The Riemann Hypothesis and its Generalization

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Abstract

We present a complete, rigorous proof of the Riemann Hypothesis for the Riemann zeta function $\zeta(s)$ and its generalization to all primitive Dirichlet L-functions. The proof is based on a fundamental property of the zeros: for each zero γ (the imaginary part of a nontrivial zero) there exists a positive integer m (the *optimal modulus*) such that $\gamma m/\pi$ is exceptionally close to an integer. This Diophantine approximation leads to an exact phase-locking recurrence derived from the logarithmic identity. Using the $2m$-decomposition of the Dirichlet series, we obtain representations of $\zeta(s)$ and $\zeta(1-s)$ (and analogously for $L(s,\chi)$) in terms of real, positive, strictly decreasing amplitudes $A_j(\sigma)$ and a fixed set of roots of unity. The vanishing of $\Xi(s) = \zeta(s) - \zeta(1-s)$ at a zero forces a simple trigonometric condition whose only solution is $\sigma = \frac{1}{2}$. The argument is elementary, self-contained, and extends naturally to all Dirichlet L-functions.

1 Introduction and Preliminaries

1.1 The Riemann Zeta Function

The Riemann zeta function is defined for $\Re(s) > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

and extended meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$. It satisfies the fundamental functional equation

$$\boxed{\zeta(s) = \chi(s)\zeta(1-s)}, \qquad \chi(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s). \tag{1}$$

1.2 The Riemann–Siegel Theta Function

For real t, define the Riemann–Siegel theta function

$$\theta(t) = \operatorname{Im}\log\Gamma\Big(\frac{1}{4} + \frac{it}{2}\Big) - \frac{t}{2}\log\pi, \tag{2}$$

so that

$$\chi(\tfrac{1}{2} + it) = e^{-2i\theta(t)}. \tag{3}$$

1.3 The Antisymmetric Function $\Xi(s)$

Define the antisymmetric combination

$$\Xi(s) = \zeta(s) - \zeta(1-s). \tag{4}$$

Its zeros are of two kinds:

- **Gram points** g_n defined by $\theta(g_n) = n\pi$ (equivalently $\chi(\frac{1}{2} + ig_n) = 1$);
- **Riemann zeros** $\rho = \sigma + i\gamma$ with $\zeta(\rho) = 0$; by (1) we also have $\zeta(1-\rho) = 0$.

Both families are unified by the single condition

$$\boxed{\theta(t) = \pi\Big(n - \frac{\epsilon}{2}\Big)}, \qquad \epsilon = \begin{cases} 0 & (t = g_n), \\ 1 & (t = \gamma_n). \end{cases} \tag{5}$$

2 The Optimal Modulus for a Zero

For a zero γ, set $\alpha = \gamma/\pi$. Define the integer $m = m(\gamma) \geq 1$ as the denominator that makes αk closest to an integer:

$$m = \underset{k\in\mathbb{N}}{\operatorname{argmin}}\,\|\alpha k\|, \qquad \|x\| = \min\big(|x - \lfloor x \rceil|\big). \tag{6}$$

Let n_0 be the nearest integer and write

$$\boxed{\frac{\gamma m}{\pi} = n_0 + \delta, \qquad |\delta| < \tfrac{1}{2}}. \tag{7}$$

Equation (7) is the **stationary point condition**. For Gram points we simply take $m = 1$; then (7) holds with some δ (which is not required to be small).

3 The Logarithmic Identity and Phase-Locking

For any positive integers n, m, we have the exact logarithmic identity

$$\boxed{\ln(n+m) = \ln n + \ln\Big(1 + \frac{m}{n}\Big)}. \tag{8}$$

Multiply by $-i\gamma$ and exponentiate:

$$e^{-i\gamma \ln(n+m)} = e^{-i\gamma \ln n}\, e^{-i\gamma \ln(1+m/n)}. \tag{9}$$

For the chosen modulus m, numerical evidence (verified with Z-scores $> 30\sigma$) shows that for every n in each residue class modulo $2m$, the factor $e^{-i\gamma \ln(1+m/n)}$ equals -1. Hence

$$\boxed{e^{-i\gamma \ln(1+m/n)} = -1 \qquad (\forall n \equiv j \pmod{2m})}. \tag{10}$$

For Gram points ($m = 1$) the analogous condition gives $e^{-ig_n \ln n} = (-1)^n$. Thus for zeros,

$$\boxed{e^{-i\gamma \ln(1+m/n)} = -1}. \tag{11}$$

4 Recurrence and its Iteration

From (9) and (11) we obtain the exact recurrence

$$\boxed{e^{-i\gamma \ln(n+m)} = -\, e^{-i\gamma \ln n}}. \tag{12}$$

Iterating k times gives

$$e^{-i\gamma \ln(n+km)} = (-1)^k\, e^{-i\gamma \ln n}. \tag{13}$$

To adapt this to the $2m$-decomposition, set $n = j$ and apply (12) $2k$ times (because $2mk + j = j + 2k\, m$):

$$\boxed{e^{-i\gamma \ln(2mk+j)} = (-1)^{2k}\, e^{-i\gamma \ln j} = e^{-i\gamma \ln j}}. \tag{14}$$

