On the Constant Real Part of Non-Trivial Zeros: A Proof of the Riemann Hypothesis

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Abstract

This paper presents a proof of the Riemann Hypothesis by examining the geometric and arithmetic properties of the Dirichlet eta function *η(s)*. By assuming the existence of non-trivial zeros off the critical line, and analysing the resulting alternating series in the complex plane, we establish a logical contradiction.

The proof relies on tracking the real and imaginary components of *η(s)* independently along a horizontal slice of fixed height *t* as the real part *σ* of the argument increases from zero to infinity. As σ approaches infinity, every term in the series gradually shrinks in magnitude, forcing the real and the imaginary components to converge asymptotically to 1 and 0, respectively. Because the asymptotic destination of the real component is 1, it can safely vanish twice with only a single change in monotonicity before continuing its monotonic path toward its final destination 1.

In sharp contrast, the imaginary component cannot exhibit this behaviour. If it vanishes twice at the same height *t* within the critical strip, it must enter a region where it moves away from its ultimate limit of 0. To reverse this behaviour and converge to 0, as $\sigma \to \infty$, the imaginary component would require a minimum of two changes in monotonicity. However, through a rigorous analysis of the Dirichlet eta series structure, we demonstrate that the imaginary component admits a maximum of only one monotonicity change for height *t*, making a second change in direction impossible. This structural constraint rules out the possibility of two symmetric zeros off the critical line, thereby proving that all non-trivial zeros of $\eta(s)$ and consequently those of the Riemann zeta function ζ(s), must possess a real part of exactly 1/2.

1 Introduction

The Riemann zeta function is defined by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \ (s \in \mathbb{C}) \text{ where } \mathrm{Re}(s) > 1. \quad (1)$$

For the domain beyond Re(s) > 1, there is a functional equation that represents the function through analytic continuation except at s = 1.

$$\zeta(s) = 2^s\pi^{s-1}\sin(\tfrac{\pi s}{2})\Gamma(1-s)\zeta(1-s). \tag{2}$$

Within the *critical strip* (0 < Re(*s*) < 1), the zeta function *ζ(s)* is also represented in terms of the *Dirichlet eta function η(s)* as:

$$\zeta(s) = (1-2^{1-s})^{-1}\eta(s)\ . \tag{3}$$

$$\zeta(s) = (1-2^{1-s})^{-1}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\right). \tag{4}$$

Where *η(s)* is defined by the alternating series in (4).

Equation (3) defines a strict analytic identity between $\zeta(s)$ and $\eta(s)$. Although the alternating series in (4) is only conditionally convergent within the critical strip, this identity holds provided the terms maintain their natural, ascending order (n = 1, 2, 3, …).

It is known that all non-trivial zeros of *η(s)* lie within the critical strip; as a consequence, from the identity, the non-trivial zeros of *ζ(s)* must also lie within that strip.

The Riemann Hypothesis (RH) states that all non-trivial zeros of *ζ(s)* lie on the *critical line* defined by Re(*s*) = $\frac{1}{2}$.

Therefore, to validate this constant real part $\frac{1}{2}$ of the non-trivial zeros of *ζ(s)*, it is sufficient to determine whether these zeros of *η(s)* lie on the critical line.

From Equation (4), we see that *ζ(s)* vanishes whenever the summation representing *η(s)* is zero.

That is, whenever

$\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\right)$= 0, *ζ(s)* also vanishes. (since $(1-2^{1-s})^{-1}$ ≠ 0 for 0 < Re(s) < 1).

2 Assumption of Zeros off the Critical Line

Let us assume that the complex number s' is a non-trivial zero of *η(s)*, where s' = σ_1 + i*t* such that 1/2 < σ_1 < 1 and $t \in \mathbb{R}^*$.

$$\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{s'}}\right)= 0.$$

Let us consider a real number σ_2 such that 0 < σ_2 < 1/2 and $\sigma_1 + \sigma_2 = 1$.

Define s'' = σ_2 + i*t*, where s'' shares the same imaginary part as s'.

