An Inconsistency in Birkhoff's Theorem re Spherical Shells

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Abstract

According to Birkhoff's theorem the interior and exterior vacuum intervals for a spherically symmetric shell are the flat Minkowski metric and the static Schwarzschild solution respectively. Although these results are generally accepted individually, they should be considered in conjunction with each other. Here we examine the junction of the interior and exterior vacuum intervals across a thin spherical shell. The results show Birkhoff's theorem is not self consistent. The interior interval cannot be the Minkowski metric and, for a non-static shell, the overall solution cannot be static. As such, Birkhoff's theorem should not be relied upon as a basis for determining the properties of a given space-time.

Keywords: Birkhoff's theorem, Schwarzschild, relativity, static, shell

Birkhoff's theorem can be stated as "any spherically symmetric vacuum solution of Einstein's equations must be static and must agree with the Schwarzschild solution" [1, p. 299]. The Schwarzschild solution being defined as

$$\mathrm{d}s^2 = \left(1 - \frac{2\,G\,M}{c^2\,r}\right) c^2\,\mathrm{d}t^2 - \frac{1}{1 - \dfrac{2\,G\,M}{c^2\,r}}\,\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\phi^2$ and the values G and M are Newton's gravitational constant and the gravitational mass respectively. In addition, according to Birkhoff's theorem and, as a consequence of the Schwarzschild solution (1) being independent of the radius of the gravitational mass, the space-time curvature of any spherically symmetric mass is the same as if all the mass were concentrated at a central point (see, for example, [2, p. 24], [3, p. 52], and [4, p. 337]). These assertions lead directly to a corollary of Birkhoff's theorem: The interior vacuum interval of any spherically symmetric shell is the flat Minkowski metric

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{2}$$

The general argument for this corollary is that the Schwarzschild solution must hold in the interior of a hollow shell. However, to ensure that there is no singularity at the center of the shell (i.e., when $r = 0$), the mass M in the interior interval must be set to 0, thereby reducing the Schwarzschild solution (1) to the flat Minkowski interval (2). Similar arguments can be found in [1, p. 301], [4, pp. 337-338], [5, pp. 40-41], [6, p. 244], [7, pp. 230-231], and [8, pp. 289-290].

For an (infinitesimally) thin shell these arguments lead to the compound interval

$$\mathrm{d}s^2 = \begin{cases} c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 & \text{if } r \le r_m, \quad (3a) \\ \left(1 - \dfrac{2\,G\,M}{c^2\,r}\right) c^2\mathrm{d}t^2 - \dfrac{1}{1 - \dfrac{2\,G\,M}{c^2\,r}}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 & \text{if } r \ge r_m \quad (3b) \end{cases}$$

where r_m denotes the radius of the shell.[1] [2] Because both (3a) and (3b) apply at r_m, these intervals should match at this radius. However, except when $M = 0$, these intervals are notably different at this boundary. This difference leads to a number of discontinuities.

For example, as the shell is approached from the outside, a standard clock will run slower until, upon reaching the exterior surface of the shell, it would be running at a rate $\sqrt{1 - 2GM/c^2 r_m}$ the rate of a similar clock at infinity. However, in this compound interval, the rate at infinity is the same as that in the interior of the shell—including on its interior surface. Consider an outgoing photon emitted from this interior surface. For the frequency of this photon to match the value at infinity, it would need to be instantaneously blueshifted as it passed through the shell. Then, from the exterior surface to infinity, it would be redshifted such that, at infinity, it will have returned to its initial frequency. However, if this same photon is considered to have been emitted from the exterior surface, rather than the interior surface, then it would only be redshifted and would have a notably different frequency at infinity. Other accountings for this discontinuity can be found in [10, 11].

To eliminate the discontinuities in (3) we could modify the interior interval (3a), the exterior interval (3b), or both. Modifying the exterior interval to match the interior interval (3a) would require the exterior gravitational field vanish on the surface of a spherical mass. Because this is physically unrealistic, we will not consider it further.

If, instead, we relax Birkhoff's requirement that the interior vacuum interval be the Minkowskian metric (2) and simply require this interior interval be flat,

[1] Except for a reversal of the signature, this is the interval given in [9, p. 556] for a spherical shell of dust.

[2] As implied by the use of the same set of coordinate names in this, and other compound intervals, the coordinates for both the interior and exterior regions must be defined in the same manner.

then we get the interval derived in [11] for an empty shell:

$$
\mathrm{d}s^2 = \begin{cases} \left(1 - \dfrac{2\,G\,M}{c^2\,r_m}\right) c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 & \text{if } r \le r_m, \quad \text{(4a)} \\ \left(1 - \dfrac{2\,G\,M}{c^2\,r}\right) c^2\mathrm{d}t^2 - \dfrac{1}{1 - \dfrac{2\,G\,M}{c^2\,r}}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 & \text{if } r \ge r_m. \quad \text{(4b)} \end{cases}
$$

In this interval, a clock on the interior surface of the shell would run at the same rate as a clock on the exterior surface. Nevertheless, there is still a discontinuity in the radial component of this interval—i.e., "the time term of the metric is always maintained continuous, but the space term is not" [11, p. 3]. This discontinuity in the radial term leads to a new set of problems, similar to those caused by the mismatch in the time-like term.