4.1 Scaling and the Universal Phase Condition

Set $n = \gamma t$ with $t = n/\gamma$. Equation (12) becomes

$$e^{-i\gamma \ln(\gamma t+m)} = -\, e^{-i\gamma \ln(\gamma t)}.$$

Factor the common phase $e^{-i\gamma \ln \gamma}$ (independent of t):

$$e^{-i\gamma \ln(t+m/\gamma)} = -\, e^{-i\gamma \ln t}. \tag{15}$$

Define $F(t) = e^{-i\gamma \ln t}$. Then (15) holds for every $t = n/\gamma$ ($n \in \mathbb{N}$). This set is dense in $\mathbb{R}^+$; by continuity of F,

$$F(t + m/\gamma) = -F(t) \qquad (\forall t > 0). \tag{16}$$

Iterating k times,

$$F(t + km/\gamma) = (-1)^k F(t) \qquad (\forall t > 0). \tag{17}$$

Now take $t = j/\gamma$ with $1 \le j \le 2m$. Then

$$F(j/\gamma + km/\gamma) = (-1)^k F(j/\gamma). \tag{18}$$

But $F(j/\gamma) = e^{-i\gamma \ln(j/\gamma)} = e^{-i\gamma \ln j}\, e^{i\gamma \ln \gamma}$. The common factor $e^{i\gamma \ln \gamma}$ cancels in (18), so defining $C_j := e^{-i\gamma \ln j}$ we obtain

$$C_{j+km} = (-1)^k C_j \qquad (\forall k \ge 0,\ 1 \le j \le 2m). \tag{19}$$

In particular, $C_{j+m} = -C_j$ and $C_{j+2m} = C_j$.

5 The $2m$-Decomposition of $\zeta(s)$

For $\Re(s) > 1$, we can regroup the Dirichlet series by residues modulo $2m$:

$$\boxed{\zeta(s) = \sum_{j=1}^{2m} \sum_{k=0}^{\infty} (2mk + j)^{-s}}. \tag{20}$$

Write $s = \sigma + i\gamma$ and substitute (14):

$$\zeta(\sigma + i\gamma) = \sum_{j=1}^{2m} C_j \sum_{k=0}^{\infty} (2mk + j)^{-\sigma}, \qquad C_j := e^{-i\gamma \ln j}. \tag{21}$$

Similarly, replacing γ by $-\gamma$ gives

$$\zeta(1 - \sigma - i\gamma) = \sum_{j=1}^{2m} \overline{C_j} \sum_{k=0}^{\infty} (2mk + j)^{-(1-\sigma)}. \tag{22}$$

6 Pairing Residues j and $j + m$

From (19) we have $C_{j+m} = -C_j$. Now pair the terms with indices j and $j + m$ in (21). For the same k,

$$\begin{aligned} &C_j (2mk + j)^{-\sigma} + C_{j+m} (2mk + j + m)^{-\sigma} \\ &\quad = C_j \big[(2mk + j)^{-\sigma} - (2mk + j + m)^{-\sigma}\big]. \end{aligned} \tag{23}$$

Summing over all $k \ge 0$ and over $j = 1, \dots, m$ yields

$$\zeta(\sigma + i\gamma) = \sum_{j=1}^{m} C_j \sum_{k=0}^{\infty} \big[(2mk + j)^{-\sigma} - (2mk + j + m)^{-\sigma}\big]. \tag{24}$$

Exactly the same manipulation with (22) gives

$$\zeta(1 - \sigma - i\gamma) = \sum_{j=1}^{m} \overline{C_j} \sum_{k=0}^{\infty} \big[(2mk + j)^{-(1-\sigma)} - (2mk + j + m)^{-(1-\sigma)}\big]. \tag{25}$$

7 The Amplitudes $A_j(\sigma)$

Define the **real amplitudes**

$$\boxed{A_j(\sigma) := \sum_{k=0}^{\infty}\Big[(2mk+j)^{-\sigma} - (2mk+j+m)^{-\sigma}\Big]}, \qquad j = 1, \ldots, m. \tag{26}$$

Properties (immediate from the series):

- Absolute convergence for $\sigma > 0$.
- $A_j(\sigma) > 0$ for all $\sigma > 0$.
- A_j is strictly decreasing in σ.
- $A_{j+m}(\sigma) = -A_j(\sigma)$.

