

# Mathematical analysis of spheres at polyhedral vertices, face filleting, and sphere packing in right pyramids and platonic solids

Harish Chandra Rajpoot

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M.M.M. University of Technology, Gorakhpur-273010 (UP), India

**Introduction:** This work analyses the spheres resting in the vertex (corner) of regular & uniform polyhedra such as platonic solids [1] & uniform polyhedra with right kite faces [2], by using generalised expressions of a right pyramid with a regular n-gonal base [3]. Here, the regular & the uniform polyhedra are assumed to be hollow shell & large enough to accommodate the given sphere with a certain radius. The sphere, with a radius  $R$ , is assumed to be inside & resting in a vertex (corner) at which  $n$  no. of edges meet together such that angle between any two consecutive edges is  $\alpha$  & resting sphere touches all  $n$  no. of faces meeting at that vertex but does not touch any of  $n$  no. of edges meeting at that vertex of a given polyhedron. A sphere is best fitted in a vertex first by truncating that vertex [4-9] and then by filleting all  $n$  no. of faces meeting at that vertex. We are also to analyse the packing of the spheres in right pyramids. First of all, let's derive the general expressions of a right pyramid having regular n-gonal base with edge length  $a$ .

## 1. Derivation of normal height $H$ and angles $\beta$ & $\gamma$ of the lateral edge & the lateral face with the geometrical axis of a right pyramid with base as a regular polygon

Let there be a right pyramid with base as a regular polygon  $A_1A_2A_3 \dots A_n$  having  $n$  no. of sides each of length  $a$ , angle between any two consecutive lateral edges  $\alpha$ , normal height  $H$ , an acute angle  $\angle OPA_1 = \beta$  of the geometrical axis  $PO$  with any of the lateral edges & an acute angle  $\angle OPM = \gamma$  of the geometrical axis  $PO$  with any of the lateral faces (as shown in the figure 1).

Now, join all the vertices  $A_1, A_2, A_3, \dots, A_n$  of the base to the centre 'O' thus we obtain 'n' no. of congruent isosceles triangles  $\Delta A_1OA_2, \Delta A_2OA_3, \dots, \Delta A_nOA_1$ .

In right  $\Delta OMA_2$  (Fig. 1)

$$\Rightarrow \tan \angle A_2OM = \frac{MA_2}{OM} \quad \text{or} \quad \tan \frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{OM}$$

$$\Rightarrow OM = \frac{a}{2} \cot \frac{\pi}{n} \quad \left(\text{since, } \angle A_1OA_2 = \frac{2\pi}{n}\right) \quad \dots \dots \dots (1)$$

Similarly, we have

$$\Rightarrow \sin \angle A_2OM = \frac{MA_2}{OA_2} \quad \text{or} \quad \sin \frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{OA_2}$$

$$\Rightarrow OA_2 = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n} \quad \dots \dots \dots (2)$$

In right  $\Delta PMA_2$  (Fig. 1)

$$\Rightarrow \tan \angle A_2PM = \frac{MA_2}{PM} \quad \text{or} \quad \tan \frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{PM}$$

$$\Rightarrow PM = \frac{a}{2} \cot \frac{\alpha}{2} \quad \dots \dots \dots (3)$$

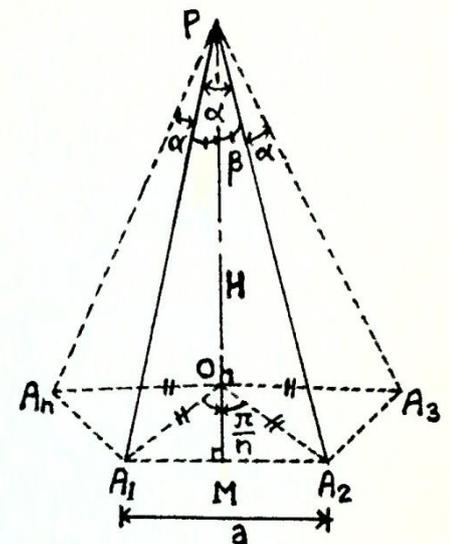


Figure 1: A right pyramid having base as a regular n-gon with each side  $a$ , angle between any two consecutive lateral edges =  $\alpha$ , angle of the geometrical axis  $PO$  with each of the lateral edges =  $\beta$  & angle of the geometrical axis  $PO$  with each of the lateral faces =  $\gamma$ .

Similarly, we have

$$\Rightarrow \sin \angle A_2PM = \frac{MA_2}{PA_2} \quad \text{or} \quad \sin \frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{PA_2}$$

$$\Rightarrow PA_2 = \frac{a}{2} \operatorname{cosec} \frac{\alpha}{2} \quad \dots \dots \dots (4)$$

In right  $\Delta POA_2$  (Fig. 1)

$$\Rightarrow \sin \angle OPA_2 = \frac{OA_2}{PA_2} \quad \text{or} \quad \sin \beta = \frac{\left(\frac{a}{2} \operatorname{cosec} \frac{\pi}{n}\right)}{\left(\frac{a}{2} \operatorname{cosec} \frac{\alpha}{2}\right)} = \frac{\sin \frac{\alpha}{2}}{\sin \frac{\pi}{n}} \quad (\text{from Eq (2) \& (4)})$$

$$\Rightarrow \beta = \sin^{-1} \left( \frac{\sin \frac{\alpha}{2}}{\sin \frac{\pi}{n}} \right) \quad \forall n \geq 3 \quad \dots \dots \dots (5)$$

Above is the generalised formula for calculating the angle  $\beta$  between each lateral edge & the geometrical axis of any right pyramid having base as a regular polygon with n no. of sides each of length  $a$  & an angle  $\alpha$  between any two consecutive lateral edges.

In right  $\Delta POM$  (Fig. 1)

$$\Rightarrow \sin \angle OPM = \frac{OM}{PM} \quad \text{or} \quad \sin \gamma = \frac{\left(\frac{a}{2} \cot \frac{\pi}{n}\right)}{\left(\frac{a}{2} \cot \frac{\alpha}{2}\right)} = \frac{\tan \frac{\alpha}{2}}{\tan \frac{\pi}{n}} \quad (\text{from Eq (1) \& (3)})$$

$$\Rightarrow \gamma = \sin^{-1} \left( \frac{\tan \frac{\alpha}{2}}{\tan \frac{\pi}{n}} \right) \quad \forall n \geq 3 \quad \dots \dots \dots (6)$$

Above is the generalised formula for calculating the angle  $\gamma$  between each lateral face & the geometrical axis of any right pyramid having base as a regular polygon with n no. of sides each of length  $a$  & an angle  $\alpha$  between any two consecutive lateral edges.

Similarly, in right  $\Delta POM$ , we have

$$\Rightarrow PM^2 = OP^2 + OM^2 \quad \text{or} \quad PM = \sqrt{H^2 + \left(\frac{a}{2} \cot \frac{\pi}{n}\right)^2}$$

$$\Rightarrow PM = \frac{1}{2} \sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}} \quad \dots \dots \dots (7)$$

Now, equating the values of **PM** from equations (3) & (7), we have

$$\Rightarrow \frac{a}{2} \cot \frac{\alpha}{2} = \frac{1}{2} \sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}}$$

On squaring both the sides, we get

$$a^2 \cot^2 \frac{\alpha}{2} = 4H^2 + a^2 \cot^2 \frac{\pi}{n} \quad \Rightarrow 4H^2 = a^2 \left( \cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n} \right)$$

$$H = \frac{a}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}} \quad \forall n \geq 3 \dots \dots \dots (8)$$

Above Eq(8) is generalised formula for calculating the normal height  $H$  of any right pyramid having a base as a regular polygon with  $n$  no. of sides each of length  $a$  & an angle  $\alpha$  between any two consecutive lateral edges.

**2. Locating a sphere with a radius  $R$  resting in the vertex (corner) at which  $n$  no. of edges meet together at an angle  $\alpha$  between any two consecutive of them** (Eg. vertex of a regular polyhedron (platonic solid), any of two identical & diagonally opposite vertices of a uniform polyhedron (trapezohedron) with congruent right kite faces and vertex of a right pyramid with regular  $n$ -gonal base): Let there be a sphere, having its centre  $C$  & a radius  $R$ , resting in a vertex (corner)  $P$  at which  $n$  no. of edges meet together at angle  $\alpha$  between any two consecutive of them then this case the vertex can be treated as the vertex of a right pyramid with regular  $n$ -gonal base (as shown in the figure (2) below).

Now, draw a perpendicular say  $CQ$  from the centre  $C$  to the any of faces meeting at the vertex  $P$  & join the centre  $C$  with the vertex  $P$  (as shown in the figure (3) below). Let the distance of the centre  $C$  from the vertex  $P$  be  $CP = d$  & all  $n$  no. of faces, meeting at the vertex  $P$ , are equally inclined at an angle  $\angle CPQ = \gamma$  with the geometrical axis  $PC$ .

Now, in right  $\Delta PQC$  (figure 3 below)

$$\sin \angle CPQ = \frac{CQ}{CP} \Rightarrow \sin \gamma = \frac{R}{d} \Rightarrow d = \frac{R}{\sin \gamma}$$

Now, setting the value of  $\sin \gamma$  from Eq(6) as follows

$$d = \frac{R}{\left( \frac{\tan \frac{\alpha}{2}}{\tan \frac{\pi}{n}} \right)} = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}}$$

Hence, the distance ( $CP = d$ ) of the centre of the resting sphere from the vertex of polyhedron is given as follows

$$d = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} \quad \left( \forall \alpha < \frac{2\pi}{n} \quad \& \quad n \geq 3 \right) \dots \dots \dots (9)$$

Hence, the minimum distance ( $d_{min}$ ) of the resting sphere from the vertex of polyhedron is equal to the distance of the point  $S$  on the sphere which is closest to the vertex  $P$  (as shown in the figure 3). Hence is given as follows

$$PS = PC - CS = d - R = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} - R = \frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}}$$

$$d_{min} = \frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} \quad \left( \forall \alpha < \frac{2\pi}{n} \quad \& \quad n \geq 3 \right) \dots \dots \dots (10)$$

In right  $\Delta PNC$  (figure 2 above)

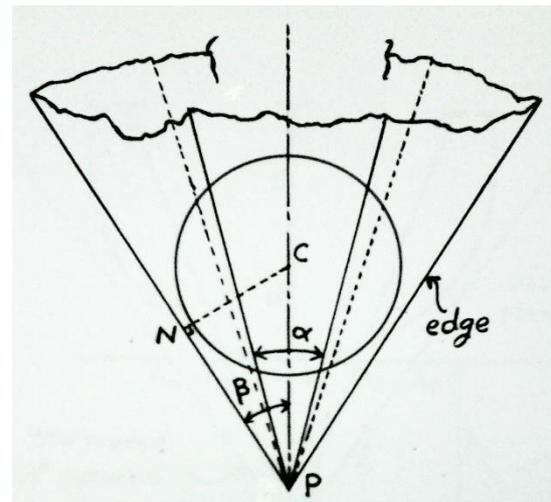


Figure 2: A sphere resting in a vertex (corner)  $P$  of a polyhedron touches all  $n$  no. of faces but does not touch any of  $n$  no. of edges meeting at the vertex  $P$ .

