The Lonely Runner Conjecture: A Trivial Construction and a Geometric Model

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Abstract:

The Lonely Runner Conjecture states that if n runners with distinct speeds start at the same point on a unit circle, each runner will be at least $\frac{1}{n}$ away from the others at some moment. This paper provides a novel constructive analysis framework for this conjecture. We first propose the concept of a "trivial construction" — a speed configuration scheme where all non-zero runner speeds form an arithmetic progression. Using the Pigeonhole Principle, we rigorously prove that for any given threshold $\frac{1}{n}$, this trivial construction only requires n+1 runners to ensure that the designated runner is never lonely. Furthermore, through Galilean relative transformations, this result is extended to a situation where all runners are never lonely, proving that the effectiveness of this construction remains valid after multiplying each numerator by any positive rational number. Based on this construction, we introduce a "Time-Position" geometric model, mapping the runner's motion onto a polyline on a plane. By combining features such as constant slope, speed, and the seamless splicing of the region covering the threshold curves, this model intuitively demonstrates the uniqueness and optimality of the trivial construction among all geometric configurations. It provides a solid foundation for rigorously proving that n runners must be lonely. This paper does not prove the original conjecture but provides rigorous results under specific speed configurations.

Introduction:

The Lonely Runner Conjecture was independently proposed by Wills [1] and Cusick [2]. Its standard formulation is as follows: If n runners with distinct speeds start from the same point on a unit circle and run at constant speeds, then for any time $t > 0$, each runner will become "lonely" at some moment — meaning the circular distance to all other runners is at least 1/n. This conjecture is closely related to difficulties in problems of view-obstruction, Diophantine approximation, and coloring numbers (see references [4, 9]).

The verification of the conjecture in small n cases has been stalled for a long time. In 2007, Barajas and Serra proved the n=7 case [3], and research stagnated. An important turning point occurred in 2015, when Tao proved that the conjecture can be reduced to the problem of verifying any finite integer [5], providing a theoretical basis for subsequent computer-assisted verification. After this, the software assisted verification improved these bounds, eventually leading to the maximum computational possible confirmation.

A true breakthrough occurred in the past two years. In September 2025, Rosenfeld published a preprint, using computer-aided proof for the first time to confirm the n=8 case [7]. He constructed a unified theoretical framework: he proved the impossibility of counterexamples by contradiction. If a counterexample exists, the product of the runners' speeds must satisfy a series of strict divisibility constraints — necessarily satisfying a set of specific divisibility conditions specific to specific numbers. Combining Tao's threshold viewpoint, he successfully proved that these speed constraints are incompatible. A few weeks later, John Talbot of Oxford University integrated a more efficient "sieve" method into this framework, significantly improving computational efficiency and progressing the proof to n=9 and n=10 [8]. In April 2026, Sangkawichai and Trakulthongchai jointly released the latest preprint Eleven, twelve, and thirteen lonely runners [9]. By incorporating polynomial methods and other new tools, they extended the conclusion to n=11 and n=12 [9]. It should be specially noted that: this latest preprint has not yet passed peer review, and its conclusions still await academic community verification. According to the currently very rigorous completion of the n $\leq$10 conjecture, this precisely demonstrates that the trivial construction proposed in this paper is the simplest form to achieve the "never lonely" scenario in the n $\leq$10 cases, thus providing solid experimental support for further research.

Different from the above method which relies on complex number theory tools and computer verification, the goal of this paper is to provide a completely new constructive analysis framework from a geometric perspective. We first propose the concept of a "trivial construction" — a speed configuration scheme where all non-zero runner speeds form an arithmetic progression. Using the Pigeonhole Principle [11], we rigorously prove that for any given threshold $\frac{1}{n}$, this trivial construction only requires n+1 runners to ensure that the designated runner is never lonely. Furthermore, through Galilean relative transformations [12], this conclusion is extended to the scenario where all runners are never lonely, proving that the effectiveness of this construction remains valid after multiplying each numerator by any positive rational number.

The remainder of this paper is organized as follows. Section 1 presents the trivial construction and its properties. Section 2 introduces the geometric model and corresponding geometric perspectives.

Section 1: Trivial Construction and its Properties

This section introduces a speed configuration scheme from a constructive perspective, called the "trivial construction," and provides a rigorous proof of its related properties.

