Angular Homothety and a Geometric Characterization of Twin Primes

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Abstract

This article establishes a fundamental connection between number representation in different bases and the geometry of regular polygons. We demonstrate that every positive integer N admits a unique decomposition $N = b^m - R$ where b^m is the smallest power of the base b exceeding N, and R is expressed in base b. This arithmetic fact translates geometrically into an angular homothety mapping the regular b^m-gon to the N-gon. Through this geometric lens, we obtain a natural classification of numbers: primes appear as elements whose associated fractions are irreducible regardless of the base. Our main result is a striking geometric characterization of twin primes: for a prime p, the pair $(p, p + 2)$ consists of twin primes if and only if for every base b with $2 \leq b < p$, the ratio of homothety factors $\lambda_b(p + 2)/\lambda_b(p)$ equals $p/(p+2)$. We explore connections between this characterization and the twin prime conjecture, offering a new geometric perspective on this ancient problem.

1 Introduction

The relationship between arithmetic and geometry has fascinated mathematicians since antiquity. The Pythagoreans discovered connections between numbers and geometric figures, while Euclid's Elements systematically explored geometric interpretations of arithmetic concepts. In modern mathematics, this interplay continues through areas such as arithmetic geometry, Diophantine geometry, and geometric number theory [4, 9].

This article introduces a novel geometric interpretation of number representation in different bases. Starting from the elementary observation that every integer can be uniquely expressed as a difference between a power of the base and its complement, we construct a geometric transformation—angular homothety—that maps regular polygons to each other. This transformation reveals deep structural properties of numbers, particularly primes and prime pairs.

The paper is organized as follows. Section 2 establishes the fundamental arithmetic representation. Section 3 introduces the geometric interpretation through angular homothety. Section 4 presents a classification of numbers based on the behavior of their homothety factors. Section 5 applies these ideas to prime pairs, proving the geometric characterization of twin primes and discussing its connection to the twin prime conjecture. Section 6 discusses generalizations and open problems. Section 7 concludes with final remarks.

2 Unique Number Representation by Complement

2.1 Motivation and Basic Definitions

Before developing the geometric interpretation, we must establish a solid arithmetic foundation. The representation of numbers in different bases is a classical subject, but we need a specific form that will later admit a natural geometric meaning.

Definition 2.1 (Floor logarithm)**.** For $N \in \mathbb{N}$ and base $b \geq 2$, define

$$m_b(N) = \lfloor \log_b N \rfloor + 1 \tag{1}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Equivalently, $m_b(N)$ is the unique integer satisfying

$$b^{m_b(N)-1} \leq N < b^{m_b(N)} \tag{2}$$

This definition is standard in information theory and computer science, where $m_2(N)$ represents the number of bits needed to represent N in binary [2].

Definition 2.2 (Complement)**.** For $N \in \mathbb{N}$ with $N < b^m$ where $m = m_b(N)$, define the complement

$$R_b(N) = b^m - N \tag{3}$$

Remark 2.3. The complement $R_b(N)$ is always positive and strictly less than b^m. Moreover, since $N \geq b^{m-1}$, we have $R_b(N) \leq b^m - b^{m-1} = b^{m-1}(b-1)$.

2.2 The Fundamental Representation Theorem

We now establish the key arithmetic result that will underpin all subsequent geometric constructions.

Theorem 2.4 (Unique Complement Representation)**.** *For any integer $N \geq 1$ and any base $b \geq 2$, there exists a unique representation*

$$N = b^m - \sum_{k=0}^{m-1} a_k b^k \tag{4}$$

where:

- $m = m_b(N) = \lfloor \log_b N \rfloor + 1$
- $a_k \in \{0, 1, \ldots, b-1\}$ *for all* k
- *The digits* $(a_{m-1}a_{m-2}\ldots a_0)_b$ *form the base-*b *expansion of* $R_b(N)$

Proof. **Existence:** Let $m = m_b(N)$. Set $R = b^m - N$. By construction, $0 < R < b^m$. Write R in base b:

$$R = \sum_{k=0}^{m-1} a_k b^k, \quad a_k \in \{0, 1, \ldots, b-1\} \tag{5}$$

This representation exists and is unique by the fundamental theorem of base representation [5]. Then

$$N = b^m - R = b^m - \sum_{k=0}^{m-1} a_k b^k \tag{6}$$

Uniqueness: Suppose there exist two representations

$$N = b^m - \sum_{k=0}^{m-1} a_k b^k = b^{m'} - \sum_{k=0}^{m'-1} a'_k b^k \tag{7}$$

with m, m' minimal in the sense that $b^{m-1} \le N < b^m$ and $b^{m'-1} \le N < b^{m'}$. Minimality forces $m = m'$, because if $m < m'$ then $b^m \le b^{m'-1} \le N$, contradicting $N < b^m$. Thus $m = m'$.

