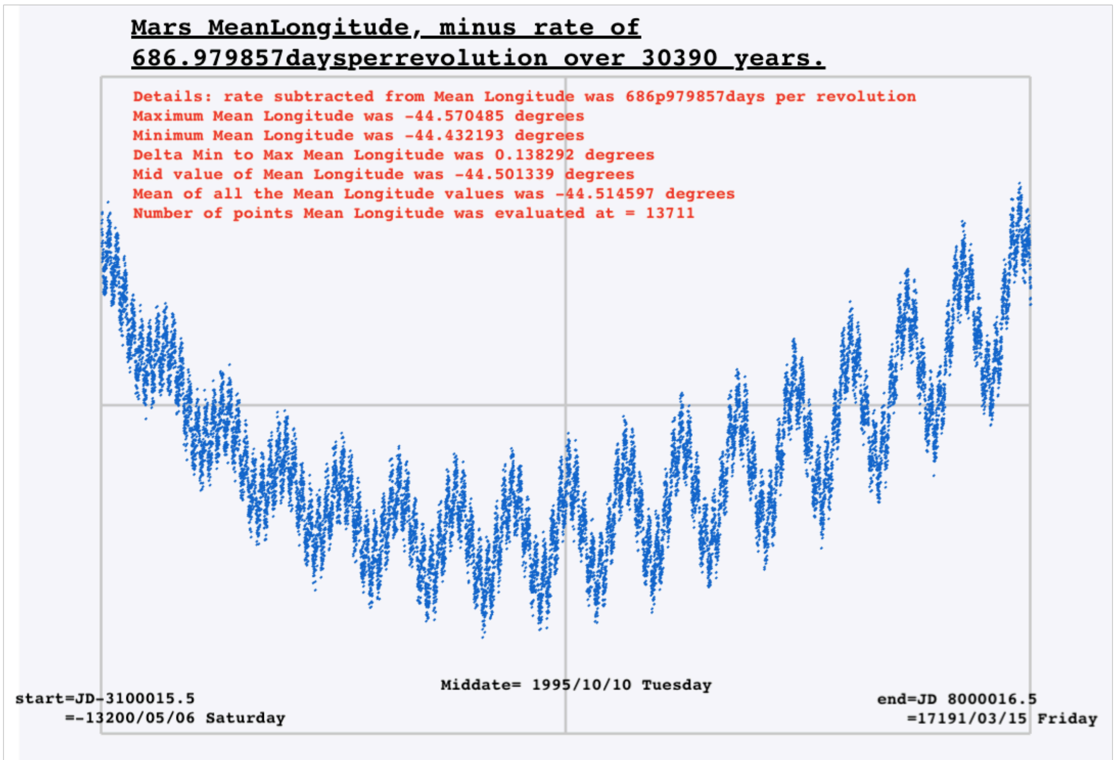
Written 2026/06/25 by Erik Gaylan Bergren

A more precise Relation between Period and Mean Distance



In this paper I show for the example Planet Mars, how a precise long-term Period can be determined, and similarly how a precise long-term Mean Distance can be determined, through the use of JPL's 30390 year ephemeris file DE441.

I then show the inadequacy of the standard Gaussian Year method commonly used to relate Period to Mean Distance, and show that modeling each Planet in the Solar System as an evenly distributed circular mass ring enables a simple integration of the force on Mars that links its Period and Distance 16 times more accurately than that common method used by; for example the Astronomical Almanac and also JPL's Horizons Web app.

As the graph on the last page shows, using JPL's 30390-year Ephemeris it is easy to get a precise Period for Mars by simply keeping track of its accumulated progress in Mean Longitude. Most of the variations from one orbit to the next cancel out in the long run. Thus by removing a linear rate of 686.979857 days per revolution Mars' Mean Longitude position stays nearly stationary over the whole 30390 years of the DE441, thus I will use that value as one of the main reliable data points in this paper.
It's quit incredible that by subtracting (without any filtering) a rate of 686.979857 days per revolution,
Mars over 30,300 years stays exactly between
having Mean Longitudes of -44.570485 and -44.432193 degrees!