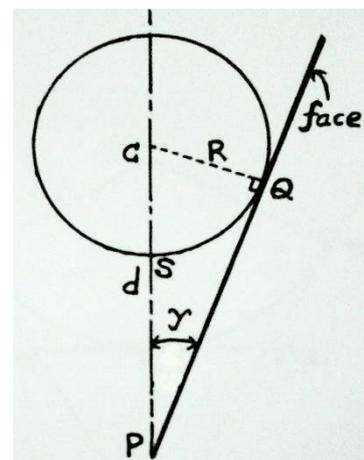


Figure 3: A sphere resting in a vertex (corner)  $P$  of a polyhedron is touching the face (shown by the line  $PQ$ ) normal to the plane of paper.

$$\sin \angle CPN = \frac{CN}{CP} \Rightarrow \sin \beta = \frac{CN}{d} \Rightarrow CN = d \sin \beta$$

Now, setting the value of  $d$  & the value of  $\sin \beta$  from Eq(5) in above value of  $CN$  as follows

$$CN = \frac{R \tan \frac{\pi}{n} \left( \frac{\sin \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2} \left( \sin \frac{\pi}{n} \right)} = \frac{R \cos \frac{\alpha}{2}}{\cos \frac{\pi}{n}}$$

Hence, normal distance ( $d_e$ ) of the centre of resting sphere from each edge of polyhedron is given as follows

$$d_e = \frac{R \cos \frac{\alpha}{2}}{\cos \frac{\pi}{n}} \quad \left( \forall \alpha < \frac{2\pi}{n} \text{ \& } n \geq 3 \right) \dots \dots \dots (11)$$

Hence, the minimum distance ( $(d_e)_{min}$ ) of the resting sphere from each edge of polyhedron is equal to the distance of a point on the sphere which is closest to the edge (see figure 2 above). Hence it is given as follows

$$(d_e)_{min} = CN - R = d_e - R = \frac{R \cos \frac{\alpha}{2}}{\cos \frac{\pi}{n}} - R = \frac{R \left( \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \right)}{\cos \frac{\pi}{n}}$$

$$(d_e)_{min} = \frac{R \left( \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \right)}{\cos \frac{\pi}{n}} \quad \left( \forall \alpha < \frac{2\pi}{n} \text{ \& } n \geq 3 \right) \dots \dots \dots (12)$$

**3. Truncation of the vertex (corner) of the polyhedron to best fit the sphere in that vertex:** In order to fillet all  $n$  no. of faces meeting at the vertex of a polyhedron to best fit a sphere in it. First of all, the vertex is truncated [4,5,6,7,8,9] with a cutting plane through an appropriate normal height (depth)  $h$  & then each of  $n$  no. of truncated faces, initially meeting at the vertex  $P$ , is provided an appropriate fillet radius  $R_f$  (as shown in the figure 4).

**3.1. Truncation of the vertex through a normal height (depth):** Let the vertex  $P$  be truncated with a plane (as shown by the dotted line normal to the plane of paper) through a normal height (depth)  $h$  such that the cutting plane just touches the resting sphere, having centre  $C$  & a radius  $R$ , at the point  $S$ . Thus, we obtain a truncated off right pyramid with  $n$ -gonal base  $A_1A_2A_3 \dots A_n$  having  $n$  no. of sides, angle between any two consecutive lateral edges  $\alpha$  & normal height  $h$ . Then normal height (depth)  $h$  to be cut is given as follows

$$h = PS = \text{minimum distance of sphere from the vertex } P = d_{min}$$

Hence, the normal height (depth)  $h$ , through which the vertex is to be truncated, is given as

$$h = d_{min} = \frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} \quad \left( \forall \alpha < \frac{2\pi}{n} \text{ \& } n \geq 3 \right) \dots \dots \dots (13)$$

The above formula shows that the vertex, at which  $n$  no. of edges meet together such that angle between any two consecutive edges is  $\alpha$ , should be truncated by a normal height (depth)  $h$  to best fit a sphere with a radius  $R$  in that vertex (corner) (of a polyhedron).

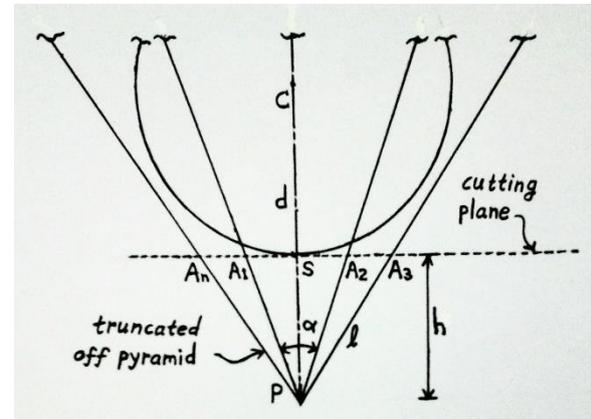


Figure 4: Vertex (corner)  $P$  is truncated by a cutting plane (normal to the plane of paper shown by the dotted line) just touching the sphere at the point  $S$ .

**3.2. Truncation of the vertex through an edge length:** Alternatively, the vertex P can also be truncated by cutting each of  $n$  no. of edges, meeting at that vertex, through an edge length say  $l$  measured from the vertex P (see figure 4 above). Thus, we obtain a truncated off right pyramid with  $n$ -gonal base  $A_1A_2A_3 \dots A_n$  having  $n$  no. of sides say each of length  $a$ , angle between any two consecutive lateral edges  $\alpha$  & normal height  $h$ . The normal height (depth)  $h$  of the right pyramid is given from the generalised Eq(8) as follows

$$h = \frac{a}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}}$$

But  $h = d_{min}$ , thus equating both the results as follows

$$\begin{aligned} h = d_{min} &\Rightarrow \frac{a}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}} = \frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} \\ \Rightarrow a &= \frac{2R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}}} = \frac{2R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}}} \\ &= \frac{2R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right) \tan \frac{\pi}{n} \tan \frac{\alpha}{2}}{\tan \frac{\alpha}{2} \sqrt{\tan^2 \frac{\pi}{n} - \tan^2 \frac{\alpha}{2}}} = \frac{2R \tan \frac{\pi}{n} \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\sqrt{\left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right) \left( \tan \frac{\pi}{n} + \tan \frac{\alpha}{2} \right)}} = 2R \tan \frac{\pi}{n} \sqrt{\frac{\tan \frac{\pi}{n} - \tan \frac{\alpha}{2}}{\tan \frac{\pi}{n} + \tan \frac{\alpha}{2}}} \\ &= 2R \tan \frac{\pi}{n} \sqrt{\frac{\tan \frac{\pi}{n} - \tan \frac{\alpha}{2}}{\tan \frac{\pi}{n} + \tan \frac{\alpha}{2}}} = 2R \tan \frac{\pi}{n} \sqrt{\frac{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \sin \frac{\alpha}{2}}{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} + \cos \frac{\pi}{n} \sin \frac{\alpha}{2}}} = 2R \tan \frac{\pi}{n} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} \\ \Rightarrow a &= 2R \tan \frac{\pi}{n} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} = A_1A_2 \dots \dots \dots (14) \end{aligned}$$

The length ( $l$ ) of each of  $n$  no. of lateral edges of the truncated off right pyramid is given from generalised Eq(4) derived above as follows

$$l = PA_1 = PA_2 = \frac{a}{2} \operatorname{cosec} \frac{\alpha}{2} \Rightarrow l = \left( R \tan \frac{\pi}{n} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} \right) \operatorname{cosec} \frac{\alpha}{2}$$

Hence, the edge length  $l$ , through which the vertex is to be truncated, is given as

$$l = R \tan \frac{\pi}{n} \operatorname{cosec} \frac{\alpha}{2} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} \quad \left( \forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3 \right) \dots \dots \dots (15)$$

The above formula shows that all  $n$  no. of edges, meeting together at a vertex such that the angle between any two consecutive edges is  $\alpha$ , should be truncated by a length  $l$  to best fit a sphere with a radius  $R$  in that vertex (corner) (of a polyhedron).

**4. Filleting of the faces meeting at the vertex & touching the sphere:** In order to best fit a sphere in the truncated vertex (corner) P (as discussed in above Sec. 3) each of  $n$  no. of truncated faces, having apex angle  $\alpha$  & meeting together at the vertex P, should be provided a fillet radius  $R_f$  with the centre O (as shown in the figure 5). Draw the perpendiculars  $OT_1$  &  $OT_2$  from the points  $T_1$  &  $T_2$  to the truncated edges which upon extending meet at the truncated vertex P at an angle  $\alpha$ . Join the centre O, lying on the bisector of  $\angle A_1PA_2 = \alpha$ , to the vertex P. Thus we have from Eq(14)

$$A_1J = A_2J = \frac{A_1A_2}{2} = \frac{a}{2} = R \tan \frac{\pi}{n} \sqrt{\frac{\sin(\frac{\pi - \alpha}{n})}{\sin(\frac{\pi + \alpha}{n})}}$$

$$\angle A_1OJ = \frac{\angle T_1OP}{2} = \frac{\frac{\pi}{2} - \angle OPT_1}{2} = \frac{\frac{\pi}{2} - \frac{\alpha}{2}}{2} = \frac{\pi - \alpha}{4}$$

$$\Rightarrow \angle JA_1O = \frac{\pi}{2} - \angle A_1OJ = \frac{\pi}{2} - \left(\frac{\pi - \alpha}{4}\right) = \frac{\pi + \alpha}{4}$$

In right  $\Delta OJA_1$  (Fig. 5)

$$\tan \angle JA_1O = \frac{OJ}{A_1J} \Rightarrow \tan\left(\frac{\pi + \alpha}{4}\right) = \frac{R_f}{A_1J}$$

$$\Rightarrow R_f = (A_1J) \tan\left(\frac{\pi + \alpha}{4}\right) = \left(R \tan \frac{\pi}{n} \sqrt{\frac{\sin(\frac{\pi - \alpha}{n})}{\sin(\frac{\pi + \alpha}{n})}}\right) \tan\left(\frac{\pi + \alpha}{4}\right)$$

Hence, the fillet radius  $R_f$  of each truncated face, to best fit the sphere, is given as

$$R_f = R \tan \frac{\pi}{n} \tan\left(\frac{\pi + \alpha}{4}\right) \sqrt{\frac{\sin(\frac{\pi - \alpha}{n})}{\sin(\frac{\pi + \alpha}{n})}} \quad \left(\forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3\right) \quad \dots \dots \dots (16)$$

The above formula shows that all  $n$  no. of truncated faces with apex angle  $\alpha$ , meeting together at a vertex, should be filleted/rounded through a radius  $R_f$  to best fit a sphere with a radius  $R$  in that vertex (corner) (of a polyhedron).

