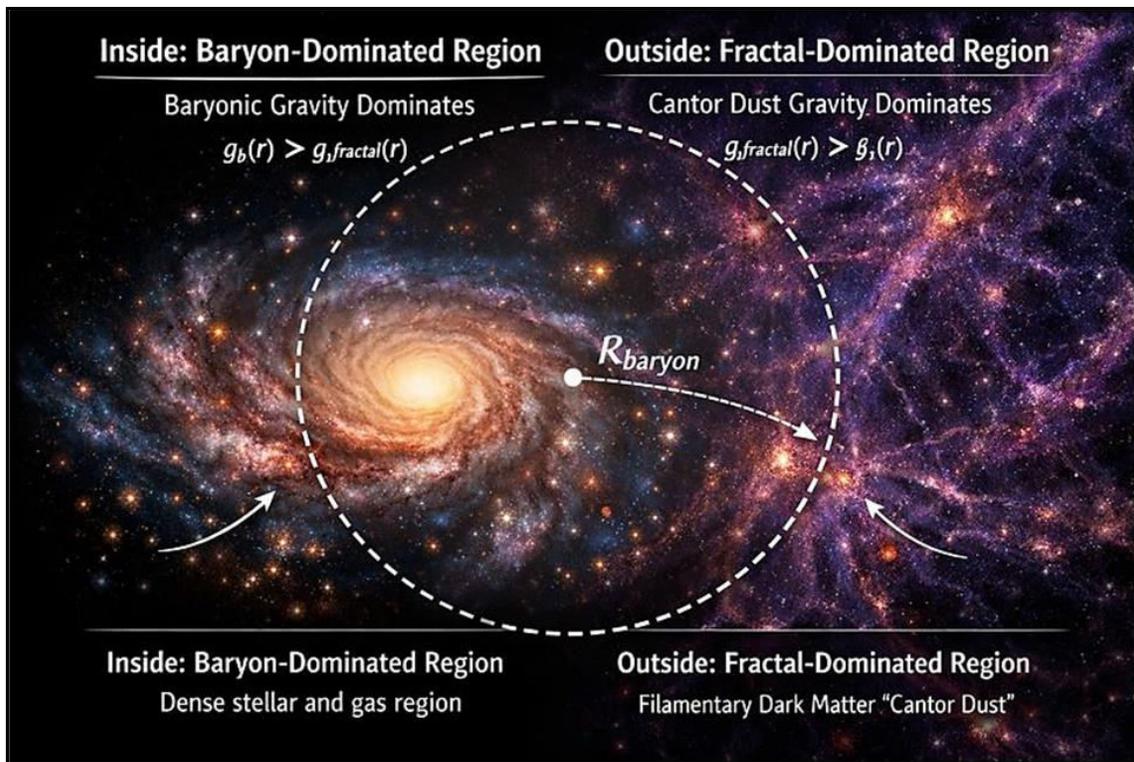

Fig. 1: Fractal-dominated vs. baryon-dominated gravitational regions.

5. Conclusions

We have presented a geometric framework in which a CD dark sector accounts for the principal phenomenology traditionally attributed to DM. By modeling the dark component as a statistically isotropic filamentary network and formulating gravity through a Poisson equation with a measure source, we have shown that the gravitationally relevant enclosed mass scales linearly with radius despite a higher-order scaling of geometric filament content. Within this framework, baryonic collapse is regulated by cooling and dissipation in a slowly varying gravitational field, leading to a dynamically selected radius at which the local acceleration reaches a universal scale determined by the fractal mass measure. This mechanism provides a natural explanation for the radial acceleration relation (RAR) and the baryonic Tully–Fisher relation (BFTR) without modifying Newtonian dynamics or introducing free interpolation functions. The same geometric principles govern lensing, suppress dynamical friction in filament-dominated halos, and offer testable predictions for halo–galaxy correlations

and cluster collisions. Our approach reframes the DM problem as a question of spatial support and measure rather than new particle species. While further work is required to embed this framework fully within relativistic cosmology and to confront precision cosmological data, the results presented here imply that a singular fractal dark sector can reproduce a broad range of observations within a unified and mathematically controlled setting. This, in turn, suggests that the geometry of the dark sector may play a more fundamental role in cosmic structure and dynamics than previously assumed.