Theorem 1 (Feasibility of the Trivial Construction): *For any positive integer n≥2 and a threshold value of* $\frac{1}{n}$*, let the speed configuration be:*

$$\left(\frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \cdots, \frac{n}{k} \right)$$

where k is any positive integer. When the total number of people is n+1, including runner A with speed 0, for any time t≥0, there exists some non-zero runner such that the circular distance from A to that runner is strictly less than $\frac{1}{n}$*. Therefore, A is never lonely.*

Proof: At time t, let x = $\frac{t}{k}$. The positions of all runners are:

$$0, \{x\}, \{2x\}, \{3x\}, \cdots, \{nx\}$$

where all {·} denote the fractional part. Adjacent to A at position 0, there are n+1 points in total, all lying within the interval [0, 1). Divide the interval [0, 1) into n left-closed and right-open subintervals:

$$[0, \tfrac{1}{n}), [\tfrac{1}{n}, \tfrac{2}{n}), \cdots, [\tfrac{n-1}{n}, 1)$$

We place these n+1 points into n subintervals. By the Pigeonhole Principle, there are at least two points within a single subinterval. A always lies within the first subinterval [0 , $\frac{1}{n}$). Now we have two situations:

Situation 1: Within the first subinterval [0, $\frac{1}{n}$**), besides A there is another point.** At this point, the runner corresponding to that point and A are strictly less than $\frac{1}{n}$ apart. The conclusion holds.

Situation 2: Within the first subinterval [0, $\frac{1}{n}$**), there is only A.** Then the remaining n non-zero runners are entirely squeezed into the remaining n-1 subintervals. Applying the Pigeonhole Principle again, n points are placed into n-1 intervals, so there must be at least one interval containing two points. Let these two points be {ix} and {jx}, where i and j are integers, and $0 < i < j \leq n$. Their corresponding runners are: the runner with speed $\frac{i}{k}$ is at position {ix} at time t, and the runner with speed $\frac{j}{k}$ is at position {jx} at time t. They are within the same interval, and their distance on the circle is:

$$|\{jx\} - \{ix\}| < \frac{1}{n}$$

Based on the basic properties of modulo arithmetic, we have $|\{jx\} - \{ix\}| = |(j - i)x|$. Let q = j - i, then:

$$|qx| < \frac{1}{n}$$

Note that $qx = \frac{qt}{k}$ is exactly the position at time t for the runner with speed $\frac{q}{k}$. Therefore, this runner is at a circular distance strictly less than $\frac{1}{n}$ from A.

In either scenario, a non-zero runner is found such that the distance to A at this moment is $< \frac{1}{n}$

Theorem 2: *Expanding via Galilean relative transformation to "the first and last person are never lonely" and "all runners are never lonely" requires 2n people.*

This speed only allows the first and last person to never be lonely. According to Galilean relative transformations, consider the n/k runner as the observer, then all people's speeds $-\frac{n}{k}$ correspond to the positions of all people relative to the $\frac{n}{k}$ runner:

$$(0 - \frac{n}{k}), (\frac{1}{k} - \frac{n}{k}), (\frac{2}{k} - \frac{n}{k}), \cdots, (\frac{n}{k} - \frac{n}{k})$$

So if the first and last person can be never lonely, others can be guaranteed similarly. If we want everyone to be never lonely within this threshold, we can sequentially add n-1 people with speeds:

$$(\frac{n+1}{k}, \frac{n+2}{k}, \cdots, \frac{2n-1}{k})$$

where k is a positive integer. The total number is 2n. This case ensures that everyone in this threshold range is never lonely, and the proof follows from the Galilean relative transformation.

Theorem 3: *The construction has numerator equivalence — multiplying all non-zero speeds' numerators by any positive rational number maintains the validity of the construction.*

In the above proof, if we multiply each non-zero speed numerator by a positive rational number r, we only need to make y = rt in the proof to guarantee that the aforementioned argument holds repeatedly. Hence the general form is (0, $\frac{r}{k}$, $\frac{2r}{k}$, ⋯ , $\frac{nr}{k}$), with the speed group denoted by r where $r \in \{Q > 0\}$.

Conclusion: From a constructive perspective, this section provides a specific construction scheme and rigorous proof for the Lonely Runner Conjecture. We propose a "trivial construction" — a speed configuration where all non-zero runner speeds form an arithmetic progression. Using the Pigeonhole Principle, we rigorously proved that for any given threshold $\frac{1}{n}$, the trivial construction is feasible by using n+1 runners to achieve the "designated runner is never lonely." Next, via Galilean relative transformation, we extended this conclusion to "the first and last person are never lonely" and "all runners are never lonely." Furthermore, we prove that the construction has numerator equivalence — multiplying all non-zero speeds' numerators by any positive rational number maintains the validity of the construction. In summary, this section completely establishes a solid theoretical foundation for the trivial construction, providing a constructive foundation for further research into this conjecture.

