

# A Transport-Diffusion Kinetic Equation

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## Abstract

This paper studies a linear kinetic equation on a periodic phase space with free transport in position and Ornstein-Uhlenbeck relaxation in velocity. The equation is formulated in the weighted Hilbert space associated with the Maxwellian equilibrium. In that setting, the paper establishes the dissipation identity, conservation of mass, semigroup well-posedness, microscopic coercivity in velocity, and exponential convergence to equilibrium on the zero-mass subspace. The spatially homogeneous problem is treated separately, where the entropy law and exponential entropy decay follow directly from the Gaussian logarithmic Sobolev inequality. A final numerical section presents a Fourier-Hermite discretization and illustrates the same relaxation mechanism at the level of decay curves, spectral localization, and density profiles.

## 1 Introduction

The purpose of this paper is to study a linear kinetic transport-diffusion equation on the phase space

$$\mathbb{T}^d \times \mathbb{R}^d,$$

where the spatial domain is periodic and the velocity domain is unbounded. The model combines free transport in the spatial variable with diffusion and linear friction in the velocity variable. In explicit form, it reads

$$\partial_t f + v \cdot \nabla_x f = \gamma \nabla_v \cdot (\nabla_v f + v f), \quad t > 0, \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad (1.1)$$

where  $\gamma > 0$  is a fixed relaxation parameter.

The analytical difficulty is structural. The transport operator is conservative and contributes no direct decay in the natural energy space. The velocity operator is dissipative, but only in the velocity variable, so the equation is not coercive in the full phase-space norm. The central problem is therefore to explain, in a precise functional framework, how dissipation confined to one family of variables is transferred to the whole phase space and yields exponential relaxation to equilibrium.

This question belongs to the theory of hypocoercivity. The general framework was developed systematically by Villani [1]. Earlier model-specific work of Hérau established exponential time decay for linear inhomogeneous kinetic equations of relaxation type [2]. A particularly effective abstract formulation for linear kinetic equations conserving mass was later given by Dolbeault,

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Mouhot, and Schmeiser [3]. The semigroup formulation used below is standard in the theory of linear evolution equations; see Engel and Nagel [4]. The microscopic coercivity estimate rests on the Gaussian Poincaré inequality for the Maxwellian measure; for that inequality and its functional-analytic background we refer to Bakry, Gentil, and Ledoux [5]. The entropy argument in the spatially homogeneous case uses the Gaussian logarithmic Sobolev inequality introduced by Gross [6]. For the spectral representation of the Ornstein–Uhlenbeck operator in Hermite variables used in the numerical section, we follow the standard presentation in Pavliotis [7].

The equation (1.1) is classical and the analytical tools used here are established ones. No claim of methodological novelty is made for the hypocoercive mechanism itself. We assemble, in a single note, a coherent treatment of this transport-diffusion model in the weighted Hilbert space naturally associated with its Maxwellian equilibrium, we record the basic structural identities in full detail, we carry out a direct hypocoercive energy estimate adapted to the present operator, and we complement the analytical results with a Fourier-Hermite computation that makes the dominant decay mechanisms visible at the level of the modal dynamics. In this sense, the paper is intended as a careful PDE note rather than as a claim of a new kinetic theory.

The argument proceeds in four steps. First, the equation is formulated in the weighted Hilbert space associated with the Maxwellian equilibrium, and the decomposition of the generator into a dissipative velocity part and a conservative transport part is made explicit. Second, the weighted dissipation law, conservation of mass, the equilibrium manifold, and the microscopic coercivity estimate are derived in a form suited to the later energy method. Third, the microscopic velocity coercivity is combined with a macroscopic Poincaré estimate and a modified energy functional to obtain exponential convergence to equilibrium on the zero-mass subspace. Fourth, a Fourier-Hermite discretization is used to illustrate numerically the relaxation mechanism identified in the analytical part of the paper.

The paper is organized as follows. Section 2 introduces the equation, the weighted setting, and the basic operator identities. Section 3 establishes semigroup well-posedness and the microscopic coercivity estimate. Section 4 proves exponential convergence to equilibrium by a direct hypocoercive energy argument. Section 5 treats the spatially homogeneous entropy law. Section 6 contains the numerical illustration.

## 2 Functional Setting and Basic Identities

### 2.1 Phase space, Maxwellian, and weighted space

Let  $d \geq 1$ . The phase space is

$$\mathbb{T}^d \times \mathbb{R}^d,$$

where  $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$  is the flat torus. The normalized Maxwellian is

$$M(v) = (2\pi)^{-d/2} e^{-|v|^2/2}, \quad v \in \mathbb{R}^d. \quad (2.1)$$

It satisfies

$$\int_{\mathbb{R}^d} M(v) \, dv = 1, \quad \nabla_v M(v) = -vM(v), \quad \nabla_v \cdot (\nabla_v M + vM) = 0.$$

The natural reference measure is therefore

$$d\mu(x, v) := dx M(v) \, dv.$$

The kinetic equation under consideration is

$$\partial_t f + v \cdot \nabla_x f = \gamma \nabla_v \cdot (\nabla_v f + vf), \quad t > 0, \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad (2.2)$$

with initial condition

$$f|_{t=0} = f_0.$$

Since  $M$  is independent of  $x$  and solves the stationary Ornstein-Uhlenbeck equation in velocity, it is a stationary solution of (2.2).

The natural Hilbert space is

$$L^2(M^{-1}) = L^2(\mathbb{T}^d \times \mathbb{R}^d, M(v)^{-1} dx dv),$$

with inner product

$$\langle f, g \rangle_{M^{-1}} = \int_{\mathbb{T}^d \times \mathbb{R}^d} f(x, v) g(x, v) M(v)^{-1} dx dv. \quad (2.3)$$

The associated norm is denoted by  $\|\cdot\|_{M^{-1}}$ .