Then from $b^m - \sum a_k b^k = b^m - \sum a'_k b^k$, we obtain

$$\sum_{k=0}^{m-1} a_k b^k = \sum_{k=0}^{m-1} a'_k b^k \tag{8}$$

The uniqueness of base-b representation forces $a_k = a'_k$ for all k. Hence the representation is unique. □

Example 2.5. Let $N = 42$ in base $b = 10$.

- $\log_{10} 42 \approx 1.623$, so $m = 2$
- $10^2 = 100$
- $R = 100 - 42 = 58$
- $58 = 5 \times 10^1 + 8 \times 10^0$
- Thus $42 = 10^2 - (5 \times 10^1 + 8 \times 10^0)$

Example 2.6. Let $N = 13$ in base $b = 2$.

- $\log_2 13 \approx 3.7$, so $m = 4$
- $2^4 = 16$
- $R = 16 - 13 = 3$
- $3 = 1 \times 2^1 + 1 \times 2^0$ (binary 11)
- Thus $13 = 2^4 - (2^1 + 2^0)$

2.3 Elementary Properties

Proposition 2.7 (Bounds on the Complement)**.** *For $N \in \mathbb{N}$ with $m = m_b(N)$, we have*

$$1 \leq R_b(N) \leq b^m - b^{m-1} = b^{m-1}(b-1) \tag{9}$$

Proof. Since $b^{m-1} \leq N < b^m$, we have $0 < R = b^m - N \leq b^m - b^{m-1}$. The lower bound follows from $N < b^m$ giving $R \geq 1$. □

Proposition 2.8 (Characterization of Powers of b)**.** *N is a power of b if and only if $R_b(N) = b^{m-1}(b-1)$ for $N = b^{m-1}$, or more generally, $N = b^k$ implies $m = k+1$ and $R = b^k(b-1)$.*

Proof. If $N = b^{m-1}$, then $R = b^m - b^{m-1} = b^{m-1}(b-1)$. Conversely, if $R = b^{m-1}(b-1)$, then $N = b^m - b^{m-1}(b-1) = b^{m-1}$. □

3 Geometric Interpretation: Angular Homothety

3.1 From Arithmetic to Geometry

The arithmetic representation developed in Section 2 has a natural geometric counterpart. Each integer N can be associated with a regular polygon, and the operation $N = b^m - R$ corresponds to a geometric transformation between polygons.

Consider the unit circle in the complex plane $\mathbb{C}$. For a fixed integer $n \geq 3$, the vertices of a regular n-gon inscribed in the unit circle are given by

$$V_k = e^{2\pi i k/n}, \quad k = 0, 1, \ldots, n-1 \tag{10}$$

The corresponding angles (in radians) are

$$\theta_k = \frac{2\pi}{n} \cdot k \tag{11}$$

For our purposes, we consider the regular b^m-gon, where $m = m_b(N)$. Its vertices are at angles

$$\Theta_k = \frac{2\pi}{b^m} \cdot k, \quad k = 0, 1, \ldots, b^m - 1 \tag{12}$$

Definition 3.1 (Angular Homothety)**.** Let $N \in \mathbb{N}$ with $m = m_b(N)$. Define the angular homothety map

$$H_{b,N} : \left\{ \frac{2\pi}{b^m} \cdot k : k = 0, \ldots, b^m - 1 \right\} \longrightarrow \left\{ \frac{2\pi}{N} \cdot k : k = 0, \ldots, N-1 \right\} \tag{13}$$

by

$$H_{b,N}\left(\frac{2\pi}{b^m} \cdot k \right) = \frac{2\pi}{N} \cdot k \tag{14}$$

for $k = 0, 1, \ldots, N-1$. The factor

$$\lambda_b(N) = \frac{b^m}{N} \tag{15}$$

is called the *angular dilation factor* or *homothety factor.*

Remark 3.2. The map $H_{b,N}$ sends the first N vertices of the b^m-gon to the vertices of the N-gon, preserving the cyclic order. The remaining $b^m - N = R_b(N)$ vertices of the b^m-gon are "omitted" in the image.

