Periodic Table of Prime Number Intervals: A Systems Approach

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Abstract

In accordance with the general systems theory of Yu.A. Urmantsev, systems of objects are constructed on the set of prime numbers. The relation between objects is taken as the difference of primes. Each prime number (except the first and the last) is assigned two normalized intervals: the left interval – the difference with the previous prime divided by 2, and the right interval – the difference with the next prime also divided by 2. The third coordinate (sequential number) records the multiplicity of a given combination of intervals. A three-dimensional array is obtained, whose projection onto the plane (x, y) reveals a strict periodicity: in the table of residues modulo 6, cells with equal non-zero residues $(x \bmod 3, y \bmod 3)$ are empty, while the remaining cells contain only 1 or 5. A lemma is proved that explains these regularities. It is shown that the systematisation does not provide a deterministic algorithm for finding the next prime, yet it reveals deep structural constraints.

Keywords: object-system, prime numbers, intervals, periodicity, modulo 6, periodic table.

Introduction

In the works of Yu.A. Urmantsev [1,2], a general approach to constructing systems of objects of any kind is proposed. In this article, this approach is applied to the set of prime numbers. The goal is to uncover hidden regularities that might be analogous to the periodic table of chemical elements. As Urmantsev wrote: *“One should try to identify the studied objects as object-systems and, without fear of any accusations, boldly construct systems of objects of the same kind. The results of such an approach will more than repay the effort”* [2].

We consider prime numbers as primary elements. The relation between them is the difference. Each prime number (except the endpoints) is characterised by three parameters: the left interval, the right interval, and the ordinal number of occurrence of that interval pair. This allows building a three-dimensional table (an analogue of the periodic table of elements). As a result, a strict periodicity modulo 6 is discovered and proved in the form of a lemma. The article contains a description of the method, the proof, the software implementation, and a discussion of practical significance.

1. Construction of the Object System

1.1. Basic Definitions

Let $P = \{p_1, p_2, p_3, \dots, p_N\}$ be the sequence of primes, with $p_1 = 2, p_2 = 3,\ p_3 = 5, \dots$. For every n from 2 to $N - 1$ we define the **normalised intervals**:

$x = (p_n - p_{n-1})/2$, $y = (p_{n-1} - p_{n-2})/2$.

Both intervals are positive integers because the differences between consecutive primes are even (except the pair (2,3), which we exclude from consideration).

1.2. Three-Dimensional Structure

Each prime p_n (except the first and the last) is assigned a tuple:

$(p_n, x_n, y_n, r_n = p_n \bmod 6, z_n)$,

where z_n is the sequential number of occurrence of the combination (x_n, y_n) in the sequence (starting from 1). In other words, if there are already $k - 1$ numbers with the same pair (x, y), then for the new number $z = k$.

We form a three-dimensional array $M[x][y]$ where each element is a vector (list) of primes having those x and y. The third coordinate z is the index inside that vector.

The maximum values of x and y are determined by the largest gap between consecutive primes in the chosen range. For a range up to 10^6, the maximal interval does not exceed 57, so the array size is about 57×57.

1.3. Two Variants of Interval Choice

- **Variant 1:** x – difference with the previous prime, y – difference with the next prime (as above).

- **Variant 2**: x – difference with the previous prime, y – difference between the previous and the one before that (both intervals look to the left).

The second variant leads to similar periodicities, but for predicting the next prime the first variant is more natural. In this article we use **Variant 1**, because it directly relates to the transition to the future prime.

2. Lemma on the Periodicity of Intervals

When constructing the table of residues $r = p_n \bmod 6$ for various pairs (x, y), the following regularities are observed, Table 1:

	1	2	3	4	5	6	7	8	9	10	11	12
1	5	1	1	0	1	1	0	1	1	0	1	1
2	5	0	5	5	0	5	5	0	5	5	0	5
3	5	1	5	5	1	5	5	1	1	5	1	5
4	0	1	1	0	1	1	0	1	1	0	1	1
5	5	0	5	5	0	5	5	0	5	5	0	5
6	5	1	1	5	1	1	5	1	5	5	1	1
7	0	1	1	0	1	1	0	1	1	0	1	1
8	5	0	5	5	0	5	5	0	5	5	0	5
9	5	1	1	5	1	5	5	1	5	5	1	1
10	0	1	1	0	1	1	0	1	1	0	1	1
11	5	0	5	5	0	5	5	0	5	5	0	5
12	5	1	1	5	1	5	5	1	1	5	1	5

Table 1

- Cells where $x \equiv 1 \pmod 3$ и $y \equiv 1 \pmod 3$ или $x \equiv 2 \pmod 3$ и $y \equiv 2 \pmod 3$ are empty (contain 0).
- In the remaining cells the value is either 1 or 5, and it is uniquely determined by x and y (except when x or y is a multiple of 3 – then both possibilities may occur).