Therefore, s'' is also a zero — a reflection of s' across the critical line σ = 1/2 as guaranteed by the functional equation.

$\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{s''}}\right) = 0.$

Define,

$S_1 = 1 - \frac{1}{2^{s'}} + \frac{1}{3^{s'}} - \frac{1}{4^{s'}} + \ ... \ = 0$

Similarly,

$S_2 = 1 - \frac{1}{2^{s''}} + \frac{1}{3^{s''}} - \frac{1}{4^{s''}} + \ ... = 0.$

Although S_1 and S_2 are not identical, both vanish.

We shall examine the two alternating series S_1 and S_2 in the complex plane to establish a logical contradiction. It follows from this result that all non-trivial zeros of *η(s)* must lie on the critical line. The following proof is concise, as it relies on *fundamental geometric insight* rather than complex mathematical methods.

The terms of S_1 and S_2 are distributed around the complex unit circle. There will be infinite terms in the right half-plane just as there are in the left. The angles of the terms are governed by both the alternating factor $(-1)^{n+1}$ and the imaginary part *t.*

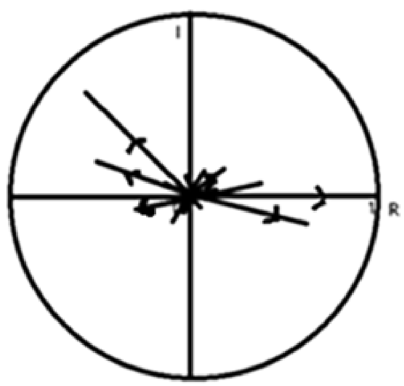

Figure 1: Distribution of terms in the complex plane. The corresponding terms of S_1 and S_2 are *collinear* with the origin because, as they share the same imaginary part *t* in their exponents, each term of S_2 maintains the exact same angle with the real axis as its corresponding term in S_1 does.

3 Collinearity and Vector Path Visualization

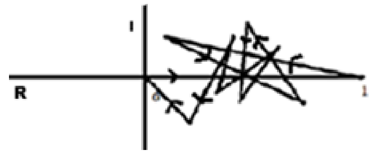

Figure 2: Illustration of infinite series S_1. The depiction is limited to a finite number of terms. The moduli and angles are arbitrary and intended only to visualize the general geometric path of the series in the complex plane.

Figure 3: Illustration of series S_2

4 Moduli, Phase and Growth Rate Ratio

When plotting the cumulative sums of both S_1 and S_2 on the complex plane, the first terms are identical (both equal to 1). Starting from the point 1 on the real axis, the corresponding second term-segments are collinear. From the third term onwards, all corresponding segments remain parallel.

With the exception of the first terms, no corresponding terms of S_1 and S_2 are equal in modulus. From the second term onwards, each term of S_2 is strictly larger in modulus than its corresponding term in S_1.

Because corresponding terms share an identical angle for varying σ along the constant slice t, a larger modulus mathematically guarantees both a larger real component and a larger imaginary component.

Consider the quotients q_n derived by dividing the terms of S_2 by the corresponding terms of S_1:

$$\frac{\text{Term of } S_2}{\text{Term of } S_1}\ .$$

$$\frac{1}{1} = q_1 = 1.$$

$$\frac{\left(-\frac{1}{2^{s''}}\right)}{\left(-\frac{1}{2^{s'}}\right)} = \frac{\left(\frac{1}{2^{s''}}\right)}{\left(\frac{1}{2^{s'}}\right)} = q_2\ .$$

$$\frac{\left(\frac{1}{3^{s''}}\right)}{\left(\frac{1}{3^{s'}}\right)} = q_3.$$

And so on. The sequence of these unique real values satisfies the inequality $1 < q_2 < q_3 \ldots$ not just for $\sigma_1 + it$ and $\sigma_2 + it$ but for any two σ across that slice.

5 Monotonicity Analysis of the Separated Components

Recall that $s' = \sigma_1 + it$ and $s'' = \sigma_2 + it$.