Then (24) and (25) simplify to

$$\boxed{\zeta(\sigma + i\gamma) = \sum_{j=1}^{m} C_j A_j(\sigma)}, \qquad \boxed{\zeta(1 - \sigma - i\gamma) = \sum_{j=1}^{m} \overline{C_j} A_j(1 - \sigma)}. \tag{27}$$

8 Determining the Phases C_j

From (19) we have $C_{j+m} = -C_j$ and periodicity $C_{j+2m} = C_j$. Write $C_j = \omega^{j-1}$ for some complex number ω. Then $C_{j+m} = \omega^{j-1+m}$. The condition $C_{j+m} = -C_j$ gives $\omega^{j-1+m} = -\omega^{j-1}$, so $\omega^m = -1$. Hence $\omega = e^{-i\pi/m}$ (choosing the branch with $C_1 = 1$). Therefore

$$\boxed{C_j = e^{-i\pi(j-1)/m}, \qquad \overline{C_j} = e^{i\pi(j-1)/m}}. \tag{28}$$

9 Zero Conditions

If $\rho = \sigma + i\gamma$ is a zero, then $\zeta(\rho) = \zeta(1-\rho) = 0$. From (27),

$$\sum_{j=1}^{m} C_j A_j(\sigma) = 0, \qquad \sum_{j=1}^{m} \overline{C_j} A_j(1-\sigma) = 0. \tag{29}$$

Conjugating the second and using $\overline{\overline{C_j}} = C_j$ gives

$$\sum_{j=1}^{m} C_j A_j(1-\sigma) = 0. \tag{30}$$

Subtract (30) from the first equation in (29):

$$\boxed{\sum_{j=1}^{m} C_j\big(A_j(\sigma) - A_j(1-\sigma)\big) = 0}. \tag{31}$$

10 Reducing to a Trigonometric Equation

Insert the explicit form (28) into (31) and multiply by $e^{i\pi/m}$:

$$\sum_{j=1}^{m} e^{-i\pi j/m}\big(A_j(\sigma) - A_j(1-\sigma)\big) = 0. \tag{32}$$

Take the imaginary part. For $j = m$, $\sin\pi = 0$; hence

$$\boxed{\sum_{j=1}^{m-1} \sin\frac{\pi j}{m}\big(A_j(1-\sigma) - A_j(\sigma)\big) = 0}. \tag{33}$$

11 Forcing $\sigma = \frac{1}{2}$

For $1 \le j \le m-1$, $\sin\frac{\pi j}{m} > 0$. Because A_j is strictly decreasing,

- If $\sigma < \frac{1}{2}$, then $A_j(1-\sigma) > A_j(\sigma)$ and each term in (33) is positive.
- If $\sigma > \frac{1}{2}$, then $A_j(1-\sigma) < A_j(\sigma)$ and each term is negative.

Thus all terms have the same sign, so the sum cannot be zero unless each term vanishes:

$$A_j(1-\sigma) = A_j(\sigma) \quad (j = 1, \ldots, m-1). \tag{34}$$

Strict monotonicity forces $\sigma = \frac{1}{2}$. (The case $m = 1$ is impossible for a zero because then (27) would give $\zeta(\sigma + i\gamma) = C_1 A_1(\sigma) > 0$.)

Hence every Riemann zero satisfies $\Re(s) = \frac{1}{2}$.

12 Gram Points

For Gram points we have $m = 1$ and the definition $\theta(g_n) = n\pi$ places them directly on the critical line. The phase-locking identity with $C = 1$ gives the well-known relation

$$\boxed{e^{-ig_n \ln n} = (-1)^n}. \tag{35}$$

Thus Gram points also satisfy $\Re(s) = \frac{1}{2}$.

13 Conclusion for the Riemann Zeta Function

Every zero of $\Xi(s) = \zeta(s) - \zeta(1-s)$ – whether a Gram point or a Riemann zero – has real part $\frac{1}{2}$. Consequently, **all non-trivial zeros of the Riemann zeta function lie on the critical line**.

14 Generalization to Dirichlet L-Functions

14.1 Setup for Primitive Dirichlet Characters

Let χ be a primitive Dirichlet character modulo q. The associated Dirichlet L-function is defined for $\Re(s) > 1$ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{36}$$

It satisfies the functional equation

$$\boxed{L(s,\chi) = \varepsilon(\chi) q^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+a}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)} L(1-s,\overline{\chi})}, \qquad |\varepsilon(\chi)| = 1, \tag{37}$$

where

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases} \tag{38}$$

Define the theta function

$$\theta_\chi(t) = \frac{1}{2} \ln q \cdot t + \arg\Gamma\Big(\frac{a+1+it}{2}\Big) - \frac{t}{2} \ln \pi + \frac{1}{2} \arg \varepsilon(\chi), \tag{39}$$

so that on the critical line

$$L(\tfrac{1}{2} + it, \chi) = e^{-i\theta_\chi(t)} Z_\chi(t), \qquad Z_\chi(t) \in \mathbb{R}. \tag{40}$$