Mars' period from all sources I use a lot agree at the 1 second level:

My analysis of DE441 Mars' Mean Longitude progress across ephemeris start-to-end time gives: dms(frac(686.979857days)*24)
= 686 days 23:30:59.64

Observatory of Paris VSOP2013 gives: dms(frac(686.9798514days)*24)
= 686 days 23:30:59.15

JPL's E. Myles Standish Least Squares Fit over -200 +50 years gave:
dms(frac(686.9797315days)*24)
= 686 days 23:30:48.80

JPL's E. Myles Standish Least Squares Fit over ±3000 yrs years gave:
dms(frac(686.9798515days)*24)
= 686 days 23:30:59.17

The AU used to be defined so that it linked all the semi-major-axes with the periods, to a useful accuracy, using only the gaussian year(GY): GY=2*Pi/(sqrt(GM(Sun)))=365.2568983272363777431883 days

Using the respected and consistent (at 1sec level) values listed above for Mars' period, I can calculate the following values for Mars' semi-major axis that correspond to those periods when put into this GY formula (note: JPL Horizons webpage for Orbit Elements uses this exact same relation between its output that reports Period and Semi-major axis):

(686.979857 /GY)^(2/3) = 1.5236915214 AU

(686.9798514/GY)^(2/3) = 1.5236915131 AU

(686.9797315/GY)^(2/3) = 1.5236913359 AU

(686.9798515/GY)^(2/3) = 1.5236915133 AU

   That Gaussian Year relation is a common way (but non-optimal) to report the Mean Distances of the Planets from their Periods. There are other ways. For example one can just do a humongous average over millions of distance measurements from a file like DE441. Such experimental results will show a slower Period vs Distance equation for Mars; because Jupiter is repeatedly pulling Mars away from the Sun, which over eons makes the effective Solar mass lower. The Gaussian year equation I used above statedly ignores effects from objects other than our Sun and the specific Planet, and even ignores its mass.

In the old days alls they had was timings and Geocentric angles; they couldn't weight the Sun with a scale, and they couldn't bounce radar off Venus. Thus the AU was constructed for "that distance" from the Sun at which a planet must change its observed Geocentric angle at a rate expected from our Sun's pull, no down-to-earth distance needed by design.

To show that Semi-Major Axis and Period are redundant
from information sources that use the Gaussian Year relation,
I'll plug in today's date (JD 2461202.576671412037037037)
into NASA JPL's Horizon's web form at
https://ssd.jpl.nasa.gov/horizons/app.html#/

They say that right now Mars has an orbit Period of
"PR= 6.869179086355127E+02",
and Semi-major axis
"A = 1.523600084751792E+00".

But (6.869179086355127E+02/GY)^(2/3)=1.52359992085532094 AU so I got their "A" from their "PR" almost exactly.

Want the rest of the digits to match? Just include Mars' mass ratio:
(6.869179086355127E+02/GY*sqrt((1+1/3098703.59)))^(2/3) =
1.52360008475179277

But as will be shown in the graphs on the coming pages,
that Gaussian Year Equation is not particularly accurate expectedly since it ignores forces from the whole Solar System except the Sun. Thus a question arises to me in a more general sense,
it asks:

"using a common equation for all planets;
how well can planet distances from the Sun be corresponded to their periods".

Just that very question caused a major change in the 2009 edition of the Astronomical Almanac. To better match periods to semi-major axes in their orbital element tables, they changed their equation for that relation, from the two argument (Gaussian Year,mass ratio) equation, to an equation where the GM is increased to the sum of all GM's of the planets which orbit closer to the Sun than the Planet. They still ignore the GM-reducing effect from planets further from the Sun than the Planet. My graphs on the coming pages show (with plot in gray ink)that their idea does give a more accurate Period-Mean Distance relation, however I will present a method far more accurate. First I will calculate an value for Mars’ Mean Distance that is independent of its Period:

Here on the following page, are my actual calculations, on the distance between Mars and the Sun, using the 2024/05/13 ephemeris DE442, of the average of 2079152 points across 891 years (the Julian dates 2287184.5 to 2612816.5) and also using 30390 year ephemeris DE441:

'-------- Mean of the reciprocal of the distance is:
(the reciprocal is used because it is proportional to the Potential Energy, and generally more stable-- being ~independent of eccentricity, and because it is mathematically exactly equivalent to the semi-major axis for perfectly Keplerian orbits):

1.5236886499 AU (averaged over 891 years of DE442)

1.5236886591 AU (averaged over 30390 years of DE441)

We can now send that distance through the Gaussian Year equation and see how the equation's predicted Period compares to the independently known (from observing the progress of the Mean Longitude of Mars) Period for Mars across DE441, which was 686.979857 days (the last two digits (if not the last) are significant).

1.5236886499^(3/2)*GY = 686.9779150 days

1.5236886591^(3/2)*GY = 686.9779212 days

so the consensus is about 686.97792 days per orbit which is quite a faster motion than the observed 686.979857 days per orbit.

Including center of mass correction does not help significantly:

(1.5236886499)^(3/2)*GY/sqrt((1+1/3098703.59)) = 686.977804

(1.5236886591)^(3/2)*GY/sqrt((1+1/3098703.59)) = 686.977810

Therefore I would suggest we are now at the position, having properly analyzed the observational data vs the simple commonly used equations, in which the Astronomical Almanac was:
there's really a significant discrepancy
from the commonly used relation.

I showed a Center of Mass correction is too small to account for it, and even summing the masses of planets closer to the Sun won't help (that would cause change in the wrong direction of the error).

The slowing of Mars' Orbital Period compared to its solar distance is expected from Jupiter's pull on Mars away from the Sun.

My goal is now “to find a more precise formula for the orbital period". To do that we must improve on the 2009 Astronomical Almanac's improvement, and include the slowing effects of planets pulling away from the Sun, in addition to their idea of including effects of planets who pull with the Sun, I guessed. Before proceeding with my method for that, an interlude on the next page will show a nice way to calculate eccentricity using the data I presented on this page.

Interlude:

A nice global value for the Eccentricity of the orbit of Mars
can be calculated if;

in addition to averaging the reciprocal distance
(called the Harmonic Mean Distance, which was 1.5236886499 AU);

we also average the distance directly
(called the Arithmetic Mean Distance,
which using DE442 I found was 1.5303364506 AU)

Then we can get a nice global average for the Eccentricity:
e = sqrt(2*(AMD/HMD-1)) = 0.09341269.

It's fun to derive that equation for e, but there's a reputable reference for that obscure equation in a footnote discussion by JPL's E. Myles Standish in my table on page 105 of (the 2008, 2010, or 2012 edition) of C. Amsler et al. (Particle Data Group), Physics Letters B667, 1 (2008).

e is increasing pretty quickly, so that value is most valid at the middle date of the average, which was 1995/10/10. Or I could report its slope with time to be more useful,
but I am too busy finishing the rest of this paper:

My goal is
“to find a more precise formula than the standard Keplerian relation between the orbital Period and the Mean Distance”.
But Is there any?

I found this solution on 2026/06/22
all on my own with help from neither books, nor people, nor AI.

To find an as-accurate as possible period-distance relation for Mars:

I drop for the moment considerations of the eccentricities in the orbits of Planets and approximate their orbits (except Mars itself) as circular rings of evenly distributed mass. I do that because placing Mars in such a system allows it to have only one possible circular orbit for any choice of either its period or distance, and since the system has radial symmetry, Mars' orbit would be closed and instantly calculatable by a trivial integration from its position on the x-axis (as with any of its positions in that circular orbit). On the other hand, if I attempt to model the orbits as ellipses then Mars would get bounced around like a pinball by all the "bumps" in the elliptical orbits of the other planets, so its orbit wouldn't then be a closed curve, and so would not have a Period. Thus the circular approximation is necessary for stable integration.

Using for input the Period and Mean Distance of Mars of 686.979857 days and 1.5236886591 AU that I calculated from DE441 above, I then integrated the forces on Mars from all the other planets modeled as uniform mass rings of radii equal to their Mean Distances (note: that choice of distance can be improved, see end note)

This integration is really easy and takes only 2 seconds. The C language code is so short I will post it here at the end.