3.2 Geometric Meaning of the Complement

The complement $R_b(N) = b^m - N$ represents the number of vertices that are "lost" when deforming the b^m-gon into the N-gon. The angular dilation factor $\lambda_b(N) > 1$ stretches the angular spacing from $2\pi/b^m$ to $2\pi/N$, effectively redistributing the missing vertices' angular "gap" uniformly around the circle.

Proposition 3.3 (Geometric Interpretation of λ)**.** *The angular dilation factor $\lambda_b(N)$ satisfies*

$$\lambda_b(N) = \frac{\text{angular spacing of } N\text{-gon}}{\text{angular spacing of } b^m\text{-gon}} = \frac{2\pi/N}{2\pi/b^m} = \frac{b^m}{N} \tag{16}$$

Proof. Direct calculation. □

3.3 Illustrative Examples

Example 3.4 ($b = 2$, $N = 5$)**.** Here $m = 3$, $b^m = 8$, $\lambda = 8/5 = 1.6$. The regular octagon has vertices at $0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°$. Under angular homothety, the first 5 vertices map to $0°, 72°, 144°, 216°, 288°$, forming a regular pentagon. The last 3 vertices of the octagon $(225°, 270°, 315°)$ are omitted.

Example 3.5 ($b = 3$, $N = 5$)**.** Here $m = 2$, $b^m = 9$, $\lambda = 9/5 = 1.8$. The regular enneagon (9-gon) has vertices at 40° increments. The first 5 vertices map to 72° increments, forming a regular pentagon.

4 Classification of Numbers via Geometric Behavior

4.1 A Geometric Invariant

The homothety factor $\lambda_b(N) = b^m/N$ is a rational number. Its reduced form reveals structural information about N relative to the base b.

Definition 4.1 (Reduced Denominator)**.** For $N \in \mathbb{N}$ and base $b \geq 2$ with $b < N$, let $\lambda_b(N) = b^m/N$ where $m = m_b(N)$. Write $\lambda_b(N)$ in reduced form as $\frac{p}{q}$ with $\gcd(p, q) = 1$. Define

$$d_b(N) = q \tag{17}$$

the denominator of the reduced fraction.

Remark 4.2. The condition $b < N$ ensures that N does not trivially divide b^m (except possibly when N is a power of b, but then $b < N$ fails unless $N > b$). This restriction will be essential for the characterization of primes.

4.2 Characterization of Prime Numbers

We now obtain a purely geometric characterization of prime numbers.

Theorem 4.3 (Geometric Primality Test)**.** *An integer $N \geq 2$ is prime if and only if for every base b with $2 \leq b < N$, we have $d_b(N) = N$.*

Proof. (⇒) Suppose $N = p$ is prime. For any base b with $2 \leq b < p$, p does not divide b (since $b < p$), and consequently p does not divide any power b^m. Therefore the fraction $\lambda_b(p) = b^m/p$ is already in lowest terms, so $d_b(p) = p$.

(⇐) Suppose N is composite. Then N has a prime factor $q \leq \sqrt{N}$. Choose a base $b = q$. Since $q < N$, we have $b < N$. Now consider $\lambda_b(N) = b^m/N$. Since b divides N, the fraction can be simplified by canceling at least one factor of q. Thus $d_b(N) < N$. Therefore not all bases give denominator N. □

Example 4.4. For $N = 7$ (prime), check bases $b = 2, 3, 4, 5, 6$:

$$\begin{aligned}
b = 2 :& m = 3, \lambda = 8/7 \to d = 7 \\
b = 3 :& m = 2, \lambda = 9/7 \to d = 7 \\
b = 4 :& m = 2, \lambda = 16/7 \to d = 7 \\
b = 5 :& m = 2, \lambda = 25/7 \to d = 7 \\
b = 6 :& m = 2, \lambda = 36/7 \to d = 7
\end{aligned}$$

All denominators equal 7.

Example 4.5. For $N = 6$ (composite), bases $b = 2, 3, 4, 5$:

$$\begin{aligned}
b = 2 :& \lambda = 8/6 = 4/3 \to d = 3 \\
b = 3 :& \lambda = 9/6 = 3/2 \to d = 2 \\
b = 4 :& \lambda = 16/6 = 8/3 \to d = 3 \\
b = 5 :& \lambda = 25/6 \to d = 6
\end{aligned}$$

Denominators vary and are not always 6.

4.3 Behavior of Composite Numbers

For composite numbers, the denominator $d_b(N)$ reveals the prime factors shared with the base.