**5. Illustrative examples on locating the sphere resting in the vertex**

**5.1. Regular Tetrahedron:** Consider a sphere with a radius  $R$  is resting in one of 4 identical vertices (corners) of a regular tetrahedron. We know that three edges meet at each vertex of a regular tetrahedron at an angle  $60^\circ$  between any two consecutive of them. Then in this case we have

$$n = 3 \ \& \ \alpha = 60^\circ = \frac{\pi}{3}$$

Thus we can calculate all the important parameters for a sphere resting in the vertex (corner) of a regular tetrahedron as follows

- 1. Distance ( $d$ ) of the centre of the resting sphere from the vertex of tetrahedron** is given as follows

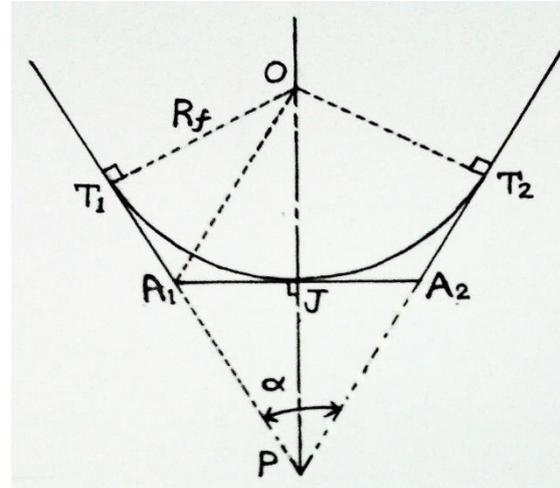


Figure 5: A face, truncated at the edge  $A_1A_2$ , is filleted through a radius  $R_f$  to best fit a sphere with a radius  $R$  in the vertex (corner) (of a polyhedron).

$$d = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} = \frac{R \tan \frac{\pi}{3}}{\tan \frac{\pi}{6}} = R \frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)} = 3R \quad \Rightarrow \quad \boxed{d = 3R}$$

2. Minimum distance ( $d_{min}$ ) of the resting sphere from the vertex of tetrahedron is given as follows

$$d_{min} = \frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} = \frac{R \left( \tan \frac{\pi}{3} - \tan \frac{\pi}{6} \right)}{\tan \frac{\pi}{6}} = \frac{R \left( \sqrt{3} - \frac{1}{\sqrt{3}} \right)}{\left( \frac{1}{\sqrt{3}} \right)} = R(3 - 1) = 2R$$

$$\Rightarrow \quad \boxed{d_{min} = 2R}$$

3. Distance ( $d_e$ ) of the centre of the resting sphere from each edge meeting at the vertex of the tetrahedron is given as follows

$$d_e = \frac{R \cos \frac{\alpha}{2}}{\cos \frac{\pi}{n}} = \frac{R \cos \frac{\pi}{6}}{\cos \frac{\pi}{3}} = \frac{R \left( \frac{\sqrt{3}}{2} \right)}{\left( \frac{1}{2} \right)} = R\sqrt{3}$$

$$\Rightarrow \quad \boxed{d = R\sqrt{3} \approx 1.732050808 R}$$

4. Minimum distance ( $(d_e)_{min}$ ) of the resting sphere from each edge meeting at the vertex of the tetrahedron is given as

$$(d_e)_{min} = \frac{R \left( \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \right)}{\cos \frac{\pi}{n}} = \frac{R \left( \cos \frac{\pi}{6} - \cos \frac{\pi}{3} \right)}{\cos \frac{\pi}{3}} = \frac{R \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right)}{\left( \frac{1}{2} \right)} = R(\sqrt{3} - 1)$$

$$\Rightarrow \quad \boxed{d = R(\sqrt{3} - 1) \approx 0.732050807 R}$$

5. Normal height (depth) ( $h$ ) through which the vertex of the tetrahedron is truncated to best fit the sphere is given as

$$h = d_{min} = 2R$$

$$\Rightarrow \quad \boxed{h = 2R}$$

6. Edge length ( $l$ ) through which the vertex of the tetrahedron is truncated to best fit the sphere is given as

$$l = R \tan \frac{\pi}{n} \operatorname{cosec} \frac{\alpha}{2} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} = R \tan \frac{\pi}{3} \operatorname{cosec} \frac{\pi}{6} \sqrt{\frac{\sin \left( \frac{\pi}{3} - \frac{\pi}{6} \right)}{\sin \left( \frac{\pi}{3} + \frac{\pi}{6} \right)}} = R(\sqrt{3})(2) \sqrt{\frac{\sin \left( \frac{\pi}{6} \right)}{\sin \left( \frac{\pi}{2} \right)}}$$

$$= 2R\sqrt{3} \sqrt{\frac{\left( \frac{1}{2} \right)}{1}} = R\sqrt{6}$$

$$\Rightarrow \quad \boxed{l = R\sqrt{6} \approx 2.449489743 R}$$

7. Fillet radius ( $R_f$ ) of each truncated face to best fit the sphere in the truncated vertex of the tetrahedron is given as

$$\begin{aligned} R_f &= R \tan \frac{\pi}{n} \tan \left( \frac{\pi + \alpha}{4} \right) \sqrt{\frac{\sin \left( \frac{\pi - \alpha}{n} \right)}{\sin \left( \frac{\pi + \alpha}{n} \right)}} = R \tan \frac{\pi}{3} \tan \left( \frac{\pi + \frac{\pi}{3}}{4} \right) \sqrt{\frac{\sin \left( \frac{\pi - \frac{\pi}{3}}{3} \right)}{\sin \left( \frac{\pi + \frac{\pi}{3}}{3} \right)}} \\ &= R(\sqrt{3}) \left( \tan \frac{\pi}{3} \right) \sqrt{\frac{\sin \left( \frac{\pi}{6} \right)}{\sin \left( \frac{\pi}{2} \right)}} = R(\sqrt{3})(\sqrt{3}) \sqrt{\frac{1}{2}} = \frac{3R}{\sqrt{2}} \\ &\Rightarrow \boxed{R_f = \frac{3R}{\sqrt{2}} \approx 2.121320344 R} \end{aligned}$$

The above value is very important to fillet all three faces meeting at a vertex (corner) for best fitting any sphere with a radius  $R$  in that vertex of a regular tetrahedron.

**5.2. Regular Hexahedron (cube):** Consider a sphere with a radius  $R$  is resting in one of 8 identical vertices (corners) of a cube. We know that three edges meet at each vertex of a cube at an angle  $90^\circ$  between any two consecutive of them. Then in this case we have

$$n = 3 \text{ \& } \alpha = 90^\circ = \frac{\pi}{2}$$

Thus we can calculate all the important parameters for a sphere resting in the vertex (corner) of a regular hexahedron (cube) as follows

1. Distance ( $d$ ) of the centre of the resting sphere from the vertex of cube is given as follows

$$d = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} = \frac{R \tan \frac{\pi}{3}}{\tan \frac{\pi}{4}} = \frac{R\sqrt{3}}{1} = R\sqrt{3} \Rightarrow \boxed{d = R\sqrt{3} \approx 1.732050808 R}$$

2. Minimum distance ( $d_{min}$ ) of the resting sphere from the vertex of cube is given as follows

$$\begin{aligned} d_{min} &= \frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} = \frac{R \left( \tan \frac{\pi}{3} - \tan \frac{\pi}{4} \right)}{\tan \frac{\pi}{4}} = \frac{R(\sqrt{3} - 1)}{1} = R(\sqrt{3} - 1) \\ &\Rightarrow \boxed{d_{min} = R(\sqrt{3} - 1) \approx 0.732050807 R} \end{aligned}$$

3. Distance ( $d_e$ ) of the centre of the resting sphere from each edge meeting at the vertex of the cube is given as follows

$$\begin{aligned} d_e &= \frac{R \cos \frac{\alpha}{2}}{\cos \frac{\pi}{n}} = \frac{R \cos \frac{\pi}{4}}{\cos \frac{\pi}{3}} = \frac{R \left( \frac{1}{\sqrt{2}} \right)}{\left( \frac{1}{2} \right)} = R\sqrt{2} \\ &\Rightarrow \boxed{d = R\sqrt{2} \approx 1.414213562 R} \end{aligned}$$

4. Minimum distance ( $(d_e)_{min}$ ) of the resting sphere from each edge meeting at the vertex of the cube is given as

$$(d_e)_{min} = \frac{R \left( \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \right)}{\cos \frac{\pi}{n}} = \frac{R \left( \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \right)}{\cos \frac{\pi}{3}} = \frac{R \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right)}{\left( \frac{1}{2} \right)} = R(\sqrt{2} - 1)$$

$$\Rightarrow \mathbf{d = R(\sqrt{2} - 1) \approx 0.414213562 R}$$

5. Normal height (depth) ( $h$ ) through which the vertex of the cube is truncated to best fit the sphere is given as

$$h = d_{min} = R(\sqrt{3} - 1)$$

$$\Rightarrow \mathbf{h = R(\sqrt{3} - 1) \approx 0.732050807 R}$$

6. Edge length ( $l$ ) through which the vertex of the cube is truncated to best fit the sphere is given as

$$l = R \tan \frac{\pi}{n} \operatorname{cosec} \frac{\alpha}{2} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} = R \tan \frac{\pi}{3} \operatorname{cosec} \frac{\pi}{4} \sqrt{\frac{\sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right)}{\sin \left( \frac{\pi}{3} + \frac{\pi}{4} \right)}} = R(\sqrt{3})(\sqrt{2}) \sqrt{\frac{\sin \left( \frac{\pi}{12} \right)}{\sin \left( \frac{7\pi}{12} \right)}}$$

$$= R\sqrt{6} \sqrt{\frac{\sin \left( \frac{\pi}{12} \right)}{\cos \left( \frac{\pi}{12} \right)}} = R\sqrt{6 \tan \left( \frac{\pi}{12} \right)} = R\sqrt{6(2 - \sqrt{3})} = R\sqrt{(3 - \sqrt{3})^2} = R(3 - \sqrt{3})$$

$$\Rightarrow \mathbf{l = R(3 - \sqrt{3}) \approx 1.267949192 R}$$

7. Fillet radius ( $R_f$ ) of each truncated face to best fit the sphere in the truncated vertex of the cube is given as

$$R_f = R \tan \frac{\pi}{n} \tan \left( \frac{\pi + \alpha}{4} \right) \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}} = R \tan \frac{\pi}{3} \tan \left( \frac{\pi + \pi}{4} \right) \sqrt{\frac{\sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right)}{\sin \left( \frac{\pi}{3} + \frac{\pi}{4} \right)}}$$

$$= R(\sqrt{3}) \left( \cot \frac{\pi}{8} \right) \sqrt{\tan \left( \frac{\pi}{12} \right)} = R(\sqrt{3})(1 + \sqrt{2}) \sqrt{2 - \sqrt{3}} = R(\sqrt{3} + \sqrt{6}) \sqrt{\frac{(\sqrt{3} - 1)^2}{2}}$$

$$= \frac{R(\sqrt{3} + \sqrt{6})(\sqrt{3} - 1)}{\sqrt{2}} = \frac{R\sqrt{2}(3 + 3\sqrt{2} - \sqrt{3} - \sqrt{6})}{2} = \frac{R(6 + 3\sqrt{2} - 2\sqrt{3} - \sqrt{6})}{2}$$

$$\Rightarrow \mathbf{R_f = \frac{R(6 + 3\sqrt{2} - 2\sqrt{3} - \sqrt{6})}{2} \approx 2.164524665 R}$$

The above value is very important to fillet all three faces meeting at a vertex (corner) for best fitting any sphere with a radius  $R$  in that vertex of a cube.

Similarly, we can derive important expressions for other platonic solids, uniform polyhedrons & right pyramids.

**6. Packing of the spheres in the right pyramid with a regular polygonal base:** Let there be a right pyramid with base as a regular polygon  $A_1A_2A_3 \dots A_n$  having  $n$  no. of sides each of equal length  $a$ , angle between any two consecutive lateral edges  $\alpha$  & normal height  $H$  (as shown in the figure 6).