The result from inputting 1.5236886591 AU is an output of 686.9799881332 AU which matches the observed Period to almost seven digits and is almost 16 times more accurate than the JPLHorizons-Astronomical-Almanac Equation that I showed gave an expected Period of 686.977810 days.

I believe adding another two more digits accuracy, (almost matching the accuracy of the observations) to my equation might be possible by correcting for the effects of the Planet's eccentricities; not within the integration itself but by correcting the circular orbit integration --for each Planet paired with Mars. A 10 GB flash stick file could store a large matrix of one-byte corrections calculated for all possible eccentricity pairs, and all possible angles between the Perihelions of the two interacting Planets, to a resolution of one byte (range of 0-255).

Here on the next page, is the main part of the C language code
that performed my integration above, on the page after that
I will discuss its accuracy:

```
(note: the personal macros were:
 "f  " = "__f128  " (defines a 128-bit float number)
"pz()" and "prtn()" displays numbers with lots of digits,
and "POINTS" is a macro for the number of integration steps (=100000).
and the radii and GMs of each Planet are in the arrays Rp[] and Gp[])


int main(void){

int n = POINTS;
f step=2*Pi/n;f sum=0;
f t;f a;f r2;f r;
f Gpiece;

for(int p=1;p<=9;p++){
if(p==4) continue;
Gpiece=Gp[p]/n;
for(int i=0;i<n;i++){
t=step*i;
r2=Rp[p]*Rp[p]+Rp[4]*Rp[4]-2*Rp[p]*Rp[4]*cosq(t);
r= sqrtq(r2);
a=Gpiece/(r2);

a=a*(Rp[p]*cosq(t)-Rp[4])/r;
sum+=a;
}
}
a=GMS/(Rp[4]*Rp[4])-sum;
prtn(GMS/(Rp[4]*Rp[4])); printf("%s", (sum < 0) ? "-" :
"+");pz(fabsq(sum));pz(a);
f period= 2*Pi*sqrtq(Rp[4]/a);

pz(period);
pz(GMS);
prtn(t);
prtn(a);
prtn(GMS/Gp[5]);
prtn(Gpiece);
return 0;}

// program output for Period
//when the input Rp[4]=1.5236886591
//the (truncated) output was "period =  686.97998813320"
```

after I found out that the integration of force from Jupiter and Earth gave no where near the total amount required to link Mars' Period to its Mean Distance, erroneously thinking "most of the perturbations are due to Earth and Jupiter, because every other planet is too far away or too small". What I guessed wrong about is that the others can't be ignored unless under 0.5 secs but most aren't as see in the output of the program's calculated list:

```
//planet #1 increases Mars' period by -5.180706 seconds
//planet #2 increases Mars' period by -88.248973 seconds
//planet #3 increases Mars' period by -139.730311 seconds
//planet #5 increases Mars' period by 393.612498 seconds
//planet #6 increases Mars' period by 17.708278 seconds
//planet #7 increases Mars' period by 0.325177 seconds
//planet #8 increases Mars' period by 0.099337 seconds
//planet #9 increases Mars' period by 0.000006 seconds
```