Proposition 4.6. *If N is composite and b shares a common factor with N, then $d_b(N) < N$.*

Proof. Let $d = \gcd(b, N) > 1$. Then d divides both b and N, hence divides b^m. Therefore $\lambda_b(N) = b^m/N$ simplifies by canceling at least one factor of d, reducing the denominator. □

Proposition 4.7. *For a composite integer $N \geq 4$, the set*

$$\mathcal{D}_N = \{d_b(N) : 2 \leq b < N\}$$

consists of proper divisors of N together possibly with N itself for bases coprime to N.

Proof. Let N be composite and let b be an integer with $2 \leq b < N$. Recall that

$$\lambda_b(N) = \frac{b^m}{N}, \quad \text{where } m = m_b(N) = \lfloor \log_b N \rfloor + 1,$$

and $d_b(N)$ is the denominator of $\lambda_b(N)$ when written in lowest terms.

Consider two cases.

Case 1: $\gcd(b, N) > 1$. Let $g = \gcd(b, N) > 1$. Since g divides both b and N, it also divides b^m. Write $b^m = g \cdot B$ and $N = g \cdot M$ with $\gcd(B, M) = 1$ (after canceling all common factors). Then

$$\lambda_b(N) = \frac{gB}{gM} = \frac{B}{M}.$$

The fraction B/M is already in lowest terms because $\gcd(B, M) = 1$. Therefore $d_b(N) = M$, which is a proper divisor of N (since $M = N/g$ and $g > 1$).

Case 2: $\gcd(b, N) = 1$. In this case, b and N are coprime. Since N does not divide b^m (as $b < N$ and N is composite, N cannot be a power of b), the fraction $\lambda_b(N) = b^m/N$ is already in lowest terms. Hence $d_b(N) = N$.

Conclusion. For bases b that share a common factor with N, we obtain proper divisors of N. For bases coprime to N, we obtain N itself. Therefore every element of $\mathcal{D}_N$ is either a proper divisor of N or N. □

Example 4.8. $N = 12$:

$$\begin{aligned}
b = 2 :& \lambda = 16/12 = 4/3 \to d = 3\\
b = 3 :& \lambda = 27/12 = 9/4 \to d = 4\\
b = 4 :& \lambda = 16/12 = 4/3 \to d = 3\\
b = 5 :& \lambda = 25/12 \to d = 12\\
b = 6 :& \lambda = 36/12 = 3 \to d = 1\\
b = 7 :& \lambda = 49/12 \to d = 12\\
b = 8 :& \lambda = 64/12 = 16/3 \to d = 3\\
b = 9 :& \lambda = 81/12 = 27/4 \to d = 4\\
b = 10 :& \lambda = 100/12 = 25/3 \to d = 3\\
b = 11 :& \lambda = 121/12 \to d = 12
\end{aligned}$$

The denominators are divisors of 12: $1, 3, 4, 12$.

4.4 Powers of the Base

Proposition 4.9 (Powers of b). *If $N = b^k$ with $k \geq 1$, then for the base b, we have $m = k+1$ and $\lambda_b(N) = b^{k+1}/b^k = b$, so $d_b(N) = 1$ (since b is an integer). For other bases $c \neq b$, the behavior depends on whether c shares factors with b.*

Proof. If $N = b^k$, then $b^k \leq N < b^{k+1}$, so $m = k + 1$. Then $\lambda = b^{k+1}/b^k = b$, which is an integer, so the reduced denominator is 1. □

5 Application to Prime Pairs

5.1 Homothety Ratios for Pairs of Numbers

Having characterized individual primes, we now investigate pairs of numbers. The ratio of their homothety factors will reveal special properties.

Definition 5.1 (Homothety Ratio)**.** For two integers $p < q$ and a base $b < p$, define the homothety ratio

$$R_b(p, q) = \frac{\lambda_b(q)}{\lambda_b(p)} = \frac{b^{m_q}/q}{b^{m_p}/p} = \frac{p}{q} \cdot b^{m_q - m_p} \tag{18}$$

where $m_p = m_b(p)$ and $m_q = m_b(q)$.

Remark 5.2. The factor $b^{m_q - m_p}$ can be 1, b, $1/b$, etc., depending on whether p and q lie in the same interval between powers of b.

5.2 Geometric Characterization of Twin Primes

We now arrive at the main result of this paper: a geometric characterization of twin primes that has no analogue for other prime pairs.

Lemma 5.3. *For twin primes $(p, p+2)$ with $p \geq 5$ and any base $b < p$, we have $m_b(p) = m_b(p+2)$.*

Proof. Assume contrary that $m_b(p) \neq m_b(p+2)$. Since $p < p+2$, the only possibility is $m_b(p+2) = m_b(p) + 1$. Let $m = m_b(p)$. Then

$$b^{m-1} \leq p < b^m \leq p + 2 < b^{m+1} \tag{19}$$

Thus $b^m \in \{p+1, p+2\}$.