It is convenient to factor out the Maxwellian. If

$$f(x, v) = M(v) g(x, v), \quad (2.4)$$

then

$$\|f\|_{M^{-1}}^2 = \int_{\mathbb{T}^d \times \mathbb{R}^d} |g(x, v)|^2 M(v) dx dv.$$

Hence the map  $f \mapsto g = f/M$  is an isometry from  $L^2(M^{-1})$  onto  $L^2(\mathbb{T}^d \times \mathbb{R}^d, d\mu)$ .

## 2.2 Generator decomposition

Define

$$Af := \gamma \nabla_v \cdot (\nabla_v f + v f), \quad Bf := -v \cdot \nabla_x f, \quad (2.5)$$

so that the full generator is

$$L := A + B. \quad (2.6)$$

The operator  $A$  acts only in the velocity variable and is dissipative. The operator  $B$  acts only in the spatial variable and is conservative. This decomposition is the starting point of the analysis.

## 2.3 Dissipation identity and skew-symmetry

**Proposition 2.1.** *For every smooth function  $f$  that is periodic in  $x$  and rapidly decaying in  $v$ ,*

$$\langle Af, f \rangle_{M^{-1}} = -\gamma \int_{\mathbb{T}^d \times \mathbb{R}^d} M(v) \left| \nabla_v \left( \frac{f}{M} \right) \right|^2 dx dv. \quad (2.7)$$

*In particular,  $A$  is symmetric and non-positive in  $L^2(M^{-1})$ .*

*Proof.* Write  $f = Mg$ . Since  $\nabla_v M = -vM$ , one obtains

$$\nabla_v f + v f = \nabla_v (Mg) + v Mg = M \nabla_v g.$$

Therefore

$$Af = \gamma \nabla_v \cdot (M \nabla_v g).$$

Using the weighted inner product and integrating by parts in  $v$  gives

$$\langle Af, f \rangle_{M^{-1}} = \gamma \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_v \cdot (M \nabla_v g) g dx dv = -\gamma \int_{\mathbb{T}^d \times \mathbb{R}^d} M(v) |\nabla_v g|^2 dx dv,$$

which is exactly (2.7). □

**Proposition 2.2.** *For every smooth function  $f$  that is periodic in  $x$  and rapidly decaying in  $v$ ,*

$$\langle Bf, f \rangle_{M^{-1}} = 0. \quad (2.8)$$

*In particular,  $B$  is skew-symmetric in  $L^2(M^{-1})$ .*

*Proof.* Since  $M$  depends only on  $v$ ,

$$\langle Bf, f \rangle_{M^{-1}} = - \int_{\mathbb{T}^d \times \mathbb{R}^d} (v \cdot \nabla_x f) f M(v)^{-1} dx dv = - \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} v \cdot \nabla_x (f^2) M(v)^{-1} dx dv.$$

The integral vanishes by periodicity in the spatial variable.  $\square$

**Corollary 2.3.** *For every smooth function  $f$  that is periodic in  $x$  and rapidly decaying in  $v$ ,*

$$\langle Lf, f \rangle_{M^{-1}} = -\gamma \int_{\mathbb{T}^d \times \mathbb{R}^d} M(v) \left| \nabla_v \left( \frac{f}{M} \right) \right|^2 dx dv. \quad (2.9)$$

*In particular,  $L$  is dissipative on its natural core.*

## 2.4 Conservation of mass and equilibrium

**Proposition 2.4.** *Let  $f$  be a smooth solution of (2.2). Then*

$$\frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t, x, v) dx dv = 0. \quad (2.10)$$

*Proof.* Integrate (2.2) over  $\mathbb{T}^d \times \mathbb{R}^d$ . The transport term vanishes by periodicity in  $x$ . The velocity divergence term vanishes by integration by parts in  $v$ .  $\square$

If

$$m_0 = \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) dx dv,$$

then the corresponding equilibrium state is

$$f_\infty(x, v) = \frac{m_0}{(2\pi)^d} M(v). \quad (2.11)$$

It is therefore natural to work on the zero-mass subspace

$$\mathcal{H}_0 = \left\{ f \in L^2(M^{-1}) : \int_{\mathbb{T}^d \times \mathbb{R}^d} f(x, v) dx dv = 0 \right\}. \quad (2.12)$$

Exponential convergence to equilibrium is meaningful precisely after restriction to this subspace.

## 3 Semigroup Well-Posedness and Microscopic Coercivity

### 3.1 Semigroup well-posedness

**Theorem 3.1.** *Let  $\mathcal{D} = C^\infty(\mathbb{T}^d; \mathcal{S}(\mathbb{R}^d))$ . The operator  $L = A + B$ , initially defined on  $\mathcal{D}$ , is closable in  $L^2(M^{-1})$ . Its closure, still denoted by  $L$ , generates a strongly continuous contraction semigroup  $(e^{tL})_{t \geq 0}$  on  $L^2(M^{-1})$ . Consequently, for every  $f_0 \in L^2(M^{-1})$ , the Cauchy problem (2.2) admits a unique global mild solution*

$$f(t) = e^{tL} f_0 \in C([0, \infty); L^2(M^{-1})),$$

and

$$\|e^{tL} f_0\|_{M^{-1}} \leq \|f_0\|_{M^{-1}}, \quad t \geq 0. \quad (3.1)$$

*Proof.* The space  $\mathcal{D}$  is dense in  $L^2(M^{-1})$  and is stable under both  $A$  and  $B$ . By Corollary 2.3, the operator  $L$  is dissipative on  $\mathcal{D}$ :

$$\langle Lf, f \rangle_{M^{-1}} \leq 0, \quad f \in \mathcal{D}.$$

The velocity part  $A$  is the weighted Ornstein-Uhlenbeck operator. In the Gaussian representation  $f = Mg$ , it is unitarily equivalent to  $\gamma(\Delta_v - v \cdot \nabla_v)$  on  $L^2(\mathbb{T}^d \times \mathbb{R}^d, dx M dv)$ , and this operator is known to be maximally dissipative on the Schwartz-periodic core. The transport part  $B = -v \cdot \nabla_x$  is skew-symmetric on  $\mathcal{D}$  by Proposition 2.2. Since  $\mathcal{D}$  is a common invariant core and  $B$  is a conservative first-order perturbation of the dissipative part, the standard perturbation theorem for maximally dissipative generators applies; see [4, Ch. III, §2]. Therefore the closure of  $L$  is maximally dissipative in  $L^2(M^{-1})$ . The Lumer–Phillips theorem then yields a strongly continuous contraction semigroup on  $L^2(M^{-1})$ . The representation of mild solutions and the estimate (3.1) follow immediately.  $\square$

*Remark 3.2.* Theorem 3.1 is a classical well-posedness statement for this operator class. It is included here in order to fix the functional setting used in the later coercivity and hypocoercivity arguments. The genuinely model-specific part of the analysis begins with the weighted identities in Section 2 and the explicit energy estimate in Section 4.