One of these new problems can be seen if one considers a shell with a coordinate thickness Δr. In the interval (4) (as well as (3)) the measured thickness of the shell is Δr when using the interior portion of the interval, but $\Delta r/\sqrt{1 - 2GM/c^2 r}$ when measured using the exterior radial term.[3] In the limit as $\Delta r \to dr$ these estimated thicknesses become more accurate, but also more disparate, until, when $\Delta r = dr$, the thickness is both dr and $dr/\sqrt{1 - 2GM/c^2 r_m}$ simultaneously.

A related problem occurs if we consider the path of a photon as it passes through a thin shell defined according to (4). As the interior space-time is flat in this interval, the radial and tangential speeds of light are the same everywhere in this region. In addition, the tangential speed of light on the interior and exterior surface of the shell must be the same because, in this interval, we have specifically set the time-like term to be the same on both surfaces and because the interior and exterior circumferences of a thin shell must be the same. However, the radial and tangential speeds of light in the Schwarzschild interval (1) are not the same. This is directly implied in [7, pp. 230-231] where they state

> By a simple change of the radial coordinate ... we can recast the Schwarzschild metric ... into the following so-called *isotropic form* (which makes the coordinate speed of light direction-independent).

Therefore the path of a photon crossing the shell at an angle (i.e., a non radial geodesic) cannot be smooth and a beam of light would exhibit a notable discontinuity in its direction as it crossed the shell.[4] This is demonstrated in Figure 1 where we have plotted the path of a photon crossing a thin shell, radius $r_m = 2.2\,G\,M/c^2$, at an interior angle of $\frac{1}{4}\pi$.

[3] Given this exterior radial term is a function of the mass and radius of the shell, the actual measured thickness will be a function of both its radius and its density profile.

[4] As stated in [12, p. 3] "At the surface of the sphere it must be [pressure]= 0, and there the functions f together with their first derivatives must reach with continuity the values ... that hold outside the sphere."

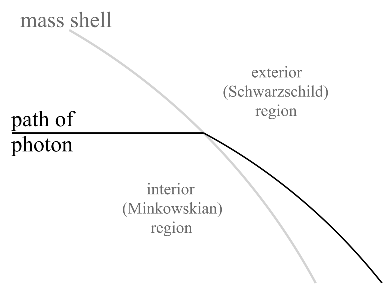

Figure 1: The path of a photon crossing a thin shell where, according to Birkhoff's theorem, the interior interval is given by the Minkowski interval $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$ and the exterior interval is the classic Schwarzschild solution $\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2\mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$ (i.e., equations (3a) and (3b) in the main text). In this example we have set the radius of the shell at 1.1 times its Schwarzschild radius and the interior angle at which the photon crosses the shell at $\frac{1}{4}\pi$. This clearly shows the path is not smooth and, as such, Birkhoff's theorem with respect to the interior of a thin shell is not consistent with its requirement that the exterior interval be the Schwarzschild solution. Note: this graphic applies equally to Eq. (4a) where the interior interval has been changed to $\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$

This discontinuity cannot be rectified by modifying the radial term in the interior portion of the interval. For the interval inside the shell to remain flat, the spatial components must be of the form $C(dr^2 + r^2\mathrm{d}\Omega^2)$ where C is some non-zero constant. Because C modifies both the radial and tangential terms of the interval equally, either one or both of the interior radius, or the circumference of the shell, cannot match the corresponding exterior values. If, instead of requiring the interior interval be of the form $C(dr^2 + r^2\mathrm{d}\Omega^2)$, we simply set the radial term to $\left(1 - 2GM/c^2 r_m\right)^{-1}$ so that it matches the exterior value, then the radial and tangential speeds of light in the interior of the shell would not be the same and the interior interval would not be flat. These results, however, suggest a possible solution.