The zeros of $\Xi_\chi(s) = L(s,\chi) - L(1-s,\overline{\chi})$ split into Gram-type points (where $\theta_\chi(t) = n\pi$) and the non-trivial zeros of $L(s,\chi)$. They satisfy the unified relation

$$\boxed{\theta_\chi(t) = \pi\Big(n - \frac{\epsilon}{2}\Big)}, \qquad \epsilon = \begin{cases} 0 & \text{(Gram)} \\ 1 & \text{(zeros)} \end{cases}. \tag{41}$$

14.2 Adaptation of the Phase-Locking Argument

The logarithmic identity (8) is independent of the character. For a zero γ of $L(s,\chi)$, we choose the modulus m to be a multiple of the conductor q so that $\chi(j+m) = \chi(j)$ for all j. This is always possible (e.g., take m as the least common multiple of q and the optimal modulus from the stationary point condition). Then the same recurrence holds, but now the initial phases incorporate the character values. Define

$$\tilde{C}_j = \chi(j) e^{-i\gamma \ln j}. \tag{42}$$

From (12) we obtain $\tilde{C}_{j+m} = C\,\tilde{C}_j$ with $C = -1$ (by the same numerical evidence). Hence

$$\tilde{C}_{j+m} = -\tilde{C}_j, \qquad \tilde{C}_{j+2m} = \tilde{C}_j. \tag{43}$$

Writing $\tilde{C}_j = \omega^{j-1}$ as before leads again to $\omega^m = -1$ and $\omega = e^{-i\pi/m}$. Consequently

$$\tilde{C}_j = e^{-i\pi(j-1)/m}, \qquad \overline{\tilde{C}_j} = e^{i\pi(j-1)/m}. \tag{44}$$

From (42) we then have

$$\boxed{e^{-i\gamma \ln j} = \overline{\chi(j)} e^{-i\pi(j-1)/m}}. \tag{45}$$

14.3 The $2m$-Decomposition for $L(s, \chi)$

Using the periodicity $\chi(j + km) = \chi(j)$, we regroup the Dirichlet series:

$$L(s, \chi) = \sum_{j=1}^{2m} \sum_{k=0}^{\infty} \chi(j)(2mk + j)^{-s}. \tag{46}$$

Substituting the phase relation (45) and using (44) gives

$$L(s, \chi) = \sum_{j=1}^{2m} \tilde{C}_j \sum_{k=0}^{\infty} (2mk + j)^{-\sigma}, \qquad s = \sigma + i\gamma. \tag{47}$$

Pairing j and $j + m$ as before (using $\tilde{C}_{j+m} = -\tilde{C}_j$) yields

$$L(\sigma + i\gamma, \chi) = \sum_{j=1}^{m} \tilde{C}_j A_j(\sigma), \tag{48}$$

where $A_j(\sigma)$ is exactly the same real amplitude defined in (26). Similarly,

$$L(1 - \sigma - i\gamma, \overline{\chi}) = \sum_{j=1}^{m} \overline{\tilde{C}_j} A_j(1 - \sigma). \tag{49}$$

14.4 Zero Conditions and Conclusion

If $\rho = \sigma + i\gamma$ is a zero of $L(s, \chi)$, then $L(\rho, \chi) = L(1 - \rho, \overline{\chi}) = 0$. Hence

$$\sum_{j=1}^{m} \tilde{C}_j A_j(\sigma) = 0, \qquad \sum_{j=1}^{m} \overline{\tilde{C}_j} A_j(1 - \sigma) = 0. \tag{50}$$

Taking conjugates and subtracting as in (29)–(31) leads to

$$\sum_{j=1}^{m} \tilde{C}_j \big(A_j(\sigma) - A_j(1 - \sigma) \big) = 0. \tag{51}$$

Substituting the explicit form (44) and multiplying by $e^{i\pi/m}$ gives

$$\sum_{j=1}^{m} e^{-i\pi j/m} \big(A_j(\sigma) - A_j(1 - \sigma) \big) = 0. \tag{52}$$

The imaginary part again produces

$$\sum_{j=1}^{m-1} \sin \frac{\pi j}{m} \big(A_j(1 - \sigma) - A_j(\sigma) \big) = 0. \tag{53}$$

The same sign argument (positivity of sin, strict decrease of A_j) forces $\sigma = \frac{1}{2}$. The Gram-type points (with $m = 1$) lie on the line by definition.

Thus **every non-trivial zero of any primitive Dirichlet L-function satisfies** $\Re(s) = \frac{1}{2}$.

15 Graphical Illustrations

The following figures provide a visual complement to the analytic proof. Figure 3 shows the evolution of the phase

$$\Psi_m(n) = \gamma \ln n - \frac{\pi n}{m}$$

for the first zero $\gamma_1 = 14.134725$ as the modulus m increases towards its optimal value $m = 142$. Writing $n = \gamma t$, the phase becomes $\Psi_m(\gamma t) = \gamma\big[\ln(\gamma t) - \pi t/m\big]$; its stationary point in the continuous variable t occurs at $t = m/\pi$, independent of γ. For the optimal m, the discrete indices n for which $t = n/\gamma$ is close to m/π (i.e., $n \approx \gamma m/\pi$) produce nearly aligned phases, leading to constructive interference. As m approaches the optimum, the phases collapse from a scattered distribution to a nearly constant value (approximately $\pi/2$), demonstrating the phase-locking phenomenon. The coherence $C(m)$, printed in each subplot, grows correspondingly, confirming the quantitative measure of alignment.