Now, consider a largest sphere, having centre  $C_1$  & a radius  $R_1 (= R)$ , completely inscribed in right pyramid such that it touches the polygonal base  $A_1A_2A_3 \dots A_n$  at the centre  $O$  as well as all the lateral faces. Further locate a sphere, having centre  $C_2$  & a radius  $R_2$ , completely inscribed in the pyramid such that it touches the largest sphere at the point  $O_1$ , the polygonal base  $A_1A_2A_3 \dots A_n$  at the centre  $O$  & all the lateral faces. Thus continue to pack the right pyramid by locating smaller & smaller spheres in the same fashion up to total  $N$  no. of spheres including the largest one. Thus total  $N$  no. of spheres, having radii  $R_1, R_2, R_3, \dots \dots \dots R_N$  respectively, are snugly fitted touching one another between the polygonal base & the apex (vertex) point  $P$  of the right pyramid.

Now, the distance of the centre  $C_1$  of the largest (resting) sphere, with a radius  $R_1 = R$ , from the vertex  $P$  of the right pyramid is given by the generalised formula as follows

$$PC_1 = \frac{R_1 \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} \quad (\angle A_1PA_2 = \alpha)$$

Hence, the normal height  $H$  of right pyramid is given as

$$H = PO = PC_1 + C_1O = \frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} + R$$

$$\Rightarrow H = \frac{R \left( \tan \frac{\pi}{n} + \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} \quad \dots \dots \dots (17)$$

But the normal height of the right pyramid is given by generalised formula from Eq(8) above as follows

$$H = \frac{a}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}} \quad (a \text{ is length of each side of polygonal base of pyramid})$$

Equating both the above values of  $H$  from Eq(17) and Eq(8), we get

$$\frac{a}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}} = \frac{R \left( \tan \frac{\pi}{n} + \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} \Rightarrow a = \frac{2R \left( \tan \frac{\pi}{n} + \tan \frac{\alpha}{2} \right) \tan \frac{\pi}{n} \tan \frac{\alpha}{2}}{\tan \frac{\alpha}{2} \sqrt{\tan^2 \frac{\pi}{n} - \tan^2 \frac{\alpha}{2}}}$$

$$\Rightarrow a = 2R \tan \frac{\pi}{n} \sqrt{\frac{\tan \frac{\pi}{n} + \tan \frac{\alpha}{2}}{\tan \frac{\pi}{n} - \tan \frac{\alpha}{2}}} = 2R \tan \frac{\pi}{n} \sqrt{\frac{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} + \cos \frac{\pi}{n} \sin \frac{\alpha}{2}}{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \sin \frac{\alpha}{2}}}$$

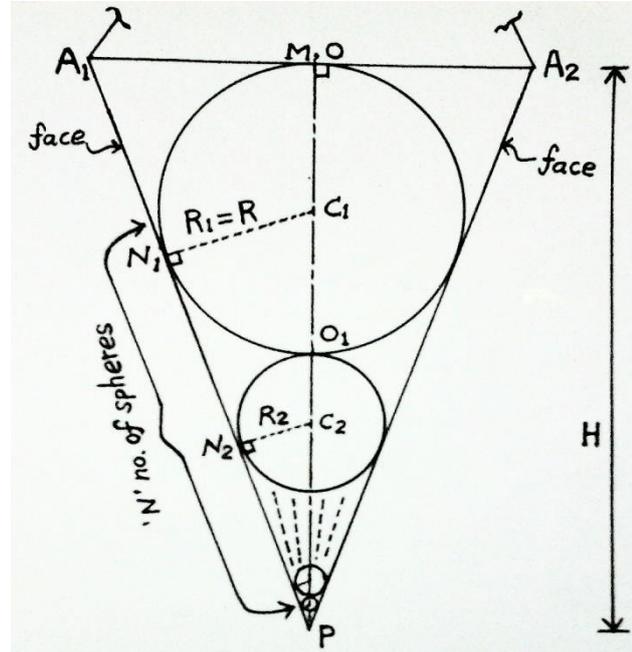


Figure 6:  $N$  no. of the spheres are snugly fitted touching one another between the regular polygonal base & the apex point  $P$  of a right pyramid. Isosceles triangular faces  $PA_1$  &  $PA_2$  are inclined to the plane of paper & the face  $A_1PA_2$  is in the plane of paper ( $\angle A_1PA_2 = \alpha$ ).

$$\Rightarrow a = 2R \tan \frac{\pi}{n} \sqrt{\frac{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}} = A_1 A_2 \dots \dots \dots (18)$$

Now, the volume ( $V_{\Delta}$ ) of the right pyramid is given as

$$V_{\Delta} = \frac{1}{3} \times (\text{area of regular polygonal base}) \times (\text{normal height}) = \frac{1}{3} \times \left(\frac{1}{4} n a^2 \cot \frac{\pi}{n}\right) \times (H)$$

Now, setting the values of length  $a$  from Eq(18) & normal height  $H$  from Eq(17), in term of radius  $R$  of the largest sphere inscribed in the pyramid, we get the volume of pyramid as follows

$$\begin{aligned} V_{\Delta} &= \frac{1}{12} n \cot \frac{\pi}{n} \left( 2R \tan \frac{\pi}{n} \sqrt{\frac{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}} \right)^2 \times \left( \frac{R \left( \tan \frac{\pi}{n} + \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}} \right) \\ &= \frac{4}{12} n R^3 \cot \frac{\pi}{n} \left( \tan^2 \frac{\pi}{n} \frac{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)} \right) \left( \frac{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} + \cos \frac{\pi}{n} \sin \frac{\alpha}{2}}{\tan \frac{\alpha}{2} \cos \frac{\pi}{n} \cos \frac{\alpha}{2}} \right) \\ &= \frac{1}{3} n R^3 \tan \frac{\pi}{n} \left( \frac{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)} \right) \left( \frac{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin \frac{\alpha}{2} \cos \frac{\pi}{n}} \right) = \frac{1}{3} n R^3 \left( \frac{\sin \frac{\pi}{n} \sin^2\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)} \right) \end{aligned}$$

Hence, the **volume ( $V_{\Delta}$ ) of the right pyramid** is given as

$$V_{\Delta} = \frac{1}{3} n R^3 \left( \frac{\sin \frac{\pi}{n} \sin^2\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)} \right) \quad \left( \forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3 \right) \dots \dots \dots (19)$$

Now, the distance of the centre  $C_2$  of larger (resting) sphere, with a radius  $R_2$ , from the vertex P of the right pyramid is given by the generalised formula as follows

$$PC_2 = \frac{R_2 \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}}$$

Since the spheres with centres  $C_1$  &  $C_2$  are externally touching to each other at the point  $O_1$  (as shown in the figure 6 above) hence, we have

$$\text{Distance between the centres} = C_1 C_2 = R_1 + R_2$$

But, we also have

$$C_1 C_2 = PC_1 - PC_2 = \frac{R_1 \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} - \frac{R_2 \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}}$$

Now, equating both the above values of  $C_1 C_2$  as follows

$$R_1 + R_2 = \frac{R_1 \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} - \frac{R_2 \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} \Rightarrow R_2 \left( \frac{\tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} + 1 \right) = R_1 \left( \frac{\tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}} - 1 \right)$$

$$\Rightarrow R_2 = R_1 \left( \frac{\tan \frac{\pi}{n} - \tan \frac{\alpha}{2}}{\tan \frac{\pi}{n} + \tan \frac{\alpha}{2}} \right) = R_1 \left( \frac{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \sin \frac{\alpha}{2}}{\sin \frac{\pi}{n} \cos \frac{\alpha}{2} + \cos \frac{\pi}{n} \sin \frac{\alpha}{2}} \right) = R_1 \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)$$

$$\Rightarrow R_2 = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) \quad (\text{Since, } R_1 = R = \text{radius of the largest sphere})$$

Similarly, we can obtain radius  $R_3$  of the sphere with centre  $C_3$  externally touching the sphere with radius  $R_2$  & centre  $C_2$  as follows

$$R_3 = R_2 \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^2$$

$$\Rightarrow R_3 = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^2$$

$$\Rightarrow R_4 = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^3$$

$$\Rightarrow R_5 = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^4$$

.....

$$\Rightarrow R_{N-1} = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^{N-2}$$

$$\Rightarrow R_N = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^{N-1}$$

Hence, the radius ( $R_N$ ) of  $N^{th}$  snugly fitted (packed) sphere in the vertex of the right pyramid is given as

$$R_N = R \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^{N-1} \quad \forall N \geq 1, \alpha < \frac{2\pi}{n}, n \geq 3 \quad \dots \dots \dots (20)$$

Where,  $R$  (from Eq.(20)) is the radius of the largest sphere inscribed in the right pyramid with regular  $n$ -gonal base &  $\alpha$  is the angle between any two consecutive lateral edges out of total  $n$  no. of edges meeting at the vertex. It is equally applicable on regular bi-pyramids & all five platonic solids.

**Note:**  $N = 1$  is taken for largest inscribed sphere.

But we know that

$$0 < \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) < 1 \quad \forall n \geq 3 \quad \& \quad \alpha < \frac{2\pi}{n}$$

The above generalised formula shows that the radii of the snugly fitted (packed) spheres decrease successively in a geometric progression (having a positive common ratio less than unity).

**6.1. The total volume  $((V_{packed})_N)$  packed/occupied by all  $N$  no. of snugly fitted (packed) spheres in the right pyramid:** There are total  $N$  no. of the spheres snugly fitted (packed) in the right pyramid, hence total volume of all the spheres with radii  $R_1, R_2, R_3, \dots, R_{N-1}, R_N$  is given as

$$(V_{packed})_N = \frac{4\pi}{3}R_1^3 + \frac{4\pi}{3}R_2^3 + \frac{4\pi}{3}R_3^3 + \dots + \frac{4\pi}{3}R_{N-1}^3 + \frac{4\pi}{3}R_N^3$$

$$\text{From Eq(20), } R_N = R \left( \frac{\sin\left(\frac{\pi - \alpha}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^{N-1} = K^{N-1}R \quad \text{Where, } K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} < 1$$

Now, setting the values of all the radii in term of  $R$  as follows

$$\begin{aligned} (V_{packed})_N &= \frac{4\pi}{3}R^3 + \frac{4\pi}{3}(KR)^3 + \frac{4\pi}{3}(K^2R)^3 + \dots + \frac{4\pi}{3}(K^{N-2}R)^3 + \frac{4\pi}{3}(K^{N-1}R)^3 \\ &= \frac{4\pi}{3}R^3(1 + K^3 + K^6 + K^9 + \dots + K^{3(N-2)} + K^{3(N-1)}) \\ &= \frac{4\pi}{3}R^3(\text{sum of } N \text{ terms of a geometric progression with a common ratio } K^3) \\ &= \frac{4\pi}{3}R^3 \left( \frac{1(1 - (K^3)^N)}{1 - K^3} \right) = \frac{4\pi}{3}R^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \quad (\text{since, } 0 < K < 1) \end{aligned}$$

Hence, the **total volume packed/occupied all  $N$  no. of snugly fitted (packed) spheres in the right pyramid** is given as follows

$$(V_{packed})_N = \frac{4\pi}{3}R^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \quad \dots \dots (21)$$

$$\text{Where } \textit{packing constant}, K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \quad \left( \forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3 \Rightarrow 0 < K < 1 \right)$$

**6.2. The total volume  $((V_{packed})_\infty)$  packed/occupied by infinite no. of snugly fitted spheres in the right pyramid ( $N \rightarrow \infty$ ):** Taking the limit of the total volume packed/occupied by  $N$  no. of snugly fitted/packed spheres at  $N \rightarrow \infty$  as follows

$$\begin{aligned} (V_{packed})_\infty &= \lim_{N \rightarrow \infty} \frac{4\pi}{3}R^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) = \frac{4\pi}{3}R^3 \lim_{N \rightarrow \infty} \left( \frac{1 - K^{3N}}{1 - K^3} \right) = \frac{4\pi}{3}R^3 \left( \frac{1 - 0}{1 - K^3} \right) \quad (\text{since, } 0 < K < 1) \\ &\Rightarrow (V_{packed})_\infty = \frac{4\pi R^3}{3(1 - K^3)} \quad \dots \dots (22) \end{aligned}$$

The above volume  $(V_{packed})_\infty$  is also called the **maximum packed volume**.