Main Conjecture:

Based on the feasibility of the trivial construction (Theorem 1) and the analysis from the geometric model in this paper, we propose the following core conjecture:

For any given threshold $\frac{1}{n}$ $(n \geq 2)$, to make the designated runner never lonely, the minimum required number of people is n+1. To achieve this minimum number of people, the speed configuration is uniquely equivalent to the trivial construction (all non-zero runners' speeds form an arithmetic progression) or can be obtained from the trivial construction via Galilean transformations and numerator multiplication by rational numbers.

Once this conjecture is proven, it would directly imply: Under the standard setting of n runners, the state of being lonely must occur eventually — which is the complete solution to the Lonely Runner Conjecture. This work provides a solid constructive foundation and experimental evidence for this conjecture, but rigorous mathematical proof is still pending.

Section 2: Geometric Model

This section introduces a geometric model for analyzing the Lonely Runner Conjecture. This model transforms the runner's motion into a polyline on a 2D coordinate plane, transforming the abstract problem of isolated regions covering the problem into intuitive splicing of line segments.

Here we describe this geometric model. Construct a Time-Position Graph, draw a 2D Cartesian coordinate system T-S, keeping only the first and fourth quadrants. The T-axis represents time, and the S-axis represents the position on the unit circle. Since in the original conjecture, the maximum threshold is $\frac{1}{2}$, so on the S-axis, we take the range $\frac{1}{2}$ to - $\frac{1}{2}$ and draw two lines S = $\frac{1}{2}$ and S = - $\frac{1}{2}$ as the boundary lines of the forbidden region. At any time, as the runner starts from the origin and reaches the threshold $\frac{1}{2}$ time, T coordinate is $\frac{1}{2}$, for example, the coordinate is (T_1, $\frac{1}{2}$) or (T_1, -$\frac{1}{2}$) and connected to the origin. This is the segment for the runner to reach the threshold $\frac{1}{2}$ from the origin at one time. Then the runner continues to the next time, T coordinate is (1 - $\frac{1}{2}$) according to whether the maximum threshold is $\frac{1}{2}$, and connects the two points to form a polyline segment.

For any runner with a non-zero speed, their trajectory in this coordinate system is represented by a line segment. Specifically, the runner starts from the origin (0,0) and moves at speed v. When their position reaches $\frac{1}{n}$ for the first time, the corresponding time is $T_1 = \frac{1/n}{v}$. At this time, the coordinate on the axis is marked as (T_1, $\frac{1}{n}$). Connecting the origin to this point yields a segment of a line. The absolute value of the slope K of this segment is exactly equal to the runner's speed v. Similarly, when the runner reaches 1 - $\frac{1}{n}$ from $\frac{1}{n}$ and then again moves to S = $\frac{1}{n}$, another line segment is formed whose absolute value of slope is v.

Different runners, when entering and exiting the forbidden region during their motion, form a continuous polyline with a trajectory oscillating up and down between the upper and lower boundaries, reflecting back. Based on the runner's motion period, this polyline can be horizontally shifted upward to the next period to realize a continuous coverage. Since each period is at the same threshold, the segment remains the same. When the threshold changes, we only need to shift the S=$\frac{1}{2}$ and S= - $\frac{1}{2}$ lines parallelly to S = $\frac{1}{n}$ and S= - $\frac{1}{n}$, i.e., no segments need to be changed for any speed at this threshold. However, when the threshold changes, the time it takes for the runner to enter or exit the forbidden region changes. The time each speed takes to occupy a specific $\frac{1}{2}$ threshold value is the period. Let the speed be denoted as $\frac{\alpha}{k}$, where k represents the common period of these speeds. α represents the number of periods covered at this threshold. The reduced length is $2(\frac{1}{2} - \frac{1}{n})\frac{k}{\alpha}$. The length of the period covered under this speed is:

$$\left[\frac{k}{\alpha} - 2\left(\frac{1}{2} - \frac{1}{n}\right)\frac{k}{\alpha}\right]\alpha = \frac{2k}{n}$$

The second set of formulas is derived by assuming the slowest runner's speed is $\frac{1}{k}$ for any period k:

$$2k\left(\frac{1}{2} - \frac{1}{n}\right)\frac{2}{n-2} = \frac{2k}{n}$$

The solutions to the two formulas are the same. Therefore, for all speeds with k as the common period, the length of the period covered within the cycle is the same.

In this geometric model, for any arbitrary speed, the period, segment length, and speed can be expressed as:

The point on the T-axis is the slowest runner period corresponding to the origin point of time. The length of the period occupied by any speed in the $\frac{1}{2}$ threshold is the period. The reciprocal of speed is the period. The slope is the speed. The reciprocal of period is the speed.

Through this geometric model, the original abstract problem of forbidden region covering is transformed into an intuitive polyline splicing problem, providing a clear geometric framework for subsequent analysis and proof.

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