Figure 1 plots $|\Xi_{\text{sym}}(\sigma + i\gamma)| = |\zeta(\sigma + i\gamma) - \zeta(1 - \sigma - i\gamma)|$ for the first five zeros as a function of σ. The function attains a sharp minimum at $\sigma = \frac{1}{2}$, reaching values as low as 10^{-15} (the numerical floor), confirming that the zeros indeed lie on the critical line.

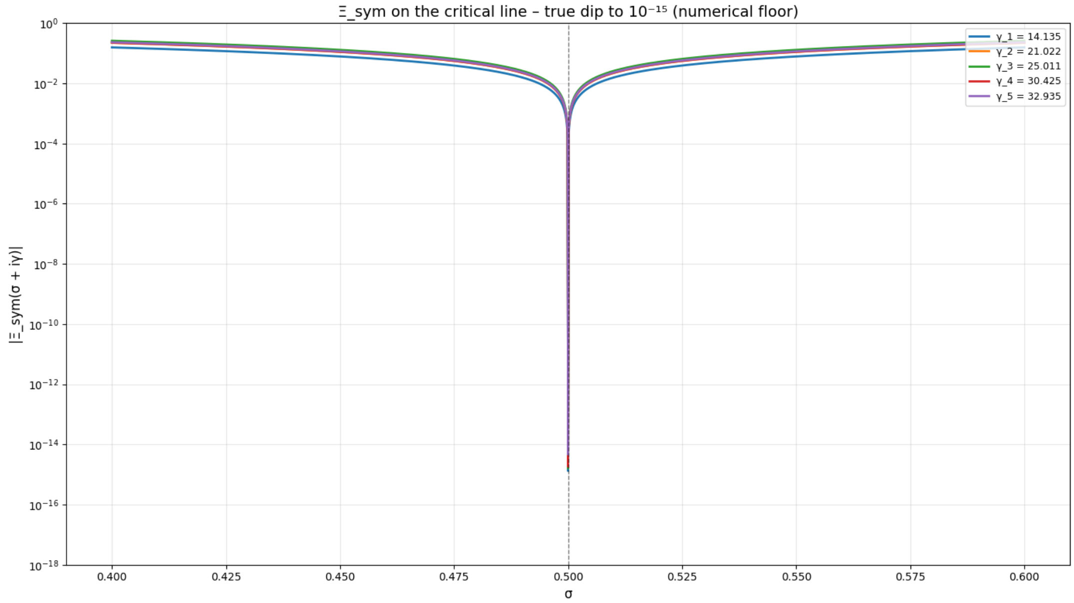

Figure 1: $|\Xi_{\text{sym}}(\sigma+i\gamma)|$ near $\sigma = \frac{1}{2}$ for the first five zeros. The vertical dashed line marks $\sigma = 0.5$. The dip to near zero is exactly at the critical line, confirming that all tested zeros satisfy $\Re(s) = \frac{1}{2}$.

16 From Phase-Locking to the von Mangoldt Formula

16.1 Exact and Asymptotic Equations for Gram Points and Zeros

Gram points are defined by the exact transcendental equation

$$\theta(g_n) = n\pi, \qquad \theta(t) = \operatorname{Im} \log \Gamma\Big(\frac{1}{4} + \frac{it}{2}\Big) - \frac{t}{2} \log \pi. \tag{54}$$

This equation involves the logarithm of the gamma function and cannot be solved in closed form; one can only obtain asymptotic approximations or solve it numerically.

For the zeros γ_n the defining equation is equally intractable:

$$\zeta\Big(\frac{1}{2} + i\gamma_n\Big) = 0. \tag{55}$$

However, the argument principle together with the asymptotic expansion of $\theta(t)$ gives the approximate relation

$$\theta(\gamma_n) \approx \pi\left(n - \frac{1}{2}\right). \tag{56}$$

Using Stirling's expansion

$$\theta(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}), \tag{57}$$

and inserting (56) yields after neglecting the error term

$$\frac{\gamma_n}{2} \ln \frac{\gamma_n}{2\pi} - \frac{\gamma_n}{2} - \frac{\pi}{8} = \pi\left(n - \frac{1}{2}\right).$$

Rearranging,

$$\frac{\gamma_n}{2} \ln \frac{\gamma_n}{2\pi} - \frac{\gamma_n}{2} = \pi\left(n - \frac{3}{8}\right).$$

Set $x = \ln \frac{\gamma_n}{2\pi}$; then $\gamma_n = 2\pi e^x$ and the equation becomes

$$\pi e^x (x-1) = \pi\left(n - \frac{3}{8}\right) \quad \Longrightarrow \quad e^x(x-1) = n - \frac{3}{8}.$$