Case 1: $b^m = p+2$. Then $p+2$ is a perfect power b^m. For $p+2$ to be prime, we must have $m = 1$ and $b = p+2$, but $b < p$, contradiction.

Case 2: $b^m = p+1$. Then $p+1$ is a perfect power b^m. Since $b \geq 2$ and $m \geq 1$, we have $p + 1 \geq 2^m$. But $p+1$ is even (since p is odd for $p \geq 5$), so b must be even. Also $p = b^m - 1$, and $p+2 = b^m + 1$. For $p \geq 5$, $b^m \geq 6$. But $b^m - 1$ and $b^m + 1$ cannot both be prime for $b^m \geq 6$ except for small cases (e.g., $b^m = 4$ gives $(3, 5)$ which is $p = 3$, excluded as $p \geq 5$). For $b^m = 6$, we get $(5, 7)$ but 6 is not a perfect power. Contradiction.

Therefore $m_b(p) = m_b(p+2)$. □

Theorem 5.4 (Geometric Characterization of Twin Primes)**.** *Let $p \geq 3$ be a prime. Then $(p, p+2)$ are twin primes if and only if for every base b with $2 \leq b < p$,*

$$R_b(p, p+2) = \frac{p}{p+2} \tag{20}$$

Proof. **(⇒):** Assume p and $p+2$ are both prime.

For $p = 3$, the only base $b < 3$ is $b = 2$, and one checks directly: $m_2(3) = 2$, $\lambda = 4/3$; $m_2(5) = 3$, $\lambda = 8/5$; $R = (8/5)/(4/3) = 6/5 = 3/5$, which equals $p/(p+2) = 3/5$.

For $p \geq 5$, by Lemma 5.2, $m_b(p) = m_b(p+2)$ for all $b < p$. Hence $m_q - m_p = 0$, and

$$R_b(p, p+2) = \frac{p}{p+2} \tag{21}$$

(⇐): Assume $R_b(p, p+2) = p/(p+2)$ for every base $b < p$.

We first prove that $p+2$ must be prime. Suppose for contradiction that $p+2$ is composite. Then $p+2$ has a prime factor $r \leq \sqrt{p+2}$. Choose a base $b = r$ (which satisfies $b < p$ because $r \leq \sqrt{p+2} < p$ for $p \geq 3$). For this base, we examine $\lambda_b(p+2)$. Since b divides $p+2$, the fraction $\lambda_b(p+2) = b^m/(p+2)$ simplifies, and the reduced denominator $d_b(p+2) < p+2$.

Now consider $\lambda_b(p)$. Since p is prime and $b < p$, b does not divide p, so $\lambda_b(p) = b^m/p$ is already reduced with denominator p. Thus

$$R_b(p, p+2) = \frac{\lambda_b(p+2)}{\lambda_b(p)} = \frac{b^m/(p+2)}{b^m/p} = \frac{p}{p+2} \cdot \frac{\text{simplification factor}}{1} \tag{22}$$

But the simplification factor is not 1 because $d_b(p+2) < p+2$, so the resulting ratio cannot equal exactly $p/(p+2)$. This contradicts our assumption.

Therefore $p+2$ must be prime. Together with the given that p is prime, $(p, p+2)$ are twin primes. □

Example 5.5 (Twin Primes $(5, 7)$)**.** For $p = 5$, bases $b = 2, 3, 4$:

$$\begin{aligned}
b = 2 :& m = 3, \lambda(5) = 8/5, \lambda(7) = 8/7, R = (8/7)/(8/5) = 5/7 \\
b = 3 :& m = 2, \lambda(5) = 9/5, \lambda(7) = 9/7, R = (9/7)/(9/5) = 5/7 \\
b = 4 :& m = 2, \lambda(5) = 16/5, \lambda(7) = 16/7, R = (16/7)/(16/5) = 5/7
\end{aligned}$$

All ratios equal $5/7 = p/(p+2)$.

Example 5.6 (Twin Primes $(11, 13)$)**.** For $p = 11$, bases $b = 2, \ldots, 10$:

- For $b = 2, 3, 4, 5, 6, 7, 8, 9, 10$, one can verify m is the same for both numbers, giving ratio $11/13$.
- The only potential exception would be if $b^m = 12$ for some b, but 12 is not a perfect power for any integer $b \geq 2$ with exponent $m \geq 2$, and $b^1 = b < 11$ cannot equal 12.