We now state and prove a strict lemma.

Lemma.

Let $p_{n-2}, p_{n-1}, p_n -$ be three consecutive primes, $p_n > 3$.

Denote

$x = (p_n - p_{n-1})/2$, $y = (p_{n-1} - p_{n-2})/2$.

Then:

1. **Forbidden equal non-zero residues:**

   If $x \equiv y$ (mod 3) then necessarily $x \equiv y \equiv 0$ (mod 3).

   (That is, $x \equiv 1,\ y \equiv 1$ or $x \equiv 2,\ y \equiv 2$ cannot happen.)
2. **Left-interval rule (x):**

- If $x \equiv 1\ (mod\ 3)$, то $p_n \equiv 5\ (mod\ 6)$.
- If $x \equiv 2\ (mod\ 3)$, то $p_n \equiv 1\ (mod\ 6)$.

3. **Right-interval rule (**y**)**:

- If y ≡ 1 (mod 3), then p_n ≡ 1 (mod 6).
- If y ≡ 2 (mod 3), then p_n ≡ 5 (mod 6).

Rules 2 and 3 are consistent: for any admissible pair (x, y) they give the same value.

Proof.

All primes greater than 3 are of the form 6k+1 or 6k+5. Let

$a = p_{n-2}$ mod 6 $\in \{1,5\}$,

$b = p_{n-1}$ mod 6 $\in \{1,5\}$,

$c = p_n$ mod 6 $\in \{1,5\}$.

Compute $x = (p_n - p_{n-1})/2$.
Write $p_{n-1} = 6B + b$, $p_n = 6C + c$. Then
$p_n - p_{n-1} = 6(C - B) + (c - b)$

Since the difference is even, $c - b$ is even. Three cases are possible:

$b = \mathrm{c} \Rightarrow \mathrm{c} - \mathrm{b} = 0 \Rightarrow \mathrm{x} = 3(\mathrm{C} - \mathrm{B}) \equiv 0$ (mod 3);
$\mathrm{b} = 1, \mathrm{c} \Rightarrow 5 \Rightarrow\ \mathrm{c} - \mathrm{b} = 4 \Rightarrow \mathrm{x} = 3(\mathrm{C} - \mathrm{B}) + 2 \equiv 2$ (mod 3);
$\mathrm{b} = 5, \mathrm{c} \Rightarrow 1 \Rightarrow\ \mathrm{c} - \mathrm{b} = -4 \Rightarrow \mathrm{x} = 3(\mathrm{C} - \mathrm{B}) - 2 \equiv 1$ (mod 3)(так как $- 2 \equiv 1$ mod 3);

Similarly for $y = (p_{n-1} - p_{n-2})/2$:

$\mathrm{a} = \mathrm{b} \Rightarrow \mathrm{y} \equiv 0$ (mod 3);

$\mathrm{a} = 1, \mathrm{b} = 5 \Rightarrow \mathrm{y} \equiv 2 \pmod 3$;
$\mathrm{a} = 5, \mathrm{b} = 1 \Rightarrow \mathrm{y} \equiv 1 \pmod 3$;

Proof of part 1.

Suppose $x \equiv 1, y \equiv 1$. From $x \equiv 1$ we get $b = 5, c = 1$. From $y \equiv 1$ we get $a = 5, b = 1$. Contradiction: b would be both 5 and 1. The case $x \equiv 2, y \equiv 2$ is analogous. Hence equal non-zero residues are impossible.

Part 2.

$x \equiv 1 \Rightarrow \mathrm{a} = 5,\ b = 1 \Rightarrow b = 1 \Rightarrow p_n \equiv 1 \pmod 6$.
$x \equiv 2 \Rightarrow a = 1,\ b = 5 \Rightarrow p_n \equiv 5 \pmod 6$.

Part 3.

$y \equiv 1 \Rightarrow b = 5,\ c = 1 \Rightarrow p_n \equiv 5 \pmod 6$.

$y \equiv 2 \Rightarrow b = 1,\ c = 5 \Rightarrow p_n \equiv 1 \pmod 6$.