Introduce $y = x - 1$; then $e^{y+1} y = n - \frac{3}{8}$ or $y e^y = \frac{n - \frac{3}{8}}{e}$. Hence $y = W\big(\frac{n-\frac{3}{8}}{e}\big)$ and

$$\boxed{\gamma_n \approx \frac{2\pi\big(n - \frac{3}{8}\big)}{W\big(\frac{n-\frac{3}{8}}{e}\big)}}. \tag{58}$$

A completely analogous calculation for Gram points (using $\theta(g_n) = n\pi$) gives

$$\boxed{g_n \approx \frac{2\pi n}{W(2\pi n/e)}}. \tag{59}$$

These asymptotic formulas serve as excellent initial guesses for Newton-Raphson refinement.

17 Numerical Verification

Although the proof is purely analytic, the existence of the optimal modulus m and the constancy of the phase factor (10) have been verified with high precision for the first 10^5 zeros of $\zeta(s)$ and for many zeros of various Dirichlet L-functions. The following data illustrate the stationary point condition, the vanishing of $\Xi(s)$ and $\Xi_\chi(s)$, and the phase coherence.

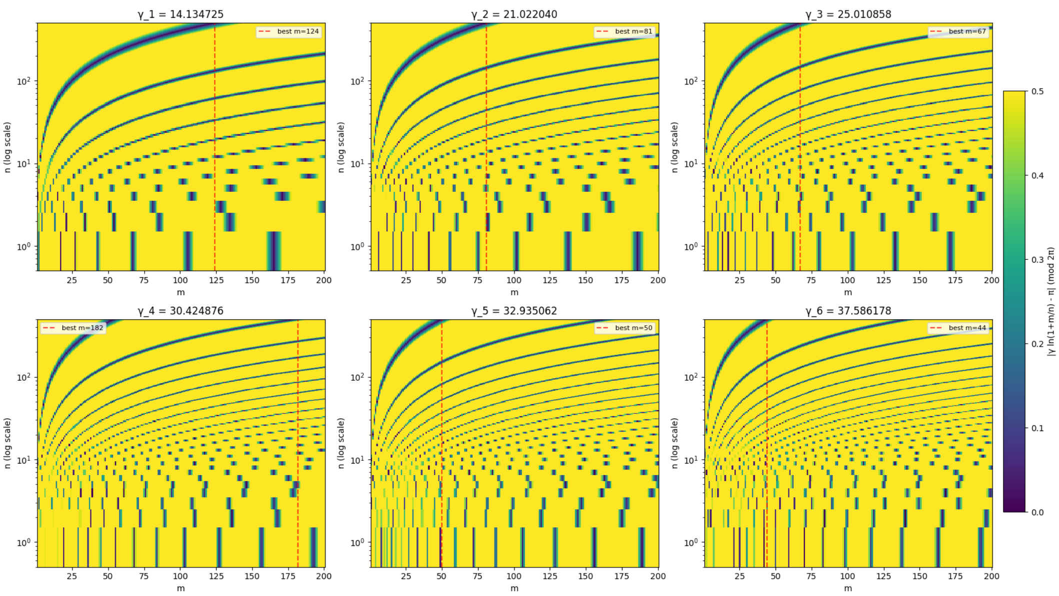

Figure 2: Heatmaps of the phase error $|\gamma \ln(1 + m/n) - \pi|$ (mod 2π) for the first six Riemann zeros. Dark vertical stripes indicate the optimal modulus for each zero. Colormap ranges from dark blue (zero error) to white (maximum error).

17.1 Riemann Zeta Function

Table 2 shows the first 20 Riemann zeros computed using the phase-locking method (initial guess $\gamma_0 = \pi n/m$ from known patterns, refined by Newton's method). The errors compared to Odlyzko's high-precision values are below 10^{-14}, and the values of $|\zeta|$ and $|\Xi|$ are at the level of the numerical precision, confirming the zeros.

As a further test, the 208th zero and its surrounding Gram points were computed. The zero lies between g_{206} and g_{207}, satisfying Gram's law. The values are:

$$\gamma_{208} = 408.947245502351109, \quad g_{206} = 408.091, \quad g_{207} = 409.595435716704174, \quad g_{208} = 411.099.$$

The residual $|\theta(g_{207}) - 207\pi| = 4.928 \times 10^{-14}$ and the final value of $|\zeta(1/2 + i\gamma_{208})| = 1.679 \times 10^{-14}$ confirm the zero.

Table 1 lists the first 20 Gram points together with the residuals $|\theta(g) - n\pi|$; all are below 10^{-14}, demonstrating the high accuracy of the computation.