### 3.2 Gaussian Poincaré inequality and velocity coercivity

**Proposition 3.3.** *Let  $h \in H^1(\mathbb{R}^d, M dv)$  satisfy*

$$\int_{\mathbb{R}^d} h(v) M(v) dv = 0.$$

*Then*

$$\int_{\mathbb{R}^d} |h(v)|^2 M(v) dv \leq \int_{\mathbb{R}^d} |\nabla_v h(v)|^2 M(v) dv. \quad (3.2)$$

*Proof.* This is the Gaussian Poincaré inequality for the standard centered Gaussian measure on  $\mathbb{R}^d$ ; see [5, Ch. 3, Sec. 3.1].  $\square$

For  $f \in L^2(M^{-1})$ , define the projection onto the local equilibrium manifold by

$$(\Pi f)(x, v) = \left( \int_{\mathbb{R}^d} f(x, w) dw \right) M(v). \quad (3.3)$$

In the Gaussian representation  $f = Mg$ , this projection is simply

$$(\Pi g)(x, v) = \int_{\mathbb{R}^d} g(x, w) M(w) dw,$$

which depends only on  $x$ .

**Proposition 3.4.** *For every smooth function  $f$  that is periodic in  $x$  and rapidly decaying in  $v$ ,*

$$-\langle Af, f \rangle_{M^{-1}} \geq \gamma \|(\text{Id} - \Pi)f\|_{M^{-1}}^2. \quad (3.4)$$

*Proof.* Write  $f = Mg$  and set

$$\bar{g}(x) = \int_{\mathbb{R}^d} g(x, v) M(v) dv.$$

Then  $\Pi f = M\bar{g}$  and

$$(\text{Id} - \Pi)f = M(g - \bar{g}).$$

By Proposition 2.1,

$$-\langle Af, f \rangle_{M^{-1}} = \gamma \int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_v g|^2 M(v) \, dx \, dv.$$

For each fixed  $x$ , the function  $g(x, \cdot) - \bar{g}(x)$  has zero Gaussian mean. Proposition 3.3 therefore yields

$$\int_{\mathbb{R}^d} |g(x, v) - \bar{g}(x)|^2 M(v) \, dv \leq \int_{\mathbb{R}^d} |\nabla_v g(x, v)|^2 M(v) \, dv.$$

Integrating in  $x$  gives

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |g - \bar{g}|^2 M(v) \, dx \, dv \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_v g|^2 M(v) \, dx \, dv.$$

Since

$$\|(\text{Id} - \Pi)f\|_{M^{-1}}^2 = \int_{\mathbb{T}^d \times \mathbb{R}^d} |g - \bar{g}|^2 M(v) \, dx \, dv,$$

the claim follows.  $\square$

## 4 Hypocoercive Exponential Convergence

We now derive exponential convergence to equilibrium on the zero-mass subspace by a direct hypocoercive energy estimate. The argument is written in the Gaussian representation, where

$$f = Mg.$$

Then (2.2) becomes

$$\partial_t g + v \cdot \nabla_x g = \gamma(\Delta_v g - v \cdot \nabla_v g). \quad (4.1)$$

The natural space is now  $L^2(\mathbb{T}^d \times \mathbb{R}^d, d\mu)$  with  $d\mu = dx M(v) dv$ .

### 4.1 Preliminary differential identities

We first record the differential identities used in the energy estimate. All integrations below are taken with respect to  $d\mu$ .

**Lemma 4.1.** *Let  $g$  be a smooth solution of (4.1). Then*

$$\frac{1}{2} \frac{d}{dt} \|g\|_{L^2(\mu)}^2 + \gamma \|\nabla_v g\|_{L^2(\mu)}^2 = 0, \quad (4.2)$$

and

$$\frac{1}{2} \frac{d}{dt} \|\nabla_x g\|_{L^2(\mu)}^2 + \gamma \|\nabla_v \nabla_x g\|_{L^2(\mu)}^2 = 0. \quad (4.3)$$

*Proof.* Multiply (4.1) by  $g$  and integrate with respect to  $\mu$ . The transport term vanishes because  $v \cdot \nabla_x$  is skew-symmetric in  $L^2(\mu)$ , and the Ornstein-Uhlenbeck part contributes

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (\Delta_v g - v \cdot \nabla_v g) g \, d\mu = -\|\nabla_v g\|_{L^2(\mu)}^2.$$

This proves (4.2).