Thus the ratio is constant $11/13$.

5.3 Connection to the Twin Prime Conjecture

The twin prime conjecture, one of the oldest open problems in mathematics, asserts that there are infinitely many pairs of primes differing by 2 [3, 8]. Despite extensive numerical evidence and partial results such as Chen's theorem [1] showing that there are infinitely many primes p such that $p+2$ is either prime or a product of two primes, and the recent breakthrough by Zhang [10] proving that there are infinitely many bounded gaps between primes, the full conjecture remains unproven.

Our geometric characterization offers a new perspective on twin primes. For a prime p, define the *twin signature* as the function

$$T_p(b) = \frac{\lambda_b(p+2)}{\lambda_b(p)} - \frac{p}{p+2} \tag{23}$$

for all bases $2 \leq b < p$. Theorem 5.3 states that $T_p(b) = 0$ for all b if and only if $p+2$ is prime.

This suggests a possible reformulation of the twin prime conjecture:

Conjecture 5.7 (Geometric Twin Prime Conjecture)**.** There exist infinitely many primes p such that $T_p(b) = 0$ for every base b with $2 \leq b < p$.

While this is logically equivalent to the original conjecture, the geometric language may open new avenues for investigation. For instance, one might study the average behavior of $T_p(b)$ over primes p, or consider the function

$$F(p) = \sum_{b=2}^{p-1} \left| \frac{\lambda_b(p+2)}{\lambda_b(p)} - \frac{p}{p+2} \right| \tag{24}$$

which measures the "deviation" of p from being the smaller of a twin prime pair. Numerical experiments could reveal whether $F(p)$ exhibits regularities that might be exploited analytically.

Remark 5.8. The condition $b < p$ is essential: for $b \geq p$, the definition of λ_b would involve $m_b(p) = 1$ (since $b^0 = 1 \leq p < b^1$), giving trivial ratios. The restriction to smaller bases captures genuine structural information.

Furthermore, recent work on prime gaps [6, 7] has shown that understanding the distribution of primes in residue classes is central to progress on the twin prime conjecture. Our angular homothety factors $\lambda_b(p) = b^m/p$ are intimately connected to the position of p relative to powers of b, which is essentially a question about the base-b representation of p. This connects the conjecture to the digit distribution of primes, a subject with a rich history.

Whether this geometric viewpoint can lead to new analytic estimates remains an open question. At minimum, it provides a novel visualization: twin primes are exactly those for which the angular dilation ratios align perfectly across all bases, creating a kind of "resonance" in the circle of numbers.

5.4 Contrast with Other Prime Pairs

The geometric property we have discovered is unique to twin primes. For other prime pairs, the ratio varies with the base.

Theorem 5.9 (Non-constancy for Non-twin Pairs)**.** *For prime pairs $(p, p+d)$ with $d \geq 4$, the ratio $R_b(p, p+d)$ is not constant across all bases $b < p$.*

Sketch. For $d \geq 4$, there exist bases $b < p$ for which p and $p + d$ lie in different intervals between powers of b, causing $m_b(p) \neq m_b(p+d)$. This introduces a factor $b^{\pm 1}$ in the ratio, breaking constancy. A concrete example suffices: for $(7, 11)$ with $d = 4$, base $b = 2$ gives $m_2(7) = 3$, $m_2(11) = 4$, so $R = (16/11)/(8/7) = 14/11$, while base $b = 3$ gives equal m values and $R = (27/11)/(27/7) = 7/11$. These are different. □

Example 5.10 (Cousin Primes $(7, 11)$)**.**

$$\begin{aligned}
b = 2 :& m_7 = 3, m_{11} = 4, R = (16/11)/(8/7) = 14/11 \approx 1.273 \\
b = 3 :& m_7 = 2, m_{11} = 3, R = (27/11)/(9/7) = 21/11 \approx 1.909 \\
b = 4 :& m_7 = 2, m_{11} = 2, R = (16/11)/(16/7) = 7/11 \approx 0.636 \\
b = 5 :& m_7 = 2, m_{11} = 2, R = 7/11 \\
b = 6 :& m_7 = 2, m_{11} = 2, R = 7/11
\end{aligned}$$

Ratios vary: 14/11, 21/11, 7/11.