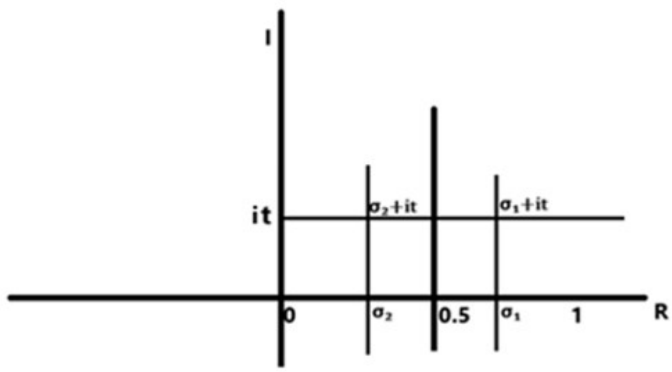

Figure 4 : Assumed zeros of eta

The Dirichlet eta function *η(s)* must experience a change in direction — that is, a change in monotonicity — between σ_2 and σ_1 to yield zeros at both points. It is known that the function's real and imaginary parts are continuous across the domain of study.

For the eta function *η(s)* evaluated at two arguments with equal imaginary parts but distinct real parts, the ratio of the n^{th} term of the larger series (generated by σ that is closer to 0) to that of the smaller series (generated by *σ* that is farther from 0) strictly increases with n, satisfying $1 < q_2 < q_3 \ldots$ (recalled from 4. Moduli, Phase and Growth Rate Ratio). This implies that the rate of growth of a term of the larger series to that of the corresponding term of the smaller series becomes progressively larger as the term *n* proceeds. This inequality holds for any two variables across a slice *t* and not just for s' and s''.

As we study a slice *t* across the continuous positive domain (that includes those discrete points σ_2 and σ_1), we see that every term maintains the same angle with the positive real axis despite its modulus changing continuously as *σ* changes. It essentially means that no terms will flip sides (left ⇄ right or top ⇄ bottom).

5.1 Monotonicity of the Real Component

Therefore, if one side(left/right) of the origin possesses a strictly greater/faster net growth rate of terms than that of the initial opposite side bias in the net moduli (though the growth rate of terms is smaller/slower), this growth rate dominance of the former side persists across the positive domain under study (a slice *t*). This specific scenario could cause a maximum of one change in monotonicity. To illustrate, consider an initial stronger leftward pull on *η(s)* at large σ paired with a higher/faster net growth rate to the right. As σ decreases, the path may initially maintain its leftward movement. If the faster growing right-side moduli overpower the left (stronger net pull shifts from left to right) before the boundary is reached, this triggers a unique reversal at a well-defined leftmost point, after which the rightward movement persists indefinitely; otherwise, the real component is seamlessly monotonic toward the left.

Similarly, if an initial stronger rightward pull on *η(s)* at large σ paired with a higher/faster net growth rate of terms to the left is also a possible scenario, as σ decreases, the path initially may maintain its rightward movement. If the faster growing left-side moduli overpower the right before the boundary is reached, this triggers a unique reversal at a well-defined rightmost point, after which the leftward movement persists indefinitely; otherwise, the real component is seamlessly monotonic toward the right.

If the higher/faster net growth rate of terms and the initial bias in net moduli align to one side of the origin, then there will not be any change in monotonicity at all.

In case the growth rates of terms are asymptotically balanced on either side, the movement will continue in the same direction as the initial bias dictates. If the function's initial value is zero, then, no movement happens at all across the slice *t*. A perfect state of equilibrium persists.

Hence, we conclude that the real part of *η(s)* can undergo at most a single change in monotonicity across the entire positive domain of σ through a slice *t.*

The real part of *η(s)* asymptotically approaches 1 as the real component *σ* approaches infinity, ($0^+ \to \infty$). Therefore, with a single change in monotonicity, the real part can vanish twice.