To prove (4.3), differentiate (4.1) with respect to  $x$ . Since  $\nabla_x$  commutes with both  $v \cdot \nabla_x$  and  $\Delta_v - v \cdot \nabla_v$ , the differentiated equation has exactly the same form with  $g$  replaced by  $\nabla_x g$ . The same computation as above gives (4.3).  $\square$

**Lemma 4.2.** *Let  $g$  be a smooth solution of (4.1). Then*

$$\frac{1}{2} \frac{d}{dt} \|\nabla_v g\|_{L^2(\mu)}^2 + \gamma \|\nabla_v^2 g\|_{L^2(\mu)}^2 + \gamma \|\nabla_v g\|_{L^2(\mu)}^2 = -\langle \nabla_x g, \nabla_v g \rangle_{L^2(\mu)}. \quad (4.4)$$

*Proof.* Differentiate (4.1) with respect to  $v$ . Using the commutators

$$[\nabla_v, v \cdot \nabla_x] = \nabla_x, \quad [\nabla_v, \Delta_v - v \cdot \nabla_v] = -\nabla_v,$$

we obtain

$$\partial_t \nabla_v g + v \cdot \nabla_x \nabla_v g + \nabla_x g = \gamma(\Delta_v \nabla_v g - v \cdot \nabla_v \nabla_v g) - \gamma \nabla_v g.$$

Taking the  $L^2(\mu)$  inner product with  $\nabla_v g$ , the transport term vanishes by skew-symmetry, the Ornstein-Uhlenbeck part gives  $-\gamma \|\nabla_v^2 g\|_{L^2(\mu)}^2$ , and the final term gives  $-\gamma \|\nabla_v g\|_{L^2(\mu)}^2$ . This yields (4.4).  $\square$

**Lemma 4.3.** *Let  $g$  be a smooth solution of (4.1). Then*

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x g, \nabla_v g \rangle_{L^2(\mu)} + \|\nabla_x g\|_{L^2(\mu)}^2 + \gamma \langle \nabla_x g, \nabla_v g \rangle_{L^2(\mu)} \\ = -2\gamma \langle \nabla_v \nabla_x g, \nabla_v^2 g \rangle_{L^2(\mu)}. \end{aligned} \quad (4.5)$$

Consequently, for every  $\eta > 0$ ,

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x g, \nabla_v g \rangle_{L^2(\mu)} + \|\nabla_x g\|_{L^2(\mu)}^2 + \gamma \langle \nabla_x g, \nabla_v g \rangle_{L^2(\mu)} \\ \leq \gamma \eta \|\nabla_v^2 g\|_{L^2(\mu)}^2 + \frac{\gamma}{\eta} \|\nabla_v \nabla_x g\|_{L^2(\mu)}^2. \end{aligned} \quad (4.6)$$

*Proof.* Differentiate the mixed product:

$$\frac{d}{dt} \langle \nabla_x g, \nabla_v g \rangle = \langle \partial_t \nabla_x g, \nabla_v g \rangle + \langle \nabla_x g, \partial_t \nabla_v g \rangle.$$

Insert the differentiated equations for  $\nabla_x g$  and  $\nabla_v g$  obtained in the proofs above. The transport contributions cancel by skew-symmetry. The commutator term contributes  $-\|\nabla_x g\|_{L^2(\mu)}^2$ . The Ornstein-Uhlenbeck terms combine to

$$\gamma \langle (\Delta_v - v \cdot \nabla_v) \nabla_x g, \nabla_v g \rangle + \gamma \langle \nabla_x g, (\Delta_v - v \cdot \nabla_v) \nabla_v g \rangle = -2\gamma \langle \nabla_v \nabla_x g, \nabla_v^2 g \rangle,$$

and the remaining commutator contribution gives  $-\gamma \langle \nabla_x g, \nabla_v g \rangle$ . This proves (4.5). The estimate (4.6) follows from Cauchy-Schwarz and Young's inequality.  $\square$

## 4.2 Control of the macroscopic component

The microscopic coercivity estimate from Proposition 3.4 controls only the orthogonal complement of the local equilibrium. To recover the full norm, one needs a spatial Poincaré estimate for the macroscopic component.

**Lemma 4.4.** *Let  $g \in L^2(\mu)$  satisfy*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} g(x, v) \, d\mu(x, v) = 0.$$

Then

$$\|g\|_{L^2(\mu)}^2 \leq 2 \|\nabla_v g\|_{L^2(\mu)}^2 + 2C_P \|\nabla_x g\|_{L^2(\mu)}^2, \quad (4.7)$$

where  $C_P$  is the Poincaré constant of the torus.

*Proof.* Let

$$\rho(x) = \int_{\mathbb{R}^d} g(x, v) M(v) \, dv.$$

Then  $\Pi g = \rho(x)$  and the zero-mass assumption implies

$$\int_{\mathbb{T}^d} \rho(x) \, dx = 0.$$

Therefore the torus Poincaré inequality gives

$$\|\rho\|_{L^2(\mathbb{T}^d)}^2 \leq C_P \|\nabla_x \rho\|_{L^2(\mathbb{T}^d)}^2.$$

By Jensen's inequality with respect to the probability measure  $M(v) \, dv$ ,

$$|\nabla_x \rho(x)|^2 = \left| \int_{\mathbb{R}^d} \nabla_x g(x, v) M(v) \, dv \right|^2 \leq \int_{\mathbb{R}^d} |\nabla_x g(x, v)|^2 M(v) \, dv.$$

After integration in  $x$ , this yields

$$\|\rho\|_{L^2(\mathbb{T}^d)}^2 \leq C_P \|\nabla_x g\|_{L^2(\mu)}^2.$$

On the other hand,

$$\|g\|_{L^2(\mu)}^2 \leq 2 \|g - \Pi g\|_{L^2(\mu)}^2 + 2 \|\Pi g\|_{L^2(\mu)}^2.$$

The first term is bounded by  $2 \|\nabla_v g\|_{L^2(\mu)}^2$  by Gaussian Poincaré in the velocity variable, and the second term equals  $2 \|\rho\|_{L^2(\mathbb{T}^d)}^2$ . Combining the two bounds gives (4.7).  $\square$

### 4.3 Modified energy and exponential decay

Define the modified energy

$$\mathcal{E}(g) = \|g\|_{L^2(\mu)}^2 + a \|\nabla_x g\|_{L^2(\mu)}^2 + b \|\nabla_v g\|_{L^2(\mu)}^2 + 2c \langle \nabla_x g, \nabla_v g \rangle_{L^2(\mu)}, \quad (4.8)$$

where  $a, b, c > 0$  are constants to be chosen.

