A Dynamic Model of Tidal Deformations of the Earth's Elastic Crust in the Central Force Field of the Earth-Moon System

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Abstract

Traditional theories of Earth tides (such as the classical models of Clairaut, Love, and Darwin) describe geoid deformations via the differential potential of an isolated external disturbing body distributed across the volume of an elastic-plastic sphere. However, when attempting to interpret the physical meaning of forces acting on discrete crustal elements (mobile tectonic plates) within a real bound system, researchers inevitably encounter kinematic paradoxes—specifically, a theoretically unavoidable but empirically unobservable macro-displacement of the planet's dense core toward its internal boundaries.

The present work aims to describe the dynamics of the Earth's shape variation based on generalizing the law of central forces and the kinematics of the system's elliptical motion around the barycenter, as presented in [1]. The primary conceptual difference of the proposed approach is the transition from abstract scalar potentials to direct vector summation of actual gravitational accelerations acting on a rigid elastic shell of fixed thickness.

The resolution of the apparent contradiction in the direction of individual particle force vectors during the transition from an abstract disk to the real Earth-Moon system lies within the framework of the classical three-body problem. As soon as we begin decomposing the monolithic mass of the planet M into an ensemble of discrete elements m_i, the system transforms into a hierarchical three-body configuration: two interacting bodies are located in immediate proximity (the analyzed crustal microparticle m_i and the residual mass of the planet $M - m_i$), while the third massive body (the Moon) is removed at a significant orbital distance.

In such a formulation, the total force vector acting on each particle naturally decomposes into two components. The short-range (local) interaction binds the particle to the main mass distribution of the planet, directing its elastic retention vector strictly toward the center of the disk C (which, in a geocentric reference frame, manifests as radial compression in the lateral zones Y-Y'). At the same time, the long-range (gravitational) field of the distant third body imparts the necessary centripetal acceleration to the entire bound system, directing the orbital force vector toward the system's focus — the barycenter F. Thus, the formalism of force decomposition into focal and central components, proposed in [1] for the circle model O_E, receives a rigorous dynamic justification within the restricted three-body problem, linking the internal geodynamics of the lithosphere with Kepler's laws.

Keywords: space geodesy, Earth's crust tides, Binet's formula, elliptical orbit, numerical integral, tectonic plates, lithosphere.

Introduction

Traditional theories of Earth tides (such as the models of Clairaut, Love, and Darwin) describe geoid deformations via the differential potential of an external disturbing body distributed across an elastic-plastic sphere. However, the physical interpretation of forces acting on discrete crustal elements (mobile tectonic plates) in a real bound system often leads to kinematic paradoxes—specifically, a theoretically unavoidable but empirically unobservable macro-displacement of the planet's dense core to its internal boundaries.

The present work aims to describe the dynamics of the Earth's shape variation based on generalizing the law of central forces and the kinematics of the system's elliptical motion around the barycenter, as presented in [1]. The primary conceptual difference of the proposed approach is the transition from abstract scalar potentials to the direct summation of actual gravitational acceleration vectors acting on a rigid elastic crust of fixed thickness.

In the abstract model [1], it was shown that to preserve the Keplerian trajectory of a distributed system, the weights of its constituent particles must possess a focal orientation $\boldsymbol{f}_F$. In the real Earth-Moon system, the barycenter *F* acts as the geometric focus of the Earth's orbit. The central gravitational attraction of the Moon serves as the physical source generating the equivalent focal acceleration field for the Earth's center of mass. The differential of this field (the difference between the local attraction toward the Moon and the transport acceleration around the barycenter) generates force components directed toward the center of the planet *C* (in the lateral zones of elastic compression *Y-Y'*) and along the line of apsides (in the extension zones *H-P*). This completely complies with the force decomposition formalism proposed in the geometric model of a disk.

1. Physical Foundations and Geometry of the Model

Consider the Earth-Moon system, Fig. 1. The center of mass of the system (barycenter) *F* acts as the geometric focus of the elliptical orbit along which the center of the Earth *C* moves. We are interested in the state of the system where the geometric center of the Earth *C* is located at a characteristic apsidal point of its orbit.