Using the coordinate change

$$r = \rho \left(1 + \frac{G\,M}{2\,c^2\,\rho}\right)^2 \tag{5}$$

the Schwarzschild solution can be transformed into the isotropic form (Eq. (6b) given below). Then, by matching the coefficients of the terms of the interior interval to the values on the exterior surface we obtain the new interval:

$$
\mathrm{d}s^2 = \begin{cases} \left(\dfrac{1-\dfrac{G\,M}{2\,c^2\,\rho_m}}{1+\dfrac{G\,M}{2c^2\,\rho_m}}\right)^2 c^2\mathrm{d}t^2 - \left(1+\dfrac{G\,M}{2\,c^2\,\rho_m}\right)^4 \left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^{\,2}\right) & \text{if } \rho \le \rho_m, \quad (6a) \\ \left(\dfrac{1-\dfrac{G\,M}{2\,c^2\,\rho}}{1+\dfrac{G\,M}{2\,c^2\,\rho}}\right)^2 c^2\mathrm{d}t^2 - \left(1+\dfrac{G\,M}{2\,c^2\,\rho}\right)^4 \left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^{\,2}\right) & \text{if } \rho \ge \rho_m \quad (6b) \end{cases}
$$

where ρ_m is defined as the radius of the shell in these new coordinates. Because we have specifically chosen the coefficients in the interior portion of the interval to be the same as the exterior coefficients at ρ_m, both the time and radial coordinates must be smooth and continuous across the shell. In addition, because both the radial and tangential terms in the interior portion of the interval use the same constant, $(1 + MG/2c^2\rho_m)^4$, the interior interval (6a) will be flat. However, this interior interval is not the simple Minkowski interval specified by Birkhoff's theorem, but rather an interval that depends on both the mass and radius of the shell.[5] Unlike when using the interval (4), a beam of light crossing a thin shell defined by the isotropic interval (6) can be both smooth and continuous.

Although the isotropic interval (6) resolves the continuity problem across the junction of a thin shell, it raises additional problems. For example, if (6a) correctly represents the interior vacuum interval of a spherically symmetric shell then, when expressed in the standard Schwarzschild coordinates, the interval (6) is

$$
\mathrm{d}s^2 = \begin{cases} \left(1 - \dfrac{2\,G\,M}{c^2\,r_m}\right) c^2\mathrm{d}t^2 - \chi^2\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^{\,2} & \text{if } r \le r_m, \quad (7a) \\ \left(1 - \dfrac{2\,G\,M}{c^2\,r}\right) c^2\mathrm{d}t^2 - \dfrac{1}{1 - \dfrac{2\,G\,M}{c^2\,r}}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^{\,2} & \text{if } r \ge r_m \quad (7b) \end{cases}
$$

where

$$
\chi = \frac{1}{2}\left(\frac{\dfrac{G\,M}{c^2\,r_m}}{1 - \dfrac{G\,M}{c^2\,r_m} - \sqrt{1 - \dfrac{G\,M}{c^2\,r_m}}}\right)\left(1 - \frac{1 - \dfrac{G\,M}{c^2\,r}}{\sqrt{1 - \dfrac{2\,G\,M}{c^2\,r}}}\right). \qquad (7c)
$$

If the Schwarzschild solution (i.e., (1) and (7b)) directly represents the exterior vacuum space-time of a spherically symmetric mass (as required by Birkhoff's

[5]Other arguments for using isotropic coordinates can be found in [13] although they replace the tangential term with $r^2(1 - 2GM/c^2r)$ in the standard Schwarzschild solution (1) rather than using the isotropic interval (6b).

theorem) then, as shown by (7a) and (7c), the interior vacuum interval of a thin shell cannot be flat.[6] In addition, because the interior vacuum intervals (6a) and (7a) are functions of the radius of the shell, a non-static shell will produce a non-static interior vacuum interval. This contradicts the common interpretation of Birkhoff's theorem with regard to a non-static shell as found in texts such as [2, p. 24], [4, p. 337], [7, pp. 230-231], [8, pp. 289-290], and [14, p. 338]. It also implies that a non-static shell can produce spherical gravitational waves in its interior but, because Birkhoff's theorem rules out spherical waves in the exterior vacuum region, these gravitational waves would not be able to penetrate the shell. This appears to be highly unlikely and brings into questions the validity of Birkhoff's theorem. Using the relative rates of clocks [10] makes a similar, but more muted, claim re a non-static shell producing a non-static vacuum metric.

In summary, Birkhoff's theorem states that the interior vacuum interval of a spherically symmetric shell is the flat Minkowski metric and the exterior interval is given by the Schwarzschild solution. As has been shown, these intervals cannot be smoothly joined across a thin shell and, based on these intervals, the geodesic of a photon crossing a shell would not be smooth and continuous. To correct this, we examined a sequence of modifications to the interval, each repairing an aspect of the previous version, until we arrived at the flat, but non-Minkowskian, interior interval in isotropic coordinates (this interior interval is only flat when described in isotropic coordinates—in standard Schwarzschild coordinates this interior interval does not appear flat, as shown by (7)). Both the isotropic and the newly determined Schwarzschild interior intervals lead to a second, equally serious problem with Birkhoff's theorem: Because the interior vacuum intervals of a hollow shell are functions of the radius of the shell, a non-static shell must produce a non-static space-time. But, if the interior interval of a non-static shell is non-static, then it is likely the exterior vacuum interval of a non-static, spherically symmetric mass would also be non-static, in direct contradiction to Birkhoff's theorem.

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