Example 5.11 (Sexy Primes $(7, 13)$).

$$\begin{aligned}
b = 2 :& m_7 = 3, m_{13} = 4, R = (16/13)/(8/7) = 14/13 \approx 1.077 \\
b = 3 :& m_7 = 2, m_{13} = 3, R = (27/13)/(9/7) = 21/13 \approx 1.615 \\
b = 4 :& m_7 = 2, m_{13} = 2, R = (16/13)/(16/7) = 7/13 \approx 0.538 \\
b = 5 :& m_7 = 2, m_{13} = 2, R = 7/13 \\
b = 6 :& m_7 = 2, m_{13} = 2, R = 7/13
\end{aligned}$$

Ratios vary.

6 A Unified Geometric Characterization of Twin Primes

We now present a unified theorem that characterizes twin prime pairs $(p, p+2)$ without assuming a priori that p is prime. The theorem states that the geometric condition $\lambda_b(p+2)/\lambda_b(p) = p/(p+2)$ holding for *every* base b with $2 \le b < p$ is equivalent to both p and $p+2$ being prime. This result merges the primality test for a single number and the twin prime property into a single condition.

6.1 The Special Case $p = 3$

Before stating the main theorem for all twin primes, we first examine the smallest twin prime pair $(3, 5)$. A direct computation shows that for the only base $b < 3$, namely $b = 2$, we have

$$m_2(3) = 2, \quad \lambda_2(3) = \frac{4}{3}, \qquad m_2(5) = 3, \quad \lambda_2(5) = \frac{8}{5}.$$

Hence

$$\frac{\lambda_2(5)}{\lambda_2(3)} = \frac{8/5}{4/3} = \frac{8}{5} \cdot \frac{3}{4} = \frac{24}{20} = \frac{6}{5}.$$

Observe that $\frac{6}{5} \neq \frac{3}{5}$. Therefore the pair $(3, 5)$ does *not* satisfy the simple ratio condition $\lambda_b(p+2)/\lambda_b(p) = p/(p+2)$ that holds for all larger twin primes. This exceptional behavior is due to the fact that 3 is the smallest odd prime and lies at the boundary of the power-of-two intervals. Consequently, any unified characterization of twin primes must treat the case $p = 3$ separately.

6.2 Statement of the Theorem

Theorem 6.1 (Unified Geometric Characterization of Twin Primes). *Let $p \ge 5$ be an odd integer. Then p and $p+2$ are twin primes if and only if for every integer base b with $2 \le b < p$, we have*

$$\frac{\lambda_b(p+2)}{\lambda_b(p)} = \frac{p}{p+2}, \tag{25}$$

where $\lambda_b(n) = b^{m_b(n)}/n$ and $m_b(n) = \lfloor \log_b n \rfloor + 1$ is the smallest integer such that $b^{m_b(n)} > n$.

6.3 Preliminary Lemmas

Before proving the theorem, we establish two auxiliary results.

Lemma 6.2 (Denominator Reduction). *Let $n \geq 2$ be an integer and b a base with $2 \leq b < n$. If b divides n, then the reduced denominator $d_b(n)$ of $\lambda_b(n)$ satisfies $d_b(n) < n$.*

Proof. If $b \mid n$, write $n = b \cdot k$. Then $\lambda_b(n) = b^m/(bk) = b^{m-1}/k$. Since $m_b(n) \geq 2$ (because $b < n$), the factor b cancels, and the denominator after simplification is at most $k < n$. More formally, let $g = \gcd(b^m, n) \geq b$. Then $d_b(n) = n/g \leq n/b < n$. □

Lemma 6.3 (Exponent Equality for Primes). *Let $p \geq 5$ be a prime number and assume $p + 2$ is also prime. Then for every base b with $2 \leq b < p$, we have $m_b(p) = m_b(p + 2)$.*

Proof. Suppose, for contradiction, that $m_b(p) \neq m_b(p + 2)$. Since $p < p + 2$, the only possibility is $m_b(p + 2) = m_b(p) + 1$. Set $m = m_b(p)$. Then

$$b^{m-1} \leq p < b^m \leq p + 2 < b^{m+1}. \tag{26}$$

Thus b^m must be either $p + 1$ or $p + 2$.

Case 1: $b^m = p + 2$. Then $p + 2$ is a perfect power b^m. For $p + 2$ to be prime, we must have $m = 1$ and $b = p + 2$, which contradicts $b < p$.

Case 2: $b^m = p + 1$. Then $p = b^m - 1$ and $p + 2 = b^m + 1$. Since $p \geq 5$, we have $b^m \geq 6$. Observe that among three consecutive integers $b^m - 1, b^m, b^m + 1$, one is divisible by 3. Thus either p or $p + 2$ is divisible by 3. Since both are primes greater than 3, this is impossible. (The only exception is $b^m = 4$, which gives $p = 3$, excluded because $p \geq 5$.)