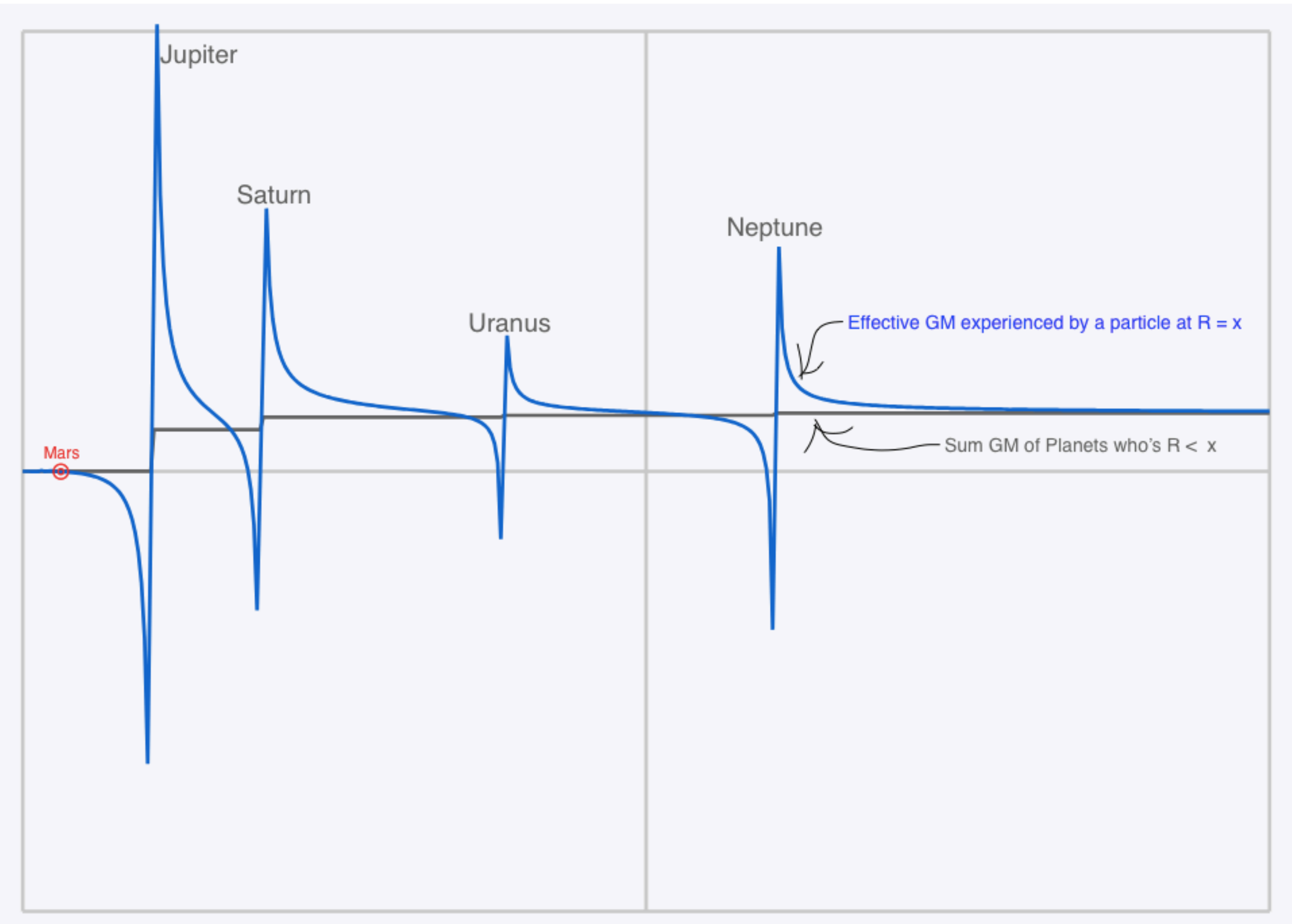
On the following last two pages of this paper , I plot what I call the "Effective GM Disk Of Our Solar System". The is produce by the C language code given in above, except I move the position of the test particle (which was Mars in the code as written) all the way from the Sun to 50 AU out from the Sun, and plot how much higher the GM force that it feels is than the GM of the Sun alone. From these graphs you can then calculate my optimal Period-Mean Distance relation by using the Effective GM in place of just the Sun's GM in the one parameter Gaussian-Year equation.