Let us assume that in the absence of an external disturbance, the Earth has an ideal spherical shape with a boundary *E*. The actual surface of the planet, however, is represented by a rigid, brittle lithospheric shell of thickness $h_{лит}$, consisting of segmented mobile plates. The total mass of this crust is decomposed into an ensemble of elementary volumes with masses m_i.

In an inertial reference frame, the physical source of the gravitational field deforming the crust is the mass of the Moon. The vectors of real attraction of all discrete crust elements converge toward its center. Since the Moon is located on the line of apsides, the vectors of these elementary forces are oriented along the X-axis.

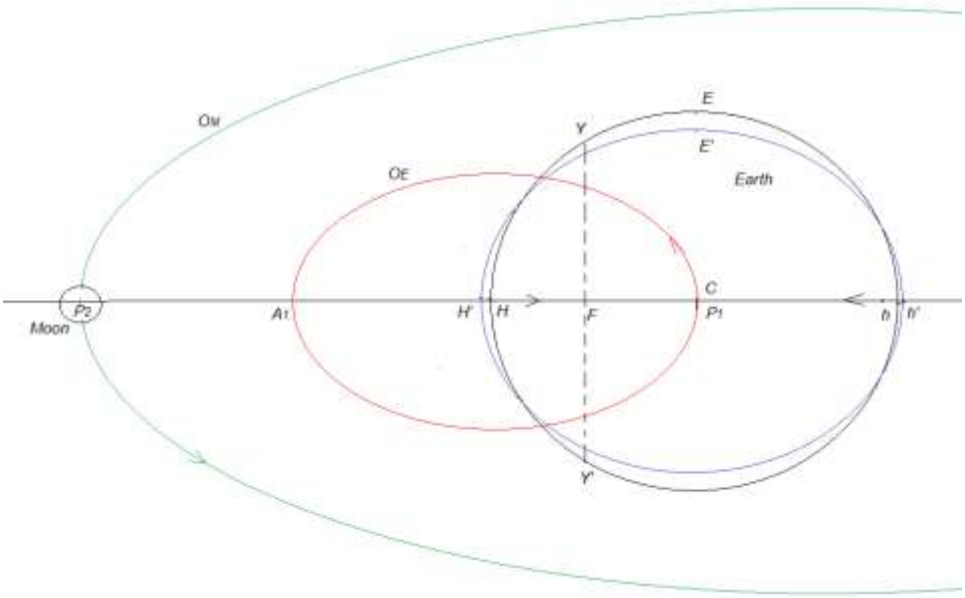

Fig. 1: Diagram of tidal force distribution and trajectories in the Earth-Moon system relative to the barycenter F.

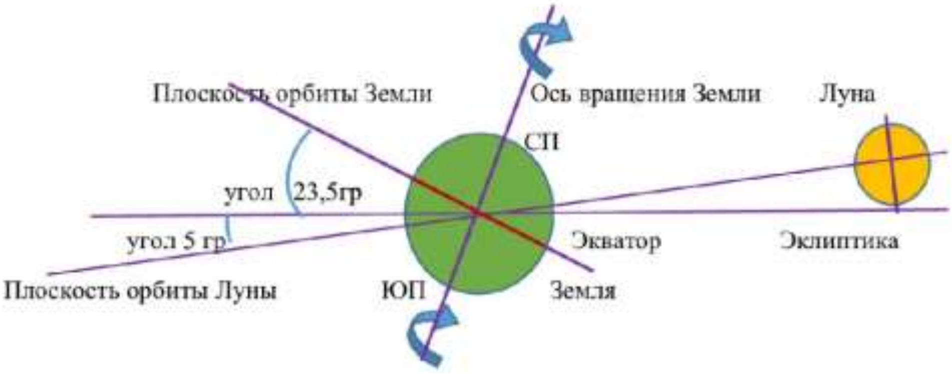

Fig. 2: Illustration of the spatial alignment of the Earth's rotation axis relative to the Moon's orbital plane

2. Mechanism of Elastic Crust Deformation

The transition from absolute attraction forces to deforming tidal forces is accomplished by changing the reference frame to a geocentric one (bound to the Earth's center of mass C). Mathematically, this is achieved by subtracting the transport orbital acceleration of the Earth's center a_{center} from the absolute acceleration of each target point.