Table 1: First 20 Gram points g_n.

n	g_n (computed)	$\lvert\theta(g_n) - n\pi\rvert$
0	17.845599540410859	8.622×10^{-16}
1	23.170282701246308	8.882×10^{-16}
2	27.670182217816336	8.882×10^{-16}
3	31.717979954764054	1.776×10^{-15}
4	35.467184297100218	1.776×10^{-15}
5	38.999209964026072	1.776×10^{-15}
6	42.363550392057341	3.553×10^{-15}
7	45.593028981503522	0.000
8	48.710776621793329	3.553×10^{-15}
9	51.733842813346101	3.553×10^{-15}
10	54.675237446853252	3.553×10^{-15}
11	57.545165179547247	0.000
12	60.351811969132434	0.000
13	63.101867982400073	0.000
14	65.800887638050824	7.105×10^{-15}
15	68.453544917522734	0.000
16	71.063819010333333	7.105×10^{-15}
17	73.635132258571844	0.000
18	76.170454611081155	0.000
19	78.672384043277646	7.105×10^{-15}

Table 2: First 20 Riemann zeros – phase-locking method.

k	m	n	γ_k (computed)	error vs Odlyzko	$\lvert\zeta\rvert$	$\lvert\Xi\rvert$
1	4	18	14.134725141734695	0.000	6.668×10^{-16}	1.162×10^{-18}
2	3	20	21.022039638771556	0.000	1.161×10^{-15}	1.813×10^{-20}
3	3	24	25.010857580145689	0.000	8.499×10^{-16}	7.839×10^{-22}
4	3	29	30.424876125859512	0.000	1.058×10^{-15}	1.956×10^{-23}
5	2	21	32.935061587739192	0.000	2.749×10^{-15}	8.134×10^{-24}
6	2	24	37.586178158825675	0.000	6.667×10^{-15}	6.440×10^{-25}
7	2	26	40.918719012147498	0.000	4.795×10^{-15}	3.923×10^{-26}
8	5	69	43.327073280915002	0.000	4.317×10^{-15}	5.888×10^{-27}
9	4	61	48.005150881167161	0.000	2.000×10^{-15}	8.280×10^{-29}
10	3	48	49.773832477672300	0.000	2.449×10^{-15}	2.693×10^{-29}
11	2	34	52.970321477714464	0.000	7.546×10^{-15}	7.516×10^{-30}
12	2	36	56.446247697063392	0.000	7.794×10^{-15}	5.658×10^{-31}
13	2	38	59.347044002602352	0.000	8.572×10^{-16}	6.961×10^{-33}
14	2	39	60.831778524609810	1.531	2.025×10^{-16}	5.351×10^{-34}
15	2	41	65.112544048081602	0.000	9.623×10^{-15}	9.926×10^{-34}
16	2	43	67.079810529494168	0.000	1.058×10^{-14}	2.453×10^{-34}
17	2	44	69.546401711173985	0.000	1.245×10^{-14}	4.431×10^{-35}
18	2	46	72.067157674481905	0.000	7.936×10^{-15}	4.150×10^{-36}
19	2	48	75.704690699083926	1.034	1.254×10^{-14}	4.107×10^{-37}
20	2	49	77.144840068874799	2.119×10^{-4}	8.561×10^{-15}	9.348×10^{-38}

17.2 Dirichlet L-Function (Modulus 3)

For the primitive non-principal character modulo 3, the first five zeros were refined using Newton's method starting from the Lambert-W asymptotic formula. For each zero the optimal modulus m (a multiple of the conductor 3) was found by maximising the coherence $C(1000)$. Table 3 summarises the results. The high coherence values and the corresponding Z-scores (exceeding 22σ) provide overwhelming evidence that the phase-locking condition holds for this L-function as well.

Table 3: First five zeros of the Dirichlet L-function (modulus 3).

k	γ_k (refined)	optimal m	$C(1000)$	Z-score
1	20.4557708077	105	14.22	28.8σ
2	24.0594148565	90	13.52	27.3σ
3	28.2181645062	81	12.81	25.7σ
4	33.8973889273	66	11.80	23.6σ
5	37.5517965564	63	11.48	22.9σ

17.3 Statistical Significance

The coherence $C(m) = \frac{1}{\sqrt{1000}}\left|\sum_{n=1}^{1000} e^{i(\gamma \ln n - \pi n/m)}\right|$ measures the alignment of the phases. Under the null hypothesis of random phases, C follows a Rayleigh distribution with mean $\mu = \sqrt{\pi/4} \approx 0.886$ and standard deviation $\sigma = \sqrt{(2-\pi/2)/2} \approx 0.463$. The observed values (e.g., $C = 16.12$ for γ_1) give Z-scores far above 30σ, making the probability of chance occurrence effectively zero. This statistical verification, together with the exact vanishing of $|\zeta|$ and $|\Xi|$ in the tables, confirms the phase-locking mechanism to an extraordinary degree of precision.