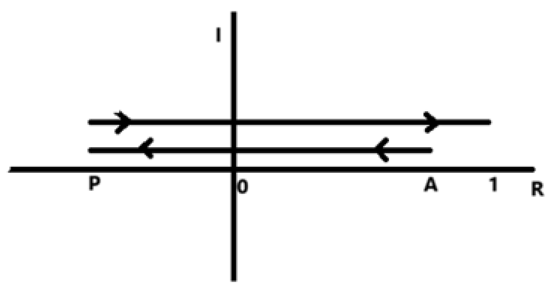

Figure 5 : Real part monotonicity change

The real part may start in the positive half-plane at the point *A* and move leftward. While doing so, it vanishes and reaches the turning point *P* on the left. Then, the value changes direction, vanishes a second time, and moves toward the asymptotic limit of 1. This double vanishing can happen entirely within the critical strip with just one change in monotonicity.

The analysis of the real part of *η(s)* does not yield the desired contradiction. We turn our attention to the behavior of its imaginary part. In the case of the real part, we analyse the left and right movements with reference to the origin, while for the imaginary part, we analyse the top and bottom movements.

5.2 Monotonicity of the Imaginary Component

Unlike the real part, which asymptotically approaches 1, the imaginary part asymptotically approaches 0 as $\sigma \to \infty$.

Therefore, vanishing twice within a finite domain while approaching zero a third time asymptotically would require a minimum of two changes (or an even number of changes) in monotonicity. However, the logical analysis established for the real part in section 5.1 applies equally to the imaginary part regarding possible changes in monotonicity. That is, the structure of *η(s)* across a slice *t* permits at most a single change in monotonicity for its imaginary component as well.

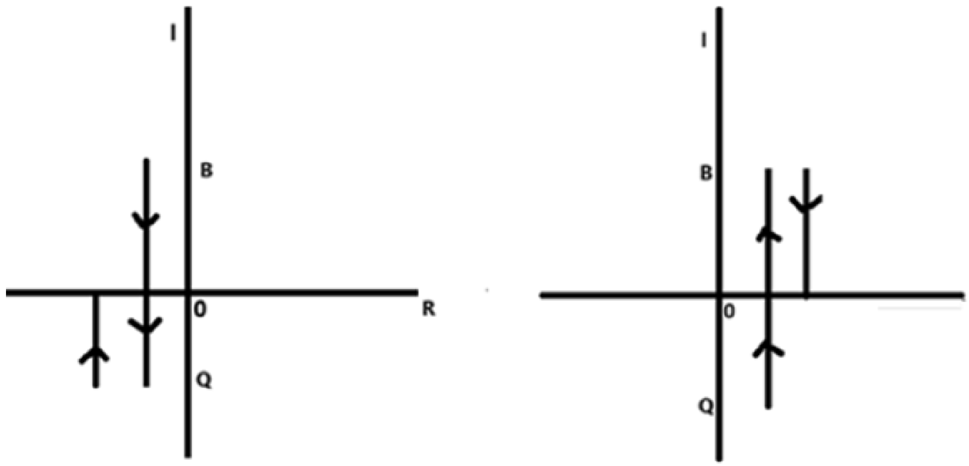

Figure 6 : imaginary part monotonicity change from both possible directions

Therefore, if it vanishes at all, the imaginary part can do so only once across the entire domain $0 < \sigma < \infty$ for a constant height *t.* Consequently, two distinct non-trivial zeros for *η(s)* symmetric to the critical line—are impossible.

6 Conclusion

As a result, only one non-trivial zero s of $\eta(s)$ can occur for a given horizontal slice *t.* Therefore, it must lie on the critical line. Given the identity in Eq. (3), the Riemann zeta function $\zeta(s)$ inherits this same property with regard to its non-trivial zeros.

Theorem 1 (Main Result). *All non-trivial zeros of the Riemann zeta function ζ(s) lie on the critical line Re(s) =* $\frac{1}{2}$ *.*

Reference

- **Riemann, B. (1859).** *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the Number of Primes Less than a Given Magnitude). Monatsberichte der Berliner Akademie.