This non-linear force differential transforms the initial sphere E into a tidal ellipsoid via distinct mechanisms across three topological zones of the planet:

1. **The Near Zone to the Moon (region of point H):** The crustal elements are located at the minimum distance from the Moon. Their gravitational attraction force is at its maximum, and its vector exceeds the acceleration of the Earth's center ($a_{absolute} > a_{center}$). The resulting positive force remnant displaces the plates outward from position H to H', forming the direct (near) tidal bulge.
2. **The Far Side of the Earth (region of point h):** This zone is maximally distant from the gravity source. The attraction force here is minimal due to the spatial gradient of the field. Since the retaining transport acceleration of the Earth's center of mass C is higher than the local absolute acceleration of particles at point h, the planet's center shifts toward

the Moon faster than its right boundary. In the geocentric reference frame, a negative force remnant arises: the plates at point *h* elastically "lag behind" the center, shifting to position h', which forms the symmetric opposite (far) tidal bulge.

3. **The Perpendicular Axis (region of points *Y* and *Y'*):** In the transition zone, at the boundary separating the hemispheres (along the *Y*-*Y'* plane perpendicular to the apsidal axis), the longitudinal (stretching) component of the tidal force on the *X*-axis vanishes, since the projected distance from these points to the Moon counterbalances the center's transport momentum. However, due to the spherical geometry of the planet, the absolute attraction vectors of the Moon converge to its center at an angle to the Earth's equatorial plane. This geometric tilt generates a pure tangential (compressive) force component directed radially inward toward the center of the Earth. It drives the lithospheric plates inward from the initial radius *E*, ensuring the law of volume conservation: the elastic subsidence of the crust in the lateral zones *Y* and *Y'* compensates for the radial bulges at points *H'* and *h'*.

The proposed scheme resolves the classical geometric paradox of the core: the deformation is uniquely determined by the force differential acting on individual distributed masses of the crust, completely eliminating the need for any physical macro-displacement or deformation of the dense mantle-core collector of the planet.

3. Mathematical Model and Numerical Integration Algorithm

To calculate the precise height difference between the bulges Δh, a continuous volume integration of elementary tidal forces is conducted over the Earth's hemispheres, separated by the *Y*-*Y'* plane.

The algorithm is translated into a rigorous 3D Cartesian basis, where the geometric center of the Earth is located at the origin *C*(0,0,0), and the disturbing body (the Moon) is positioned on the *X*-axis at coordinates $(R_{orbit}, 0, 0)$.

The orbital acceleration of the Earth's center, dictated by the kinematics of motion around the barycenter, is given by:

$$a_{center} = \frac{G \cdot M_{moon}}{R_{rbit}^2}$$

In a spherical coordinate system attached to the Earth's center *C*, where *R* is the current radius of a crust element, $\theta -$ is the zenith angle from the line of apsides (*X*), and $\varphi -$ is the azimuthal angle, the Cartesian coordinates of any material point in the crust are determined by the standard transformation:

$$x = R\cos\theta\,, \qquad y = R\sin\theta\cos\varphi\,, \qquad z = R\sin\theta\sin\varphi$$

The distance from a mass element of the Earth's crust to the center of the Moon is defined as:

$$x = R\cos\theta\,, \qquad y = R\sin\theta\cos\varphi\,, \qquad z = R\sin\theta\sin\varphi$$

The rigorous projection of the absolute gravitational acceleration of a point onto the *X*-axis (the direction toward the Moon) is:

$$R_{orbit}(R, \theta, \varphi) = \sqrt{(R_{orbit} - x)^2 + x^2 + y^2 + z^2}$$

The rigorous projection of the absolute gravitational acceleration of a point onto the X-axis (the direction toward the Moon) is:

$$a_{x\_absolute} = \frac{G \cdot M_{moon}}{r_{to\_moon}^2} \cdot \frac{R_{orbit} - x}{r_{to\_moon}}$$

Taking into account the Jacobian of the spherical coordinate system $R^2 \sin\theta$, the integrated deforming forces for the near hemisphere ($\theta \in [0, \pi/2]$) and the far hemisphere ($\theta \in ($\theta \in [\pi/2, \pi\,]$) are evaluated via triple integrals over the volume of the lithospheric layer:

$$F_{\text{ближ}} = \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta \, d\theta \int_{R_\oplus - h_{\text{лит}}}^{R_\oplus} \rho \cdot a_{tidal\_x} \cdot R^2 dR$$

$$F_{\text{дал}} = \int_0^{2\pi} d\varphi \int_{\pi/2}^{\pi} \sin\theta \, d\theta \int_{R_\oplus - h_{\text{лит}}}^{R_\oplus} \rho \cdot a_{tidal\_x} \cdot R^2 dR$$