Consistency is easily verified: e.g., the combination $x \equiv 1,\ y \equiv 2$ gives by part 2 $p_n \equiv 1$, and by part 3 (since $y \equiv 2$ gives 1) the same result. The combination $x \equiv 2, y \equiv 1$ gives $p_n \equiv 5$ by both rules.

3. Software Implementation

To test the hypotheses and build the tables, a C++ program was developed [4]. The main steps:

- Generate all primes up to a given limit using the sieve of Eratosthenes.
- For each prime p_i (except the first and last) compute $x = \frac{p_i - p_{i-1}}{2}, y = \frac{p_{i+1} - p_i}{2}, r = p_i \% 6$.
- Fill the three-dimensional array $matrix_{p[x][y]}$ (vector of structures containing *p*, *r*, and the ordinal number *z*).
- Display on screen and write to a file the table of r values for a chosen range of *x*,*y* (up to 20 for compactness).
- Compute the maximum z (the number of primes in a single cell).
- Provide a function to transform a selected cell: replace each prime by the normalised difference to the previous prime inside the same cell (first replaced by 0) and save the results to UpdatedColumn.txt.

4. Results and Discussion

4.1. Residue Table Modulo 6

Example for the upper bound 540:

Table of *p* mod 6 for (*x*,*y*):

	1	2	3	4	5	6	7
1	5	1	1	0	1	1	0
2	5	0	5	5	0	5	5
3	5	1	5	5	1	5	0
4	0	1	1	0	1	0	0
5	5	0	5	0	0	0	5
6	5	1	0	5	0	1	0
7	0	1	1	0	0	0	0

Table 2

It is evident that rows and columns whose indices are not multiples of 3 show a regular alternation of 5 and 1, and cells with equal non-zero residues $(3n+1,3m+1)$ and $(3n+2,3m+2)$ are indeed empty (0). This fully agrees with the lemma.

4.2. Analysis of a Transformed Column

For cell (1,1) (both left and right intervals equal to 1) the normalised differences between consecutive primes inside that cell were obtained:

3, 3, 12, 30, 3, 42, 18, 42, 18, 57, 90, 90, 18, 12, 105, 93, 12, 63, ...

All numbers are multiples of 3. This is a consequence of all primes in that cell having the same residue modulo 6 (for (1,1) the residue is 5). The difference of two numbers of the form $6k + 5$ gives $6(k - m)$, and after division by 2 we obtain $3(k - m)$. This is confirmed theoretically.

The transformed column is obtained by replacing each prime p in the cell by the difference with the previous number from the same cell, divided by 2; the first number is replaced by 0.

4.3. Practical Significance

The lemma allows us to discard about half of the candidates when searching for the next prime by brute force. However, it does not provide a formula to compute the next prime exactly. Thus, the systematisation reveals structural constraints but does not replace factorisation algorithms or probabilistic primality tests. This resembles the situation with the periodic table of chemical elements: it systematises known elements and predicts empty places, but it does not derive properties from first principles (that was done later by quantum mechanics).

4.4. Further Directions

- Extending to modulus 30 (8 residue classes) to uncover finer periodicities.
- Statistical analysis of the distributions of d_i inside fixed cells – possible detection of preferred intervals.
- Building a directed graph of transitions between intervals (a finite automaton) and studying its properties.

5. Conclusion

Applying the general systems theory to the set of prime numbers has allowed constructing a three-dimensional periodic table of intervals. A lemma is proved that explains the regularity in the residue table modulo 6. A computer program has been developed that builds the array and performs data transformation. Although no practical algorithm for predicting the next prime is provided, the obtained results deepen our understanding of the structural regularities of primes and may serve as a foundation for further theoretical research.

References

1. Yu.A. Urmantsev. *General Systems Theory*. Moscow: Mysl’, 1988. (In Russian)
2. Yu.A. Urmantsev. General systems theory on relations of interaction, one-sided action and interaction. In: *The Problem of Connections and Relations in Materialist Dialectics*. Moscow: Nauka, 1990, pp. 101–137. (In Russian)
3. Yu.A. Urmantsev. Symmetry of system and system of symmetry. *Computers & Mathematics with Applications*, 12(1-2), 1986, pp. 379–405.
4. PrimesXYZ, https://drive.google.com/file/d/1yIncYZVsSWJrMYGfhxPIdq5MhZ6rrmc-/view?usp=sharing