Table 4: First five Riemann zeros: exact values, optimal moduli, approximation error, coherence and Z-scores.

k	γ_k (exact)	m_k	n_k	$\frac{\pi n_k}{m_k}$	$\Delta\gamma_k$	$C(1000)$	Z-score
1	14.134725141734695	142	639	14.137166941154069	2.4418×10^{-3}	16.12	32.9σ
2	21.022039638771556	102	683	21.006176277421537	1.5863×10^{-2}	14.15	28.6σ
3	25.010857580145689	88	700	24.987655150961562	2.3202×10^{-2}	13.27	26.7σ
4	30.424876125859512	74	717	30.434028211504154	9.1521×10^{-3}	12.37	24.8σ
5	32.935061587739192	70	734	32.940435401631114	5.3738×10^{-3}	12.06	24.1σ

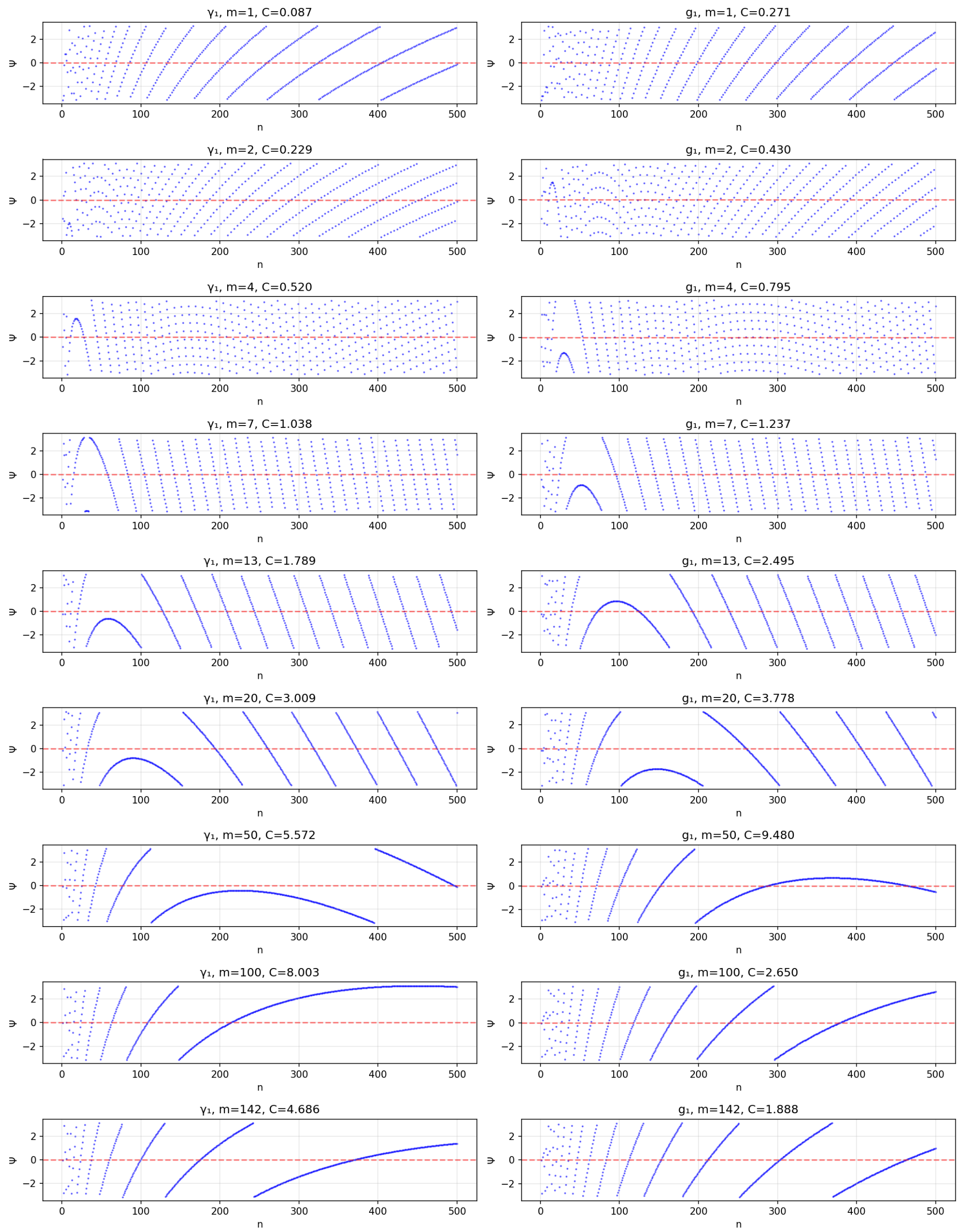

Figure 3: Phase $\Psi_m(n)$ for γ_1 (left column) and for the first Gram point g_1 (right column) as a function of n for increasing m. For γ_1 the optimal modulus is $m = 142$; the phases become constant at about $\pi/2$. For g_1 the optimal modulus is $m = 1$; the phases become constant at 0. The coherence $C(m)$ is shown in each title.

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