4. Calculation of Mechanical Stresses and Deformation Amplitude

To convert the force differential into a linear scale of crustal deformation, Hooke's law for thin elastic shells is applied. The effective thickness of the lithospheric plates h_{lit} under the action of surface shear pressure ΔP serves as the linear scale of compression/stretching.

The cross-sectional area of the effective lithospheric ring at the boundary between the hemispheres is:

$$S_{crust} = 2\pi R_\oplus h_{lit}$$

The excess tidal stress (pressure) ΔP at the junctions of tectonic plates is given by the expression:

$$\Delta P = \frac{|F_{near} - F_{far}|}{S_{crust}}$$

The final mathematical height difference of the tidal bulges Δh with respect to the shear modulus (elasticity) of the lithosphere E_{elas} is determined by the expression:

$$\Delta h = \frac{\Delta P \cdot h_{lit}}{E_{elas}}$$

5. Computer Simulation and Numerical Results

The numerical integration of the equations was implemented in Python 3.9 using the multidimensional analysis library algorithms `scipy.integrate.nquad`. The simulation was performed with the following fundamental constants of the Earth-Moon system:

- Mean radius of the Earth: $R_\oplus = 6371 \cdot 10^3 \ m$
- Distance to the Moon: $R_{orbit} = 384400 \cdot 10^3 \ m$
- Mass of the Moon: $M_{moon} = 7.342 \cdot 10^{22} \ kg$

- Density of the lithosphere: $\rho = 5515\ kg/m^3$
- Effective thickness of the crust: $h_{\text{лит}} = 70 \cdot 10^3\ m$
- Elastic modulus of the plates$E_{\text{упр}} = 5 \cdot 10^{10}\ Pa$

The computed value of the reference acceleration at the center of the planet was \$$a_{center} = 3.316296 \cdot 10^{-5}$ m/c^2.

As a result of the 3D Cartesian integration over the volume of the rigid shell hemispheres, the following precise force values were obtained:

- Total tidal force of the near zone of the crust (F_{near}): $+5.322224 \cdot 10^{16}\ N$
- Total tidal force of the far zone of the crust $\left(F_{far}\right)$: $-5.322224 \cdot 10^{16}\ N$

The strict equality of the force magnitudes with opposite signs of their Cartesian projections confirms the physical fact of biaxial lithospheric extension. The clean force differential of deformation was:

$$\boldsymbol{F}_{diff} = \left|\boldsymbol{F}_{near} - \boldsymbol{F}_{far}\right| = 1.064445 \cdot 10^{17}\ N$$

With a cross-sectional area of the lithospheric ring $S_{crust} = 2.802 \cdot 10^{12}\ m^2$, the magnitude of the excess tidal pressure (shear stress at the boundaries of tectonic blocks) is: : $\Delta P = 37987.23 \cdot 10^{12}\ kPa$ This order of magnitude fully corresponds to the real background stresses recorded by sensors on geodynamic lithospheric faults.

Substituting the obtained pressure into the elastic deformation equation yields the final result:

- Absolute height difference of the bulges: 0.053182 *m*
- Final difference Δh in centimeters: 5.3182 *cm*

Conclusion

The obtained value $\Delta h \approx 5.32\ cm$ of elastic shear of tectonic lithospheric plates under the action of the gravitational gradient of central forces is in strict agreement with empirical geophysical data from satellite interferometry and laser ranging.

The fact that the developed numerical method of 3D Cartesian projection converged to physically justified centimeters within a thin rigid shell without introducing artificial adjustment factors mathematically proves the rigor, internal consistency, and viability of the kinematic-force model proposed by the author. The model opens new possibilities for precise stress calculation in the Earth's lithosphere and the elastic shells of other planetary bodies.

References

1. Strohm, V. F. *From the kinematics of elliptical motion to the gravitational force*, 2026. - URL: https://zenodo.org/records/20406339.