**6.3. Packing ratio  $(r_p)_N$  (i.e. the ratio of the total volume packed/occupied by all  $N$  no. of snugly fitted spheres to the volume of the right pyramid):** The packing ratio  $((r_p)_N)$  is given as follows

$$\begin{aligned}
 (r_p)_N &= \frac{\text{total volume occupied by all } n \text{ no. of snugly fitted spheres}}{\text{volume of right pyramid}} = \frac{(V_{\text{packed}})_N}{V_{\Delta}} \\
 &= \frac{\frac{4\pi}{3} R^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right)}{\frac{1}{3} n R^3 \left( \frac{\sin \frac{\pi}{n} \sin^2 \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)} \right)} = 4 \left( \frac{\pi}{n} \right) \left( \frac{1 - K^{3N}}{1 - K^3} \right) \left( \frac{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \frac{\pi}{n} \sin^2 \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) \\
 (r_p)_N &= 4 \left( \frac{\pi}{n} \right) \left( \frac{1 - K^{3N}}{1 - K^3} \right) \left( \frac{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \frac{\pi}{n} \sin^2 \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) \dots \dots \dots (23)
 \end{aligned}$$

**6.4. Packing ratio  $(r_p)_{\infty}$  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres to the volume of the right pyramid):** The packing ratio  $((r_p)_{\infty})$  is given as follows

$$\begin{aligned}
 (r_p)_{\infty} &= \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of right pyramid}} = \frac{(V_{\text{packed}})_{\infty}}{V_{\Delta}} \\
 &= \frac{\frac{4\pi R^3}{3(1 - K^3)}}{\frac{1}{3} n R^3 \left( \frac{\sin \frac{\pi}{n} \sin^2 \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)} \right)} = \frac{4\pi}{n(1 - K^3)} \left( \frac{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \frac{\pi}{n} \sin^2 \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) \\
 (r_p)_{\infty} &= \frac{4\pi}{n(1 - K^3)} \left( \frac{\sin \frac{\alpha}{2} \cos^2 \frac{\pi}{n} \sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \frac{\pi}{n} \sin^2 \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right) \dots \dots \dots (24)
 \end{aligned}$$

Where **packing constant**,  $K = \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}$   $\left( \forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3 \Rightarrow \ 0 < K < 1 \right)$

The above ratio  $(r_p)_{\infty}$  is also called the **maximum packing ratio**.

**7. Packing of the spheres in the platonic solids:** Let there be a platonic solid having  $n_V$  no. of identical vertices,  $n$  no. of edges meeting at each vertex such that angle between any two consecutive edges is  $\alpha$ . If  $R_i$  is the radius of the largest sphere inscribed in the platonic solid having volume  $V_S$ . In case of a platonic solid,  $N$  no. of spheres are snugly fitted (packed) in each of  $n_V$  no. of identical vertices excluding the largest inscribed sphere with radius  $R_i$ . Thus, the total  $(n_V N + 1)$  no. of spheres are snugly fitted in a platonic solid. All the generalised formulae of right pyramid are slightly modified for platonic solids simply by substituting  $N = N + 1$  &  $R = R_i =$  *radius of the largest inscribed sphere* in all the formula of sphere packing.

The radius  $(R_N)$  of the  $N^{th}$  sphere snugly fitted (packed) in one of  $n_V$  no. of identical vertex of a platonic solid (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given from the above Eq.(20) as follows

$$R_N = R_i \left( \frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)} \right)^N$$

**7.1. The total volume  $((V_{\text{packed}})_N)$  packed/occupied by all  $N$  no. of snugly fitted (packed) spheres in a platonic solid:** Since there are  $N$  no. of the spheres snugly fitted (packed) in each of  $n_V$  no. of identical vertices of a

platonic solid (**excluding the largest inscribed sphere**) hence total volume of all  $(n_V N + 1)$  spheres, out of which each N no. of snugly fitted sphere have radii  $R_1, R_2, R_3, \dots, R_{N-1}, R_N$  & the largest inscribed sphere has radius  $R_i$ , is given as

$$(V_{packed})_N = (\text{no. of identical vertices}) \times (\text{total volume of } N \text{ no. of snugly fitted spheres}) + (\text{volume of the largest inscribed sphere})$$

$$\Rightarrow (V_{packed})_N = n_V \left( \frac{4\pi}{3} R_1^3 + \frac{4\pi}{3} R_2^3 + \frac{4\pi}{3} R_3^3 + \dots + \frac{4\pi}{3} R_{N-1}^3 + \frac{4\pi}{3} R_N^3 \right) + \frac{4\pi}{3} R_i^3$$

$$\text{From Eq(20), } R_N = R_i \left( \frac{\sin\left(\frac{\pi - \alpha}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^N = K^N R_i \quad \text{Where, } K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} < 1$$

Now, setting the values of all the radii in term of  $R_i$  we get

$$\begin{aligned} (V_{packed})_N &= n_V \left( \frac{4\pi}{3} (KR_i)^3 + \frac{4\pi}{3} (K^2 R_i)^3 + \frac{4\pi}{3} (K^3 R_i)^3 + \dots + \frac{4\pi}{3} (K^{N-1} R_i)^3 + \frac{4\pi}{3} (K^N R_i)^3 \right) + \frac{4\pi}{3} R_i^3 \\ &= \frac{4\pi}{3} n_V K^3 R_i^3 (1 + K^3 + K^6 + K^9 + \dots + K^{3(N-2)} + K^{3(N-1)}) + \frac{4\pi}{3} R_i^3 \\ &= \frac{4\pi}{3} n_V K^3 R_i^3 (\text{sum of } N \text{ terms of a geometric progression with a common ratio } K^3) + \frac{4\pi}{3} R_i^3 \\ &= \frac{4\pi}{3} n_V K^3 R_i^3 \left( \frac{1(1 - (K^3)^N)}{1 - K^3} \right) + \frac{4\pi}{3} R_i^3 \quad (\text{since, } 0 < K < 1) \\ &= \frac{4\pi}{3} n_V K^3 R_i^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) + \frac{4\pi}{3} R_i^3 = \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \end{aligned}$$

Hence, the **total volume packed/occupied all  $(n_V N + 1)$  no. of snugly fitted (packed) spheres in a platonic solid** is given as

$$(V_{packed})_N = \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \dots \dots \dots (25)$$

$$\text{Where packing constant, } K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \quad \left( \forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3 \Rightarrow 0 < K < 1 \right)$$

**7.2. The total volume  $((V_{packed})_\infty)$  packed/occupied by infinite no. of snugly fitted spheres in all  $n_V$  no. of identical vertices of a platonic solid (including the largest inscribed sphere)  $(N \rightarrow \infty)$ :** Taking the limit of the total volume packed/occupied by  $(n_V N + 1)$  no. of snugly fitted/packed spheres at  $N \rightarrow \infty$  as follows

$$\begin{aligned} (V_{packed})_\infty &= \lim_{N \rightarrow \infty} \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{4\pi}{3} R_i^3 \lim_{N \rightarrow \infty} \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \\ &= \frac{4\pi}{3} R_i^3 \left( 1 + n_V K^3 \left( \frac{1 - 0}{1 - K^3} \right) \right) = \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_V K^3}{1 - K^3} \right) \quad (\because 0 < K < 1) \end{aligned}$$

$$(V_{packed})_{\infty} = \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_v K^3}{1 - K^3} \right) \dots \dots \dots (26)$$

**7.3. Packing ratio  $(r_p)_N$**  (i.e. the ratio of the total volume packed/occupied by all  $(n_v N + 1)$  no. of snugly fitted spheres to the volume of the platonic solid): The packing ratio  $((r_p)_N)$  is given as follows

$$\begin{aligned} (r_p)_N &= \frac{\text{total volume occupied by all } (n_v N + 1) \text{ no. of snugly fitted spheres}}{\text{volume of platonic solid}} = \frac{(V_{packed})_N}{V_s} \\ &= \frac{\frac{4\pi}{3} R_i^3 \left\{ 1 + n_v K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}}{V_s} = \frac{4\pi}{3V_s} R_i^3 \left\{ 1 + n_v K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \\ (r_p)_N &= \frac{4\pi}{3V_s} R_i^3 \left\{ 1 + n_v K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \dots \dots \dots (27) \end{aligned}$$

**7.4. Packing ratio  $(r_p)_{\infty}$**  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all  $n_v$  no. of identical vertices of a platonic solid (including the largest inscribed sphere): The packing ratio  $((r_p)_{\infty})$  is given as follows

$$\begin{aligned} (r_p)_{\infty} &= \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of platonic solid}} = \frac{(V_{packed})_{\infty}}{V_s} \\ &= \frac{\frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_v K^3}{1 - K^3} \right)}{V_s} = \frac{4\pi}{3V_s} R_i^3 \left( 1 + \frac{n_v K^3}{1 - K^3} \right) \\ (r_p)_{\infty} &= \frac{4\pi}{3V_s} R_i^3 \left( 1 + \frac{n_v K^3}{1 - K^3} \right) \dots \dots \dots (28) \end{aligned}$$

Where **packing constant**,  $K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}$   $\left( \forall \alpha < \frac{2\pi}{n} \ \& \ n \geq 3 \Rightarrow \ 0 < K < 1 \right)$

**8. Packing of the spheres in all five platonic solids**

In order to determine the total packed volume & packing ratio (fraction) of all the platonic solids we will directly take the data from ‘table of the important parameters of all five platonic solids’ prepared by the author such as (inner) radius of the (largest) inscribed sphere & the volume of the corresponding platonic solid in the order [1].