**Lemma 4.5.** *There exist coefficients  $a, b, c > 0$  such that*

$$\mathcal{E}(g) \asymp \|g\|_{L^2(\mu)}^2 + \|\nabla_x g\|_{L^2(\mu)}^2 + \|\nabla_v g\|_{L^2(\mu)}^2. \quad (4.9)$$

*Proof.* Choose  $a > 0$  and  $b > 0$ , and then choose  $c > 0$  so small that

$$2c |\langle \nabla_x g, \nabla_v g \rangle| \leq \frac{a}{2} \|\nabla_x g\|_{L^2(\mu)}^2 + \frac{2c^2}{a} \|\nabla_v g\|_{L^2(\mu)}^2 \leq \frac{a}{2} \|\nabla_x g\|_{L^2(\mu)}^2 + \frac{b}{2} \|\nabla_v g\|_{L^2(\mu)}^2.$$

Then  $\mathcal{E}(g)$  is bounded above and below by positive multiples of the squared norm on the right-hand side of (4.9).  $\square$

**Theorem 4.6.** *There exist constants  $a, b, c, \lambda > 0$  and  $C > 0$ , depending only on the dimension, the torus, and the parameter  $\gamma$ , such that the following holds. Let  $g$  be a smooth solution of (4.1) with zero mean,*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} g(x, v) \, d\mu(x, v) = 0.$$

Then

$$\frac{d}{dt} \mathcal{E}(g(t)) \leq -\lambda \left( \|g\|_{L^2(\mu)}^2 + \|\nabla_x g\|_{L^2(\mu)}^2 + \|\nabla_v g\|_{L^2(\mu)}^2 \right), \quad (4.10)$$

and therefore

$$\mathcal{E}(g(t)) \leq C e^{-\lambda t} \mathcal{E}(g_0), \quad t \geq 0. \quad (4.11)$$

Equivalently, for the original unknown  $f = Mg$ ,

$$\left\| e^{tL} f_0 \right\|_{L^2(M^{-1})} \leq C e^{-\lambda t} \|f_0\|_{H_{\text{hyp}}^1}, \quad t \geq 0, \quad (4.12)$$

where  $H_{\text{hyp}}^1$  is the norm induced by (4.8).

*Proof.* Differentiate (4.8) and use Lemmas 4.1, 4.2, and 4.3. For any  $\eta > 0$ ,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(g) &\leq -2\gamma \|\nabla_v g\|^2 - 2a\gamma \|\nabla_v \nabla_x g\|^2 - 2b\gamma \|\nabla_v^2 g\|^2 - 2b\gamma \|\nabla_v g\|^2 \\ &\quad - 2c \|\nabla_x g\|^2 - 2c\gamma \langle \nabla_x g, \nabla_v g \rangle + 2c\gamma \eta \|\nabla_v^2 g\|^2 + \frac{2c\gamma}{\eta} \|\nabla_v \nabla_x g\|^2. \end{aligned}$$

Choose first  $\eta = 1$ , then choose  $c > 0$  so small that

$$2c \leq b.$$

After that choose  $a > b$ . With this choice, the positive terms involving  $\|\nabla_v^2 g\|^2$  and  $\|\nabla_v \nabla_x g\|^2$  are absorbed into the negative ones, and we obtain

$$\frac{d}{dt} \mathcal{E}(g) \leq -\gamma \|\nabla_v g\|^2 - c \|\nabla_x g\|^2 - c\gamma \langle \nabla_x g, \nabla_v g \rangle. \quad (4.13)$$

Apply Young's inequality to the mixed term:

$$c\gamma |\langle \nabla_x g, \nabla_v g \rangle| \leq \frac{c}{2} \|\nabla_x g\|^2 + \frac{c\gamma^2}{2} \|\nabla_v g\|^2.$$

Reducing  $c$  further if necessary yields

$$\frac{d}{dt} \mathcal{E}(g) \leq -\frac{\gamma}{2} \|\nabla_v g\|^2 - \frac{c}{2} \|\nabla_x g\|^2.$$

Finally, Lemma 4.4 controls  $\|g\|_{L^2(\mu)}^2$  by the two gradient terms, so

$$\frac{d}{dt} \mathcal{E}(g) \leq -\lambda \left( \|g\|_{L^2(\mu)}^2 + \|\nabla_x g\|_{L^2(\mu)}^2 + \|\nabla_v g\|_{L^2(\mu)}^2 \right)$$

for some  $\lambda > 0$ . Using the norm equivalence from Lemma 4.5, we obtain

$$\frac{d}{dt} \mathcal{E}(g) \leq -\lambda_0 \mathcal{E}(g)$$

for a possibly smaller constant  $\lambda_0 > 0$ . Grönwall's lemma yields (4.11). The estimate for  $f$  follows by the isometry between  $f$  and  $g$ .  $\square$

## 5 Spatially Homogeneous Entropy Law

In the spatially homogeneous case the transport term disappears, and the entropy structure becomes direct. Consider

$$\partial_t f = \gamma \nabla_v \cdot (\nabla_v f + v f), \quad t > 0, \quad v \in \mathbb{R}^d, \quad (5.1)$$

with  $f(t, \cdot) \geq 0$  and

$$\int_{\mathbb{R}^d} f(t, v) \, dv = 1.$$

Define the relative entropy

$$H(f|M) = \int_{\mathbb{R}^d} f(v) \log \frac{f(v)}{M(v)} \, dv \quad (5.2)$$

and the Fisher information

$$I(f|M) = \int_{\mathbb{R}^d} f(v) \left| \nabla_v \log \frac{f(v)}{M(v)} \right|^2 \, dv. \quad (5.3)$$

**Proposition 5.1.** *Let  $f$  be a smooth positive solution of (5.1). Then*

$$\frac{d}{dt}H(f(t)|M) = -\gamma I(f(t)|M). \quad (5.4)$$

*Proof.* Differentiate (5.2) with respect to time. Since the total mass is conserved, the derivative of the constant term vanishes and

$$\frac{d}{dt}H(f|M) = \int_{\mathbb{R}^d} \partial_t f \log \frac{f}{M} dv.$$

Substituting (5.1) gives

$$\frac{d}{dt}H(f|M) = \gamma \int_{\mathbb{R}^d} \nabla_v \cdot (\nabla_v f + vf) \log \frac{f}{M} dv.$$