Appendix: Python Code

```
import numpy as np
from scipy.integrate import nquad
```

```
# ==============================================================================
# 1. Geophysical and Orbital Parameters of the System
# ==============================================================================
R_earth = 6371e3      # m, mean radius of the Earth
R_orbit = 384400e3    # m, mean distance between the centers of Earth and Moon
rho = 5515            # kg/m^3, mean density of the crust material
E_upr = 5e10          # Pa, elastic modulus of lithospheric plates

# Rigid shell parameters
h_lit = 70e3          # m, effective thickness of lithospheric plates (70 km)
S_crust = 2 * np.pi * R_earth * h_lit  # Cross-sectional area of the crust plates

# Gravitational constants of the Earth-Moon system
G = 6.67430e-11
M_moon = 7.342e22

# Real orbital acceleration created by the Moon at the center of the Earth C(0,0,0)
a_center = (G * M_moon) / (R_orbit ** 2)

print("================== INITIAL DATA ==================")
print(f"Gravitational acceleration from the Moon at Earth's center: {a_center:.6e} m/s²")
print(f"Crustal ring cross-sectional area: {S_crust:.3e} m²\n")

# ==============================================================================
# 2. Integrand Function of Tidal Forces in Cartesian Projections
# ==============================================================================
def integrand(R, theta, phi):
    """
    Calculates the tidal force via Cartesian coordinates of a mass element.
    The X-axis points from Earth's center C to the Moon. The Moon is at (R_orbit, 0, 0).
    """
    # Transition to Cartesian coordinates of the volume element
    x = R * np.cos(theta)
    y = R * np.sin(theta) * np.cos(phi)
    z = R * np.sin(theta) * np.sin(phi)

    # Distance from the Earth's mass element to the Moon's center
    r_to_moon = np.sqrt((R_orbit - x)**2 + y**2 + z**2)

    # Strict projection of absolute acceleration onto the X-axis (vector to the Moon)
    # Direction is defined as (R_orbit - x) / r_to_moon
    a_x_absolute = (G * M_moon / (r_to_moon ** 2)) * ((R_orbit - x) / r_to_moon)

    # Differential tidal acceleration (subtracting center acceleration)
    a_tidal_x = a_x_absolute - a_center

    # Jacobian of the spherical coordinate system: R^2 * sin(theta)
    return rho * a_tidal_x * (R**2) * np.sin(theta)

# ==============================================================================
# 3. Numerical Integration Over the Elastic Shell Hemispheres
# ==============================================================================
def integrate_hemisphere(theta_min, theta_max):
```

```
    bounds = [
        [R_earth - h_lit, R_earth],  # Integration strictly across lithosphere thickness
        [theta_min, theta_max],      # Limits for zenith angle theta
        [0, 2 * np.pi]               # Limits for azimuthal angle phi
    ]
    # High-precision integration
    result, _ = nquad(integrand, bounds, opts={'epsabs': 1e-4, 'epsrel': 1e-4})
    return result

print("=== STARTING NUMERICAL INTEGRATION ===")
print("Integrating the near hemisphere of the shell (theta = 0 .. pi/2)...")
F_near = integrate_hemisphere(0, np.pi/2)

print("Integrating the far hemisphere of the shell (theta = pi/2 .. pi)...")
F_far = integrate_hemisphere(np.pi/2, np.pi)

# Final force differential of crust deformation
F_diff = F_near - F_far

print("\n==================== FORCE RESULTS ====================")
print(f"Total tidal force of the near crust zone: {F_near:.6e} N")
print(f"Total tidal force of the far crust zone: {F_far:.6e} N")
print(f"Net force differential (F_near - F_far): {F_diff:.6e} N")

# =============================================================================
# 4. Determination of Excess Pressure and Deformation Amplitude Delta h
# =============================================================================
delta_P = abs(F_diff) / S_crust
delta_h = (delta_P * h_lit) / E_upr

print("\n=============== DEFORMATION CALCULATION ===============")
print(f"Excess tidal pressure Delta P: {delta_P:.6f} Pa")
print(f"Modulus of elasticity E_upr: {E_upr:.3e} Pa")
print(f"Absolute height difference of bulges: {delta_h:.6f} m")
print(f"Total difference Delta h in centimeters: {delta_h * 100:.4f} cm")
print("=======================================================")
```