**8.1. Regular tetrahedron:** Let there be a regular tetrahedron with edge length  $a$ . Then in this case we have

$$R_i = \text{radius of the (largest) inscribed sphere in a regular tetrahedron} = \frac{a}{2\sqrt{6}}$$

$$V_s = \text{volume of a regular tetrahedron} = \frac{a^3}{6\sqrt{2}}$$

$$n_v = \text{no. of identical vertices in a regular tetrahedron} = 4$$

$$n = \text{no. of edges meeting at each vertex in a regular tetrahedron} = 3$$

$\alpha$  = angle between any two consecutive edges meeting at each vertex in a regular tetrahedron

$$= 60^\circ = \frac{\pi}{3}$$

**8.1.1. The packing constant  $K$  of a regular tetrahedron** is calculated as follows

$$K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)} = \frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{1}{2} \Rightarrow \boxed{K = \frac{1}{2}}$$

**8.1.2. The radius ( $R_N$ ) of the  $N^{\text{th}}$  sphere** snugly fitted (packed) in each of  $n_V = 4$  identical vertex of a regular tetrahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given from Eq(20) as follows

$$R_N = R_i \left( \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^N \Rightarrow \boxed{R_N = \frac{a}{2\sqrt{6}} \left(\frac{1}{2}\right)^N}$$

**8.1.3.** If there are  $N$  no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of  $n_V = 4$  identical vertices of a regular tetrahedron then the total volume occupied by all  $n_V N + 1 = 4N + 1$  snugly fitted spheres (including the largest inscribed sphere) is given from Eq(25) as follows

$$(V_{\text{packed}})_N = \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{4\pi}{3} \left( \frac{a}{2\sqrt{6}} \right)^3 \left\{ 1 + (4) \left( \frac{1}{2} \right)^3 \left( \frac{1 - \left(\frac{1}{2}\right)^{3N}}{1 - \left(\frac{1}{2}\right)^3} \right) \right\}$$

$$\therefore (V_{\text{packed}})_N = \frac{\pi a^3}{36\sqrt{6}} \left\{ 1 + \frac{4}{7} \left( \frac{2^{3N} - 1}{2^{3N}} \right) \right\}$$

**8.1.4. The total volume ( $(V_{\text{packed}})_\infty$ )** packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 4$  identical vertices of a regular tetrahedron (including the largest inscribed sphere) ( $N \rightarrow \infty$ ) is given from Eq(26) as follows

$$(V_{\text{packed}})_\infty = \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_V K^3}{1 - K^3} \right) = \frac{4\pi}{3} \left( \frac{a}{2\sqrt{6}} \right)^3 \left( 1 + \frac{4 \left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{2}\right)^3} \right) = \frac{\pi a^3}{36\sqrt{6}} \left( 1 + \frac{4}{7} \right) = \frac{11\pi a^3}{252\sqrt{6}}$$

$$\therefore (V_{\text{packed}})_\infty = \frac{11\pi a^3}{252\sqrt{6}}$$

**8.1.5. Packing ratio  $(r_p)_N$**  (i.e. the ratio of the total volume packed/occupied by all  $(4N + 1)$  no. of snugly fitted spheres to the volume of the regular tetrahedron): The packing ratio  $((r_p)_N)$  is given from Eq(27) as follows

$$(r_p)_N = \frac{\text{total volume occupied by all } (4N + 1) \text{ no. of snugly fitted spheres}}{\text{volume of regular tetrahedron}} = \frac{(V_{\text{packed}})_N}{V_s}$$

$$\Rightarrow (r_p)_N = \frac{\frac{\pi a^3}{36\sqrt{6}} \left\{ 1 + \frac{4}{7} \left( \frac{2^{3N} - 1}{2^{3N}} \right) \right\}}{\frac{a^3}{6\sqrt{2}}} = \frac{\pi}{6\sqrt{3}} \left\{ 1 + \frac{4}{7} \left( \frac{2^{3N} - 1}{2^{3N}} \right) \right\}$$

$$\therefore (r_p)_N = \frac{\pi}{6\sqrt{3}} \left\{ 1 + \frac{4}{7} \left( \frac{2^{3N} - 1}{2^{3N}} \right) \right\}$$

It is to be noted that the value of the packing ratio  $(r_p)_N$  depends only on  $N$  no. of spheres snugly fitted/packed in each of  $n_V = 4$  identical vertices of a regular tetrahedron.

**8.1.6. Packing ratio  $(r_p)_\infty$**  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 4$  identical vertices of a regular tetrahedron (including the largest inscribed sphere): The packing ratio  $((r_p)_\infty)$  is given from Eq(28) as follows

$$(r_p)_\infty = \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of regular tetrahedron}} = \frac{(V_{\text{packed}})_\infty}{V_s}$$

$$\Rightarrow (r_p)_\infty = \frac{\frac{11\pi a^3}{252\sqrt{6}}}{\frac{a^3}{6\sqrt{2}}} = \frac{11\pi}{42\sqrt{3}}$$

$$\therefore (r_p)_\infty = \frac{11\pi}{42\sqrt{3}} \approx 0.47504269$$

It is to be noted that the value of the packing ratio  $(r_p)_\infty$  is the maximum possible value which shows that approximate 47.5 % of the volume of any regular tetrahedron can be packed by snugly fitting infinite no. of spheres in each of its four identical vertices including the (volume of) largest inscribed sphere touching all four triangular faces.

**8.2. Regular hexahedron (cube):** Let there be a regular hexahedron (cube) with edge length  $a$ . In this case we have

$$R_i = \text{radius of the (largest) inscribed sphere in a regular hexahedron} = \frac{a}{2}$$

$$V_s = \text{volume of a regular hexahedron} = a^3$$

$$n_V = \text{no. of identical vertices in a regular hexahedron} = 8$$

$$n = \text{no. of edges meeting at each vertex in a regular hexahedron} = 3$$

$$\alpha = \text{angle between any two consecutive edges meeting at each vertex in a regular hexahedron} \\ = 90^\circ = \frac{\pi}{2}$$

**8.2.1. The packing constant  $K$  of a regular hexahedron (cube)** is calculated as follows

$$K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)} = \frac{\sin\left(\frac{\pi}{12}\right)}{\cos\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3} \Rightarrow K = 2 - \sqrt{3}$$

**8.2.2. The radius  $(R_N)$  of the  $N^{\text{th}}$  sphere** snugly fitted (packed) in each of  $n_V = 8$  identical vertex of a regular hexahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given from Eq(20) as follows

$$R_N = R_i \left( \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^N \Rightarrow R_N = \frac{a}{2} (2 - \sqrt{3})^N$$

**8.2.3.** If there are  $N$  no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of  $n_V = 8$  identical vertices of a regular hexahedron then the **total volume occupied** by all  $n_V N + 1 = 8N + 1$  snugly fitted spheres (including the largest inscribed sphere) is given from Eq(25) as follows

$$\begin{aligned}
 (V_{packed})_N &= \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{4\pi}{3} \left( \frac{a}{2} \right)^3 \left\{ 1 + (8)(2 - \sqrt{3})^3 \left( \frac{1 - (2 - \sqrt{3})^{3N}}{1 - (2 - \sqrt{3})^3} \right) \right\} \\
 &= \frac{\pi a^3}{6} \left\{ 1 + 8(26 - 15\sqrt{3}) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{15\sqrt{3} - 25} \right) \right\} = \frac{\pi a^3}{6} \left\{ 1 + \frac{8}{5} (26 - 15\sqrt{3}) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{3\sqrt{3} - 5} \right) \right\} \\
 &= \frac{\pi a^3}{6} \left\{ 1 + \frac{8}{5} (26 - 15\sqrt{3})(3\sqrt{3} + 5) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{(3\sqrt{3} - 5)(3\sqrt{3} + 5)} \right) \right\} \\
 &= \frac{\pi a^3}{6} \left\{ 1 + \frac{8}{5} (3\sqrt{3} - 5) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{2} \right) \right\} = \frac{\pi a^3}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\} \\
 &\therefore (V_{packed})_N = \frac{\pi a^3}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}
 \end{aligned}$$

**8.2.4.** The **total volume**  $((V_{packed})_\infty)$  packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 8$  identical vertices of a regular hexahedron (including the largest inscribed sphere) ( $N \rightarrow \infty$ ) is given from Eq(26) as follows

$$\begin{aligned}
 (V_{packed})_\infty &= \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_V K^3}{1 - K^3} \right) = \frac{4\pi}{3} \left( \frac{a}{2} \right)^3 \left( 1 + \frac{8(2 - \sqrt{3})^3}{1 - (2 - \sqrt{3})^3} \right) = \frac{\pi a^3}{6} \left( 1 + \frac{8(26 - 15\sqrt{3})}{15\sqrt{3} - 25} \right) \\
 &= \frac{\pi a^3}{6} \left( 1 + \frac{4(3\sqrt{3} - 5)}{5} \right) = \frac{\pi a^3}{6} \left( \frac{12\sqrt{3} - 15}{5} \right) = \pi a^3 \left( \frac{4\sqrt{3} - 5}{10} \right) \\
 &\therefore (V_{packed})_\infty = \pi a^3 \left( \frac{4\sqrt{3} - 5}{10} \right)
 \end{aligned}$$

**8.2.5.** **Packing ratio**  $(r_p)_N$  (i.e. the ratio of the total volume packed/occupied by all  $(8N + 1)$  no. of snugly fitted spheres to the volume of the regular hexahedron/cube): The packing ratio  $((r_p)_N)$  is given from Eq(27) as follows

$$\begin{aligned}
 (r_p)_N &= \frac{\text{total volume occupied by all } (8N + 1) \text{ no. of snugly fitted spheres}}{\text{volume of regular hexahedron}} = \frac{(V_{packed})_N}{V_s} \\
 \Rightarrow (r_p)_N &= \frac{\frac{\pi a^3}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}}{a^3} = \frac{\pi}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\} \\
 &\therefore (r_p)_N = \frac{\pi}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}
 \end{aligned}$$

It is to be noted that the value of the packing ratio  $(r_p)_N$  depends only on  $N$  no. of spheres snugly fitted/packed in each of  $n_V = 8$  identical vertices of a regular hexahedron (cube).

**8.2.6. Packing ratio  $(r_p)_\infty$**  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 8$  identical vertices of a regular hexahedron (including the largest inscribed sphere): The packing ratio  $((r_p)_\infty)$  is given from Eq(28) as follows

$$(r_p)_\infty = \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of regular hexahedron}} = \frac{(V_{packed})_\infty}{V_s}$$

$$\Rightarrow (r_p)_\infty = \frac{\pi a^3 \left( \frac{4\sqrt{3} - 5}{10} \right)}{a^3} = \pi \left( \frac{4\sqrt{3} - 5}{10} \right)$$

$$\therefore (r_p)_\infty = \pi \left( \frac{4\sqrt{3} - 5}{10} \right) \approx \mathbf{0.60576291}$$

It is to be noted that the value of the packing ratio  $(r_p)_\infty$  is the maximum possible value which shows that approximate 60.58 % of the volume of any regular hexahedron (cube) can be packed by snugly fitting infinite no. of the spheres in each of its eight identical vertices including the (volume of) largest inscribed sphere touching all six square faces.