Using the identity

$$\nabla_v f + vf = f \nabla_v \log \frac{f}{M},$$

and integrating by parts in  $v$ , one obtains (5.4).  $\square$

**Theorem 5.2.** *For every smooth positive probability solution of (5.1),*

$$H(f|M) \leq \frac{1}{2}I(f|M), \quad (5.5)$$

and therefore

$$H(f(t)|M) \leq e^{-2\gamma t} H(f_0|M), \quad t \geq 0. \quad (5.6)$$

*Proof.* The inequality (5.5) is the Gaussian logarithmic Sobolev inequality; see Gross [6, Thm. 1] and the discussion in [5, Ch. 5, Sec. 5.2]. Combining it with Proposition 5.1 yields

$$\frac{d}{dt}H(f(t)|M) \leq -2\gamma H(f(t)|M).$$

Grönwall's lemma gives (5.6).  $\square$

## 6 Numerical Illustration

This section records a finite-dimensional computation for the one-dimensional problem. The objective is not to establish a separate numerical analysis theorem, but to display the relaxation mechanism identified in Section 4 in a setting where the modal dynamics can be inspected directly.

The computation is performed for the conjugated unknown  $g = f/M$ , which satisfies

$$\partial_t g + v \partial_x g = \gamma (\partial_{vv} g - v \partial_v g), \quad (x, v) \in \mathbb{T} \times \mathbb{R}. \quad (6.1)$$

The spatial variable is expanded in a Fourier basis and the velocity variable in the Hermite basis associated with the Gaussian measure. In that representation the Ornstein-Uhlenbeck operator is diagonal, while multiplication by  $v$  couples only neighboring Hermite modes. After truncation one obtains a finite-dimensional linear system for the modal coefficients.

More precisely, write

$$g(t, x, v) \approx \sum_{|k| \leq K} \sum_{n=0}^N c_{k,n}(t) e^{ikx} \varphi_n(v), \quad (6.2)$$

where  $\{\varphi_n\}_{n \geq 0}$  is the orthonormal Hermite basis in  $L^2(\mathbb{R}, M(v) dv)$ . In this basis the Ornstein–Uhlenbeck operator is diagonal and the multiplication operator  $v$  is tri-diagonal. More precisely, one has

$$(\partial_{vv} - v\partial_v)\varphi_n = -n\varphi_n, \quad v\varphi_n = \sqrt{n+1}\varphi_{n+1} + \sqrt{n}\varphi_{n-1},$$

see, for example, [7, Ch. 3, Sec. 3.2]. These identities reduce (6.1) to the finite-dimensional system

$$\dot{c}_{k,n} = -\gamma n c_{k,n} - ik\sqrt{n+1}c_{k,n+1} - ik\sqrt{n}c_{k,n-1}, \quad (6.3)$$

with the conventions  $c_{k,-1} = 0$  and  $c_{k,N+1} = 0$  after truncation.

For the computation reported below we take  $\gamma = 1$ , Fourier truncation  $K = 6$ , and Hermite truncation  $N = 15$ . The initial condition is

$$g_0(x, v) = 1 + 0.25 \cos x + 0.20 v \sin x. \quad (6.4)$$

The constant term is the equilibrium state. The cosine term excites the first density mode, and the term  $v \sin x$  excites the first transport-coupled velocity mode. This makes the computation suitable for observing both direct velocity damping and transfer of dissipation to the spatial component.

Table 1 summarizes the numerical configuration used in the Fourier–Hermite approximation. The choice  $K = 6$  resolves the single nontrivial Fourier frequency present in the initial datum with a comfortable margin, while the Hermite truncation  $N = 15$  is sufficient to capture the transfer of dissipation from the first velocity moments to the density mode. On the time interval displayed below, increasing the truncation further does not modify the leading decay rate or the macroscopic profiles at the level visible in the figures.

Table 1: Parameters used in the Fourier-Hermite computation.

Relaxation parameter	$\gamma = 1$
Spatial domain	$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$
Fourier truncation	$ k  \leq 6$
Hermite truncation	modes $0, \dots, 15$
Initial datum	$g_0(x, v) = 1 + 0.25 \cos x + 0.20 v \sin x$

The first quantitative diagnostic is the weighted  $L^2$  distance from equilibrium,

$$\|f(t) - M\|_{L^2(M^{-1})}.$$

This is the natural observable associated with the decay estimate in Theorem 4.6. A semilogarithmic representation is therefore appropriate: once the short transient generated by the initial mode coupling has passed, exponential relaxation appears as an approximately affine graph. Figure 1 exhibits exactly that regime. A least-squares fit over the interval  $1 \leq t \leq 8$  gives the slope  $-1.091$ , which is stable to three decimal places under modest changes of the fitting window. In particular, the numerical trajectory is already governed by a single dominant decay scale on that interval.