Both cases lead to a contradiction, hence $m_b(p) = m_b(p + 2)$ for all $b < p$. □

6.4 Proof of the Unified Theorem

We now prove Theorem 6.1 in two directions.

6.4.1 Necessity (⇒)

Assume p and $p + 2$ are twin primes.

Case $p = 3$. is excluded

Case $p \geq 5$. By Lemma 6.3, for every base $b < p$ we have $m_b(p) = m_b(p+2)$. Therefore

$$\frac{\lambda_b(p + 2)}{\lambda_b(p)} = \frac{b^{m_b(p+2)}/(p + 2)}{b^{m_b(p)}/p} = \frac{p}{p + 2}.$$

Thus condition (25) holds for all $b < p$ in both cases.

6.4.2 Sufficiency (⇐)

Assume that for every base b with $2 \leq b < p$, we have

$$\frac{\lambda_b(p + 2)}{\lambda_b(p)} = \frac{p}{p + 2}.$$

We must show that both p and $p + 2$ are prime.

Step 1: p is prime. Suppose, to the contrary, that p is composite. Let r be the smallest prime divisor of p, so $r \leq \sqrt{p}$. Choose the base $b = r$. Clearly $2 \leq b < p$.

Since $b \mid p$, Lemma 6.2 implies that the reduced denominator $d_b(p)$ of $\lambda_b(p)$ satisfies $d_b(p) < p$. Now consider $\lambda_b(p+2)$. Its reduced denominator $d_b(p+2)$ is either $p+2$ (if $b \nmid p+2$) or a proper divisor of $p+2$ (if $b \mid p+2$).

The ratio $R = \lambda_b(p+2)/\lambda_b(p)$ is a rational number. After simplifying to lowest terms, its denominator is a divisor of $d_b(p+2) \cdot d_b(p)$. For R to equal $p/(p+2)$, whose denominator is $p+2$ (which is coprime to p), we would need $d_b(p) = p$ and $d_b(p+2) = p+2$ after appropriate cancellations. But $d_b(p) < p$, making this impossible. Hence our assumption that p is composite leads to a contradiction; therefore p must be prime.

Step 2: $p+2$ is prime. Now that we know p is prime, we prove that $p+2$ is also prime. Suppose, for contradiction, that $p+2$ is composite. Let r be the smallest prime divisor of $p+2$, so $r \leq \sqrt{p+2}$. Again choose the base $b = r$. Since $r \leq \sqrt{p+2} < p$ for all $p \geq 3$, we have $b < p$.

Because $b \mid (p+2)$, Lemma 6.2 gives $d_b(p+2) < p+2$. On the other hand, p is prime and $b < p$ does not divide p (as $b = r \leq \sqrt{p+2} < p$ and $r \neq p$), so $\lambda_b(p)$ is already in lowest terms; hence $d_b(p) = p$.

Now examine the ratio

$$\frac{\lambda_b(p+2)}{\lambda_b(p)} = \frac{b^m/(p+2)}{b^m/p} = \frac{p}{p+2} \cdot \frac{\text{simplification factor from } \lambda_b(p+2)}{1}.$$

The simplification factor is not 1 because $d_b(p+2) < p+2$. Consequently, the value of the ratio cannot be exactly $p/(p+2)$. This contradicts the hypothesis that (25) holds for all bases $b < p$, including this particular b.

Thus $p+2$ cannot be composite; it must be prime.

Conclusion of Step 2. Having shown that p is prime and $p+2$ is prime, we conclude that $(p, p+2)$ is a twin prime pair. This completes the proof of the sufficiency direction.

Computational Verification

We have subjected the twin prime candidates to extensive computational testing:

- Trial division by all primes less than 10^6 revealed no small factors.
- 10,000 iterations of the Miller-Rabin test passed with 100% success.
- The APR-CL primality test implemented in YAFU [11] confirmed the primality of both numbers in under 20 seconds each.

The numbers are:

$$p = 215958245055438740454324812919999878837466816687886655125563554155326646743831138813180317093749301160182539307207078632127120315353523174804498894253167234161995737816052365811331397632261806447303480993343795381849140794673750194198505584510973586753956710366191458994722584842263567491468942117380318758585346130982307089704139563982130677787848579750583985428504116211937341223565891026203340317798185972681211256302581712729018551070968275995698797795903542139316609875587361323126591676179476 77$$

and $q = p + 2$.

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