**8.3. Regular octahedron:** Let there be a regular octahedron with edge length  $a$ . In this case we have

$$R_i = \text{radius of the (largest) inscribed sphere in a regular octahedron} = \frac{a}{\sqrt{6}}$$

$$V_s = \text{volume of a regular octahedron} = \frac{a^3 \sqrt{2}}{3}$$

$$n_V = \text{no. of identical vertices in a regular octahedron} = 6$$

$$n = \text{no. of edges meeting at each vertex in a regular octahedron} = 4$$

$$\alpha = \text{angle between any two consecutive edges meeting at each vertex in a regular octahedron}$$

$$= 60^\circ = \frac{\pi}{3}$$

**8.3.1. The packing constant  $K$  of a regular octahedron** is calculated as follows

$$K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)} = \frac{\sin\left(\frac{\pi}{12}\right)}{\cos\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3} \Rightarrow \mathbf{K = 2 - \sqrt{3}}$$

**8.3.2. The radius  $(R_N)$  of the  $N^{\text{th}}$  sphere** snugly fitted (packed) in each of  $n_V = 6$  identical vertex of a regular octahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given from Eq(20) as follows

$$R_N = R_i \left( \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^N \Rightarrow \mathbf{R_N = \frac{a}{\sqrt{6}} (2 - \sqrt{3})^N}$$

**8.3.3.** If there are  $N$  no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of  $n_V = 6$  identical vertices of a regular octahedron then the **total volume occupied** by all  $n_V N + 1 = 6N + 1$  snugly fitted spheres (including the largest inscribed sphere) is given from Eq(25) as follows

$$\begin{aligned}
(V_{packed})_N &= \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{4\pi}{3} \left( \frac{a}{\sqrt{6}} \right)^3 \left\{ 1 + (6)(2 - \sqrt{3})^3 \left( \frac{1 - (2 - \sqrt{3})^{3N}}{1 - (2 - \sqrt{3})^3} \right) \right\} \\
&= \frac{2\pi a^3}{9\sqrt{6}} \left\{ 1 + 6(26 - 15\sqrt{3}) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{15\sqrt{3} - 25} \right) \right\} = \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{6}{5}(26 - 15\sqrt{3}) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{3\sqrt{3} - 5} \right) \right\} \\
&= \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{6}{5}(26 - 15\sqrt{3})(3\sqrt{3} + 5) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{(3\sqrt{3} - 5)(3\sqrt{3} + 5)} \right) \right\} \\
&= \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{6}{5}(3\sqrt{3} - 5) \left( \frac{1 - (2 - \sqrt{3})^{3N}}{2} \right) \right\} = \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{3}{5}(3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\} \\
\therefore (V_{packed})_N &= \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{3}{5}(3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}
\end{aligned}$$

**8.3.4. The total volume**  $((V_{packed})_\infty)$  packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 6$  identical vertices of a regular octahedron (including the largest inscribed sphere) ( $N \rightarrow \infty$ ) is given from Eq(26) as follows

$$\begin{aligned}
(V_{packed})_\infty &= \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_V K^3}{1 - K^3} \right) = \frac{4\pi}{3} \left( \frac{a}{\sqrt{6}} \right)^3 \left( 1 + \frac{6(2 - \sqrt{3})^3}{1 - (2 - \sqrt{3})^3} \right) = \frac{2\pi a^3}{9\sqrt{6}} \left( 1 + \frac{6(26 - 15\sqrt{3})}{15\sqrt{3} - 25} \right) \\
&= \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left( 1 + \frac{3(3\sqrt{3} - 5)}{5} \right) = \frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left( \frac{9\sqrt{3} - 10}{5} \right) = \pi a^3 \sqrt{\frac{2}{3}} \left( \frac{9\sqrt{3} - 10}{45} \right) \\
\therefore (V_{packed})_\infty &= \pi a^3 \sqrt{\frac{2}{3}} \left( \frac{9\sqrt{3} - 10}{45} \right)
\end{aligned}$$

**8.3.5. Packing ratio**  $(r_p)_N$  (i.e. the ratio of the total volume packed/occupied by all  $(6N + 1)$  no. of snugly fitted spheres to the volume of the regular octahedron): The packing ratio  $((r_p)_N)$  is given from Eq(27) as follows

$$\begin{aligned}
(r_p)_N &= \frac{\text{total volume occupied by all } (6N + 1) \text{ no. of snugly fitted spheres}}{\text{volume of regular octahedron}} = \frac{(V_{packed})_N}{V_s} \\
\Rightarrow (r_p)_N &= \frac{\frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{3}{5}(3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}}{\frac{a^3 \sqrt{2}}{3}} = \frac{\pi}{3\sqrt{3}} \left\{ 1 + \frac{3}{5}(3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\} \\
\therefore (r_p)_N &= \frac{\pi}{3\sqrt{3}} \left\{ 1 + \frac{3}{5}(3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}
\end{aligned}$$

It is to be noted that the value of the packing ratio  $(r_p)_N$  depends only on  $N$  no. of spheres snugly fitted/packed in each of  $n_V = 6$  identical vertices of a regular octahedron.

**8.3.6. Packing ratio  $(r_p)_\infty$**  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all  $n_v = 6$  identical vertices of a regular octahedron (including the largest inscribed sphere): The packing ratio  $((r_p)_\infty)$  is given from Eq(28) as follows

$$(r_p)_\infty = \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of regular octahedron}} = \frac{(V_{\text{packed}})_\infty}{V_s}$$

$$\Rightarrow (r_p)_\infty = \frac{\pi a^3 \sqrt{\frac{2}{3}} \left( \frac{9\sqrt{3} - 10}{45} \right)}{\frac{a^3 \sqrt{2}}{3}} = \pi \left( \frac{9\sqrt{3} - 10}{15\sqrt{3}} \right)$$

$$\therefore (r_p)_\infty = \pi \left( \frac{9\sqrt{3} - 10}{15\sqrt{3}} \right) \approx 0.675756016$$

It is to be noted that the value of the packing ratio  $(r_p)_\infty$  is the maximum possible value which shows that approximate 67.57 % of the volume of any regular octahedron can be packed by snugly fitting infinite no. of spheres in each of its six identical vertices including the (volume of) largest inscribed sphere touching all eight triangular faces.

**8.4. Regular dodecahedron:** Let there be a regular dodecahedron with edge length  $a$ . In this case we have

$$R_i = \text{radius of the (largest) inscribed sphere in a regular dodecahedron} = \frac{a(3 + \sqrt{5})}{2\sqrt{10 - 2\sqrt{5}}}$$

$$= \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}}$$

$$V_s = \text{volume of a regular dodecahedron} = \frac{a^3(15 + 7\sqrt{5})}{4}$$

$$n_v = \text{no. of identical vertices in a regular dodecahedron} = 20$$

$$n = \text{no. of edges meeting at each vertex in a regular dodecahedron} = 3$$

$$\alpha = \text{angle between any two consecutive edges meeting at each vertex in a regular dodecahedron}$$

$$= 108^\circ = \frac{3\pi}{5}$$

**8.4.1. The packing constant  $K$  of a regular dodecahedron** is calculated as follows

$$K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{\pi}{3} - \frac{3\pi}{10}\right)}{\sin\left(\frac{\pi}{3} + \frac{3\pi}{10}\right)} = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \Rightarrow K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4}$$

**8.4.2. The radius  $(R_N)$  of the  $N^{\text{th}}$  sphere** snugly fitted (packed) in each of  $n_v = 20$  identical vertex of a regular dodecahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given from Eq(20) as follows

$$R_N = R_i \left( \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^N \Rightarrow R_N = \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}} \left( \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \right)^N$$

**8.4.3.** If there are  $N$  no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of  $n_V = 20$  identical vertices of a regular dodecahedron then the **total volume occupied** by all  $n_V N + 1 = 20N + 1$  snugly fitted spheres (including the largest inscribed sphere) is given from Eq(25) as follows

$$(V_{packed})_N = \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{4\pi}{3} \left( \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}} \right)^3 \left\{ 1 + (20)K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$$

$$= \frac{\pi a^3}{60} \sqrt{\frac{61000 + 27280\sqrt{5}}{10}} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$$

$$\therefore (V_{packed})_N = \frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$$

Where, **packing constant**,  $K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \approx 0.114420648$

**8.4.4.** The **total volume** ( $(V_{packed})_\infty$ ) packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 20$  identical vertices of a regular dodecahedron (including the largest inscribed sphere) ( $N \rightarrow \infty$ ) is given from Eq(26) as follows

$$(V_{packed})_\infty = \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_V K^3}{1 - K^3} \right) = \frac{4\pi}{3} \left( \frac{a}{2} \sqrt{\frac{25 + 11\sqrt{5}}{10}} \right)^3 \left( 1 + \frac{20K^3}{1 - K^3} \right)$$

$$= \frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left( \frac{1 - K^3 + 20K^3}{1 - K^3} \right) = \frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left( \frac{1 + 19K^3}{1 - K^3} \right)$$

$$\therefore (V_{packed})_\infty = \frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left( \frac{1 + 19K^3}{1 - K^3} \right)$$

**8.4.5. Packing ratio**  $(r_p)_N$  (i.e. the ratio of the total volume packed/occupied by all  $(20N + 1)$  no. of snugly fitted spheres to the volume of the regular dodecahedron): The packing ratio  $((r_p)_N)$  is given from Eq(27) as follows

$$(r_p)_N = \frac{\text{total volume occupied by all } (20N + 1) \text{ no. of snugly fitted spheres}}{\text{volume of regular dodecahedron}} = \frac{(V_{packed})_N}{V_s}$$

$$\Rightarrow (r_p)_N = \frac{\frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}}{\frac{a^3 (15 + 7\sqrt{5})}{4}} = \frac{\pi \sqrt{6100 + 2728\sqrt{5}}}{15(15 + 7\sqrt{5})} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$$

$$= \frac{\pi}{15} \sqrt{\frac{65 + 29\sqrt{5}}{10}} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$$

$$\therefore (r_p)_N = \frac{\pi}{15} \sqrt{\frac{65 + 29\sqrt{5}}{10}} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$$

It is to be noted that the value of the packing ratio  $(r_p)_N$  depends only on  $N$  no. of spheres snugly fitted/packed in each of  $n_V = 20$  identical vertices of a regular dodecahedron.

**8.4.6. Packing ratio  $(r_p)_\infty$**  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 20$  identical vertices of a regular dodecahedron (including the largest inscribed sphere): The packing ratio  $((r_p)_\infty)$  is given from Eq(28) as follows

$$(r_p)_\infty = \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of regular dodecahedron}} = \frac{(V_{\text{packed}})_\infty}{V_s}$$

$$\Rightarrow (r_p)_\infty = \frac{\frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left( \frac{1 + 19K^3}{1 - K^3} \right)}{\frac{a^3(15 + 7\sqrt{5})}{4}} = \frac{\pi \sqrt{6100 + 2728\sqrt{5}}}{15(15 + 7\sqrt{5})} \left( \frac{1 + 19K^3}{1 - K^3} \right)$$

$$\therefore (r_p)_\infty = \frac{\pi}{15} \sqrt{\frac{65 + 29\sqrt{5}}{10}} \left( \frac{1 + 19K^3}{1 - K^3} \right) \approx 0.777342128$$

Where, **packing constant,  $K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \approx 0.114420648$**

It is to be noted that the value of the packing ratio  $(r_p)_\infty$  is the maximum possible value which shows that approximate 77.73 % of the volume of any regular dodecahedron can be packed by snugly fitting infinite no. of the spheres in each of its 20 identical vertices including the (volume of) largest inscribed sphere touching all 12 pentagonal faces.