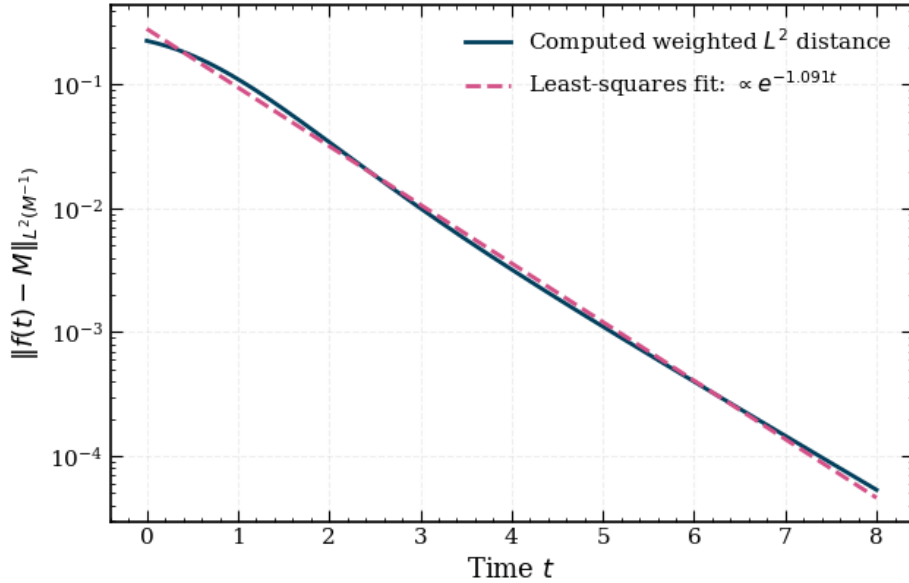


Figure 1: Semilogarithmic plot of the weighted  $L^2$  deviation from equilibrium for the Fourier-Hermite discretization of (6.1) with initial condition (6.4). The dashed line is a least-squares exponential fit over the interval  $1 \leq t \leq 8$ .

The second quantitative diagnostic concerns the spectrum of the truncated generator for the first nonzero Fourier mode. For  $k = 1$ , the matrix associated with (6.3) is

$$A_{1,N} = (a_{mn})_{0 \leq m,n \leq N}, \quad a_{nn} = -\gamma n, \quad a_{n,n+1} = -i\sqrt{n+1}, \quad a_{n,n-1} = -i\sqrt{n}.$$

Its eigenvalues determine the decay and oscillation scales present in the reduced dynamics. Figure 2 displays these eigenvalues in the complex plane for  $N = 15$ . The rightmost point of the spectrum lies at real part approximately  $-1.000$ , while the remaining eigenvalues are further to the left and therefore decay more rapidly. This is quantitatively consistent with the fitted slope in Figure 1. The role of Figure 2 is thus distinct: it does not record the decay of a particular trajectory, but identifies the spectral localization responsible for the observed long-time rate.

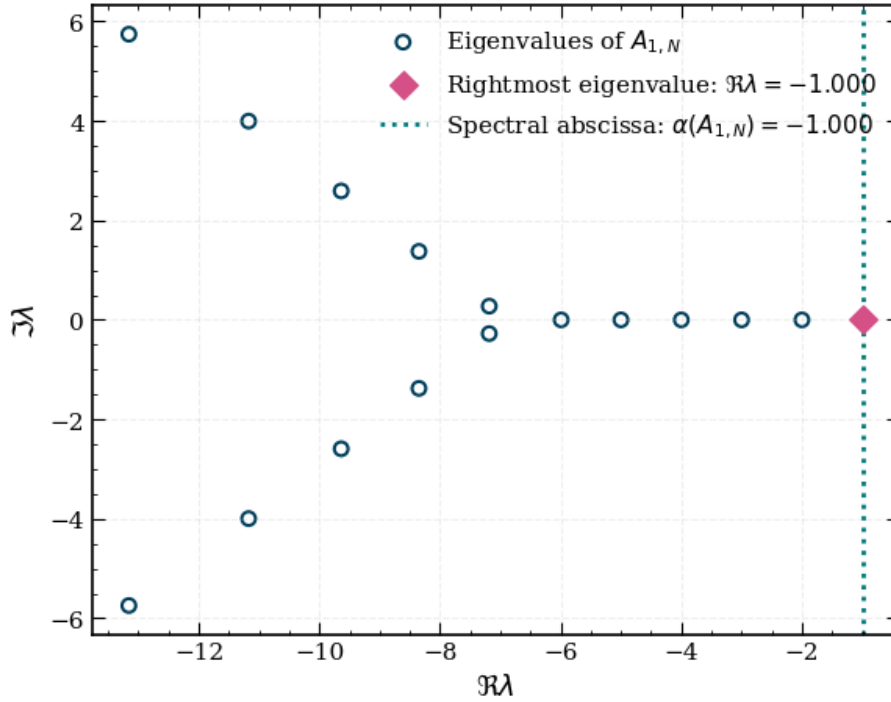


Figure 2: Spectrum of the truncated first Fourier block  $A_{1,N}$  with  $N = 15$  in the Fourier–Hermite discretization. The filled marker identifies the rightmost eigenvalue, and the dashed vertical line marks the spectral abscissa of the block.

The macroscopic density

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, v) \, dv$$

provides a third and complementary diagnostic. Since only the zeroth Hermite mode contributes to this integral, the density isolates the spatial component of the solution after averaging in velocity. Figure 3 displays the fluctuation  $\rho(t, x) - 1$  at four representative times. The dominant Fourier component remains sharply resolved throughout the computation, while its amplitude decreases monotonically from  $2.5 \times 10^{-1}$  at time  $t = 0$  to  $2.6 \times 10^{-3}$  at time  $t = 4$ . No spurious oscillation is visible at the scale of the plot. This confirms that the discretization captures the macroscopic relaxation without contamination from unresolved high-order modes.