The reader is directed to [4-12] for a wealth of background information and in-depth technical details.

References

1. https://www.researchgate.net/publication/399127390

2. https://www.researchgate.net/publication/398638254

3. https://www.researchgate.net/publication/388483112

4. F. Lelli, S. S. McGaugh, J. M. Schombert, and M. S. Pawlowski,

One Law To Rule Them All: The Radial Acceleration Relation of Galaxies,

Astrophys. J. Lett. **836**, L13 (2017).

5. C. Wheeler, P. F. Hopkins, and O. Doré,

The Radial Acceleration Relation Is a Natural Consequence of the Baryonic Tully–

Fisher Relation, Mon. Not. R. Astron. Soc. **477**, 5501 (2018).

6. J. Einasto, G. Hütsi, T. Kuutma, and M. Einasto,

Correlation Function: Biasing and Fractal Properties of the Cosmic Web,

Astron. Astrophys. **640**, A47 (2020).

7. J. E. Forero-Romero, Y. Hoffman, S. Gottloeber, A. Klypin, and G. Yepes**,**

A Dynamical Classification of the Cosmic Web,

Mon. Not. R. Astron. Soc. **396,** 1815 (2009).

8. G. De Marzo, F. Sylos Labini, and L. Pietronero**,**

Zipf's Law for Cosmic Structures: How Large Are the Greatest Structures in the

Universe? Astron. Astrophys. **651**, A114 (2021).

9. R. B. Tully and J. R. Fisher, *A New Method of Determining Distances to Galaxies,Astron. Astrophys.* **54**, 661 (1977).

10. P. J. E Peebles, *Principles of Physical Cosmology* (Princeton Univ. Press, Princeton, 1993).

11. M. Aragon-Calvo, R. van de Weygaert, B. J. T. Jones, *Multiscale Phenomenology of the Cosmic Web, arXiv:1007.0742* (2010).
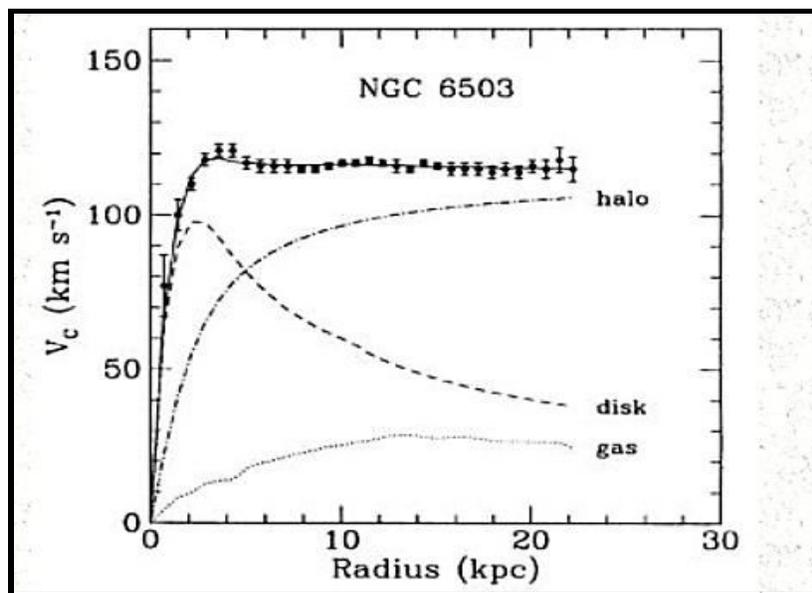
12. https://ned.ipac.caltech.edu/level5/Sept17/Freese/Freese2.html

Fig. 2. Galactic rotation curve for NGC 6503 showing disk and gas contribution plus DM halo contribution needed to match the data [12].