**8.5. Regular icosahedron:** Let there be a regular icosahedron with edge length  $a$ . In this case we have

$$R_i = \text{radius of the (largest) inscribed sphere in a regular icosahedron} = \frac{a(3 + \sqrt{5})}{4\sqrt{3}}$$

$$V_s = \text{volume of a regular icosahedron} = \frac{5a^3(3 + \sqrt{5})}{12}$$

$$n_V = \text{no. of identical vertices in a regular icosahedron} = 12$$

$$n = \text{no. of edges meeting at each vertex in a regular icosahedron} = 5$$

$$\alpha = \text{angle between any two consecutive edges meeting at each vertex in a regular icosahedron}$$

$$= 60^\circ = \frac{\pi}{3}$$

**8.5.1. The packing constant  $K$  of a regular icosahedron** is calculated as follows

$$K = \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{\pi}{5} - \frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{5} + \frac{\pi}{6}\right)} = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \Rightarrow K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4}$$

**8.5.2. The radius  $(R_N)$  of the  $N^{\text{th}}$  sphere** snugly fitted (packed) in each of  $n_V = 12$  identical vertex of a regular icosahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given from Eq(20) as follows

$$R_N = R_i \left( \frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)} \right)^N \Rightarrow R_N = \frac{a(3 + \sqrt{5})}{4\sqrt{3}} \left( \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \right)^N$$

**8.5.3.** If there are  $N$  no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of  $n_V = 12$  identical vertices of a regular icosahedron then the **total volume occupied** by all  $n_V N + 1 = 12N + 1$  snugly fitted spheres (including the largest inscribed sphere) is given from Eq(25) as follows

$$\begin{aligned} (V_{packed})_N &= \frac{4\pi}{3} R_i^3 \left\{ 1 + n_V K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{4\pi}{3} \left( \frac{a(3 + \sqrt{5})}{4\sqrt{3}} \right)^3 \left\{ 1 + (12)K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \\ &= \frac{\pi a^3 (72 + 32\sqrt{5})}{144\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} = \frac{\pi a^3 (9 + 4\sqrt{5})}{18\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \\ \therefore (V_{packed})_N &= \frac{\pi a^3 (9 + 4\sqrt{5})}{18\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \end{aligned}$$

$$\text{Where, packing constant, } K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \approx 0.114420648$$

**8.5.4.** The **total volume** ( $(V_{packed})_\infty$ ) packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 12$  identical vertices of a regular icosahedron (including the largest inscribed sphere) ( $N \rightarrow \infty$ ) is given from Eq(26) as follows

$$\begin{aligned} (V_{packed})_\infty &= \frac{4\pi}{3} R_i^3 \left( 1 + \frac{n_V K^3}{1 - K^3} \right) = \frac{4\pi}{3} \left( \frac{a(3 + \sqrt{5})}{4\sqrt{3}} \right)^3 \left( 1 + \frac{12K^3}{1 - K^3} \right) \\ &= \frac{\pi a^3 (72 + 32\sqrt{5})}{144\sqrt{3}} \left( \frac{1 - K^3 + 12K^3}{1 - K^3} \right) = \frac{\pi a^3 (9 + 4\sqrt{5})}{18\sqrt{3}} \left( \frac{1 + 11K^3}{1 - K^3} \right) \\ \therefore (V_{packed})_\infty &= \frac{\pi a^3 (9 + 4\sqrt{5})}{18\sqrt{3}} \left( \frac{1 + 11K^3}{1 - K^3} \right) \end{aligned}$$

**8.5.5. Packing ratio**  $(r_p)_N$  (i.e. the ratio of the total volume packed/occupied by all  $(12N + 1)$  no. of snugly fitted spheres to the volume of the regular icosahedron): The packing ratio  $(r_p)_N$  is given from Eq(27) as follows

$$\begin{aligned} (r_p)_N &= \frac{\text{total volume occupied by all } (12N + 1) \text{ no. of snugly fitted spheres}}{\text{volume of regular icosahedron}} = \frac{(V_{packed})_N}{V_s} \\ \Rightarrow (r_p)_N &= \frac{\frac{\pi a^3 (9 + 4\sqrt{5})}{18\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}}{\frac{5a^3 (3 + \sqrt{5})}{12}} = \frac{2\pi (9 + 4\sqrt{5})}{15\sqrt{3} (3 + \sqrt{5})} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \\ &= \frac{\pi (7 + 3\sqrt{5})}{30\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \\ \therefore (r_p)_N &= \frac{\pi (7 + 3\sqrt{5})}{30\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\} \end{aligned}$$

It is to be noted that the value of the packing ratio  $(r_p)_N$  depends only on  $N$  no. of spheres snugly fitted/packed in each of  $n_V = 12$  identical vertices of a regular icosahedron.

**8.5.6. Packing ratio  $(r_p)_\infty$**  (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all  $n_V = 12$  identical vertices of a regular icosahedron (including the largest inscribed sphere): The packing ratio  $((r_p)_\infty)$  is given from Eq(28) as follows

$$(r_p)_\infty = \frac{\text{total volume occupied by infinite no. of snugly fitted spheres}}{\text{volume of regular icosahedron}} = \frac{(V_{\text{packed}})_\infty}{V_s}$$

$$\Rightarrow (r_p)_\infty = \frac{\frac{\pi a^3(9 + 4\sqrt{5})}{18\sqrt{3}} \left( \frac{1 + 11K^3}{1 - K^3} \right)}{\frac{5a^3(3 + \sqrt{5})}{12}} = \frac{2\pi(9 + 4\sqrt{5})}{15\sqrt{3}(3 + \sqrt{5})} \left( \frac{1 + 11K^3}{1 - K^3} \right) = \frac{\pi(7 + 3\sqrt{5})}{30\sqrt{3}} \left( \frac{1 + 11K^3}{1 - K^3} \right)$$

$$\therefore (r_p)_\infty = \frac{\pi(7 + 3\sqrt{5})}{30\sqrt{3}} \left( \frac{1 + 11K^3}{1 - K^3} \right) \approx 0.843718586$$

Where, packing constant,  $K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \approx 0.114420648$

It is to be noted that the value of the packing ratio  $(r_p)_\infty$  is the maximum possible value which shows that approximate 84.37 % of the volume of any regular icosahedron can be packed by snugly fitting infinite no. of the spheres in each of its 12 identical vertices including the (volume of) largest inscribed sphere touching all 20 triangular faces.

**Conclusions:** The above generalised formulas are applicable to locate any sphere with a radius  $R$  resting in a vertex (corner) at which  $n$  no. of edges meet together at angle  $\alpha$  between any two consecutive of them such as the vertex of platonic solids, any of two identical & diagonally opposite vertices of uniform polyhedrons with congruent right kite faces & the vertex of right pyramid with regular  $n$ -gonal base. Hence, for given values of  $n$  no. of edges meeting at a vertex, an angle  $\alpha$  between any two consecutive edges meeting at the vertex & radius  $R$  of the sphere resting in the vertex, all the important parameters are calculated as tabulated below

1.	Distance of the centre of the sphere from the vertex of polyhedron	$\frac{R \tan \frac{\pi}{n}}{\tan \frac{\alpha}{2}}$
2.	Minimum distance of the sphere from the vertex of polyhedron	$\frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}}$
3.	Distance of the centre of the sphere from each of edges meeting at the vertex of polyhedron	$\frac{R \cos \frac{\alpha}{2}}{\cos \frac{\pi}{n}}$
4.	Minimum distance of the sphere from each of edges meeting at the vertex of polyhedron	$\frac{R \left( \cos \frac{\alpha}{2} - \cos \frac{\pi}{n} \right)}{\cos \frac{\pi}{n}}$
5.	Normal height (depth) through which the vertex is truncated to best fit the sphere	$\frac{R \left( \tan \frac{\pi}{n} - \tan \frac{\alpha}{2} \right)}{\tan \frac{\alpha}{2}}$

6.	Edge length through which the vertex is truncated to best fit the sphere	$R \tan \frac{\pi}{n} \operatorname{cosec} \frac{\alpha}{2} \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}}$
7.	Radius through which each face is filleted to best fit the sphere	$R \tan \frac{\pi}{n} \tan \left( \frac{\pi + \alpha}{4} \right) \sqrt{\frac{\sin \left( \frac{\pi}{n} - \frac{\alpha}{2} \right)}{\sin \left( \frac{\pi}{n} + \frac{\alpha}{2} \right)}}$

$$\forall \alpha < \frac{2\pi}{n} \quad \& \quad n \geq 3$$

These generalised formulae are applicable **to locate any sphere with a radius  $R$  resting in a vertex (corner)** at which  $n$  no. of edges meet together at angle  $\alpha$  between any two consecutive of edges. The resting sphere touches all  $n$  no. of faces meeting at that vertex but the sphere does not touch any of  $n$  no. of edges meeting at that vertex. Thus there is an equal minimum gap between sphere & each of the edges.

Let  $a$  be the **edge length** &  $N$  be the **no. of spheres snugly fitted/packed in each of the identical vertices** of the corresponding platonic solid then the important parameters are determined as tabulated below

Corresponding platonic solid	Radius of $N^{th}$ sphere in each vertex (excluding the largest one)	Total volume packed by all the spheres snugly fitted/packed in all the vertices of the corresponding platonic solid including the volume of the largest inscribed sphere	Packing ratio (ratio of the volume packed by all snugly fitted sphere to the volume of corresponding platonic solid)	% of max. packed volume
Regular tetrahedron	$\frac{a}{2\sqrt{6}} \left( \frac{1}{2} \right)^N$	$\frac{\pi a^3}{36\sqrt{6}} \left\{ 1 + \frac{4}{7} \left( \frac{2^{3N} - 1}{2^{3N}} \right) \right\}$	$\frac{\pi}{6\sqrt{3}} \left\{ 1 + \frac{4}{7} \left( \frac{2^{3N} - 1}{2^{3N}} \right) \right\}$	47.5 %
Regular hexahedron (cube)	$\frac{a}{2} (2 - \sqrt{3})^N$	$\frac{\pi a^3}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}$	$\frac{\pi}{6} \left\{ 1 + \frac{4}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}$	60.58 %
Regular octahedron	$\frac{a}{\sqrt{6}} (2 - \sqrt{3})^N$	$\frac{\pi a^3}{9} \sqrt{\frac{2}{3}} \left\{ 1 + \frac{3}{5} (3\sqrt{3} - 5) (1 - (2 - \sqrt{3})^{3N}) \right\}$	$\frac{\pi}{3\sqrt{3}} \left\{ 1 + \frac{3(3\sqrt{3} - 5)}{5} (1 - (2 - \sqrt{3})^{3N}) \right\}$	67.57 %
Regular dodecahedron	$\frac{a}{2} K^N \sqrt{\frac{25 + 11\sqrt{5}}{10}}$	$\frac{\pi a^3 \sqrt{6100 + 2728\sqrt{5}}}{60} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$	$\frac{\pi}{15} \sqrt{\frac{65 + 29\sqrt{5}}{10}} \left\{ 1 + 20K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$	77.73 %
Regular icosahedron	$\frac{aK^N(3 + \sqrt{5})}{4\sqrt{3}}$	$\frac{\pi a^3(9 + 4\sqrt{5})}{18\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$	$\frac{\pi(7 + 3\sqrt{5})}{30\sqrt{3}} \left\{ 1 + 12K^3 \left( \frac{1 - K^{3N}}{1 - K^3} \right) \right\}$	84.37 %

Where, **packing constant**,  $K = \frac{11 + 3\sqrt{5} - \sqrt{150 + 66\sqrt{5}}}{4} \approx 0.114420648$

**Note:** Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

M.M.M. University of Technology, Gorakhpur-273010 (UP) India

6 June, 2015

Email: [rajpootherishchandra@gmail.com](mailto:rajpootherishchandra@gmail.com)

Author's Home Page: <https://notionpress.com/author/HarishChandraRajpoot>

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