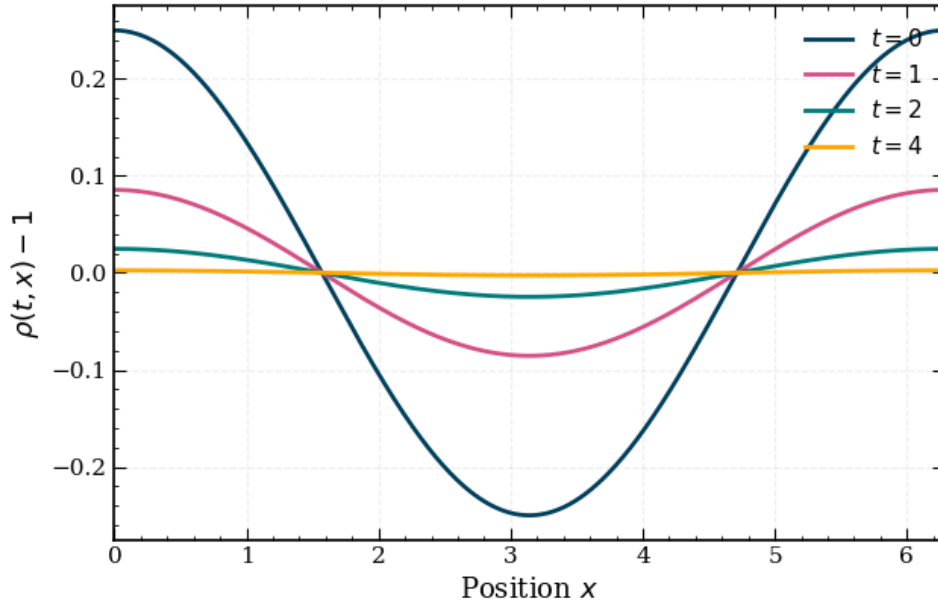


Figure 3: Snapshots of the density fluctuation  $\rho(t, x) - 1$  at times  $t = 0, 1, 2, 4$  for the Fourier-Hermite approximation. The leading spatial oscillation is preserved while its amplitude decreases steadily toward zero.

Taken together, the three diagnostics are consistent with the analytical picture developed earlier. The velocity component experiences direct Ornstein–Uhlenbeck damping, the truncated first Fourier block has its spectrum strictly in the left half-plane, and the full solution returns to equilibrium at an exponential rate after a short transient.

## 7 Conclusion

We have studied a linear transport-diffusion kinetic equation on  $\mathbb{T}^d \times \mathbb{R}^d$  in the weighted Hilbert framework associated with the Maxwellian equilibrium. The generator splits into a dissipative Ornstein-Uhlenbeck part in velocity and a conservative transport part in space. Within that framework one obtains a contraction semigroup, an explicit dissipation identity, conservation of mass, microscopic coercivity in velocity, and exponential convergence to equilibrium on the zero-mass subspace through a direct hypocoercive energy method. In the spatially homogeneous setting, the entropy law reduces to the Gaussian logarithmic Sobolev mechanism. The numerical section shows that the same relaxation pattern appears in a natural Fourier-Hermite discretization.

## Appendix: Flux formulation and geometric interpretation of transport–dissipation coupling

The purpose of this appendix is interpretive. It records a phase-space balance-law formulation of the transport-diffusion equation studied in the main text and relates that formulation to the hypocoercive mechanism used in Sections 4 and 5. No new analytical claim is introduced here. The discussion is included only to make explicit a geometric reading of the same transport–dissipation coupling already quantified by the modified energy functional.

### A.1. Phase-space flux

Consider again the kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \gamma \nabla_v \cdot (\nabla_v f + v f) \quad \text{on } \mathbb{T}^d \times \mathbb{R}^d. \quad (7.1)$$

It may be written in conservative form on phase space as

$$\partial_t f + \nabla_{x,v} \cdot \mathcal{J} = 0, \quad (7.2)$$

where  $\nabla_{x,v} = (\nabla_x, \nabla_v)$  and

$$\mathcal{J}(x, v, t) := (vf(x, v, t), -\gamma(\nabla_v f(x, v, t) + vf(x, v, t))). \quad (7.3)$$

Indeed,

$$\nabla_{x,v} \cdot \mathcal{J} = \nabla_x \cdot (vf) - \gamma \nabla_v \cdot (\nabla_v f + v f),$$

and since  $v$  is independent of  $x$ , one has  $\nabla_x \cdot (vf) = v \cdot \nabla_x f$ , so that (7.2) is equivalent to (7.1).

The two components of the flux have different structural roles. The spatial part

$$\mathcal{J}_x = vf$$

is induced by free transport and is conservative. The velocity part

$$\mathcal{J}_v = -\gamma(\nabla_v f + v f)$$

contains both diffusion and linear friction and therefore encodes the dissipative contribution.

### A.2. Local balance law

Let  $D \subset \mathbb{T}^d \times \mathbb{R}^d$  be a bounded domain with piecewise  $C^1$  boundary and outward unit normal  $n = (n_x, n_v)$ . Integrating (7.2) over  $D$  and applying the divergence theorem in phase space, one obtains

$$\frac{d}{dt} \int_D f(t, x, v) \, dx \, dv = - \int_{\partial D} \mathcal{J}(t, x, v) \cdot n(x, v) \, dS(x, v). \quad (7.4)$$

Thus the variation of mass inside  $D$  is determined by the net phase-space flux through  $\partial D$ . Formula (7.4) is the local counterpart of the global mass conservation identity proved in Proposition 3.3.

### A.3. Relation with hypocoercive transfer

The estimates of Sections 4 and 5 show that the operator is dissipative only in the velocity variable, while the spatial transport part is skew-symmetric in the weighted energy space. The flux formulation makes the same mechanism visible at the level of phase-space geometry.

More precisely, the velocity component  $\mathcal{J}_v$  acts directly on the dissipative variable and is responsible for the local smoothing and damping encoded by the Ornstein–Uhlenbeck operator. The transport component  $\mathcal{J}_x$  does not dissipate energy by itself, but it redistributes phase-space density along spatial directions. Because the equation couples  $x$  and  $v$  through the streaming term  $v \cdot \nabla_x f$ , the effect of velocity dissipation is not confined to purely velocity-dependent modes. It is transferred to the full phase-space dynamics through transport.

This is the geometric content of hypocoercivity in the present setting: dissipation is generated in one family of directions and propagated to the remaining directions by the conservative part of the flow.

#### A.4. Relation with the modified energy functional

In Section 5 the decay estimate is obtained from a modified energy containing the mixed term

$$\langle \nabla_x g, \nabla_v g \rangle_{L^2(dx \, d\mu_v)}.$$

That term has a direct interpretation in the flux picture. It measures, at the level of derivatives, the coupling between spatial transport and velocity dissipation. In particular, it quantifies the fact that the conservative part of the dynamics continually converts spatial variations into velocity variations, where coercive control is available.

It is the analytic expression of the same transport–dissipation coupling that appears geometrically in the phase-space flux (7.3). The appendix therefore does not alter the proof of exponential convergence, but only records an equivalent geometric reading of the mechanism already established in the main text.

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