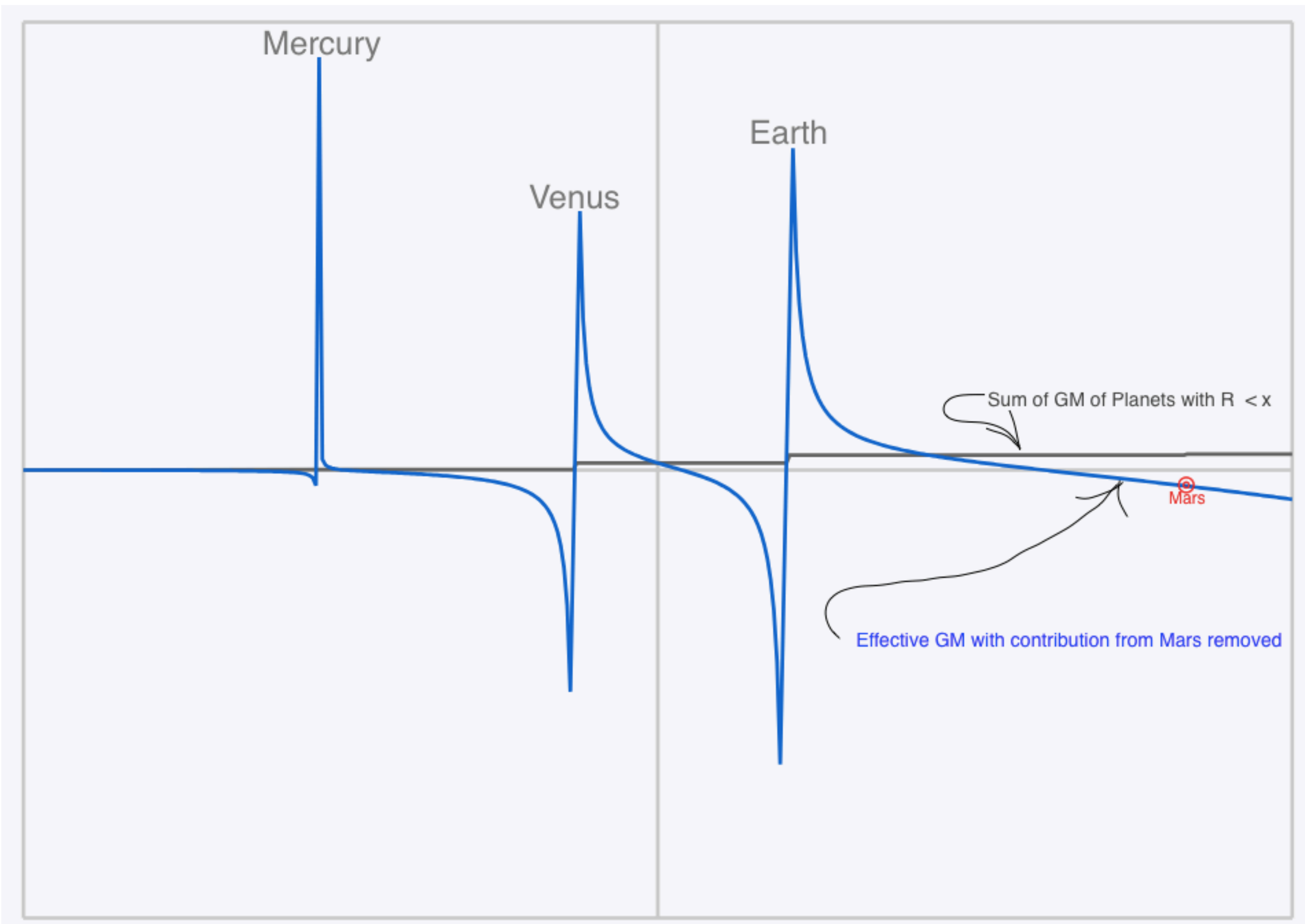
The first graph is zoomed out to show the outer Solar System as far as 50 AU. Pluto has too tiny of an effect to see.

The accuracy you would get using the common Period-Mean Distance relation is shown by the x-axis.

The accuracy you would get by using the Astronomical Almanac's technique of using GM=sum of GM of all Planets interior to the test particle, is shown by the grey curve.

The experimental DE441 equivalent-GM value that corresponds to what I calculated above for the Period and Mean Distance of Mars, is plotted as a red circle labeled "Mars". Notice that my "Effective GM Disk" model (in Blue ink) fits those experimental values for Mars so well that the error can't hardly be seen.


Jupiter
Saturn
Uranus
Neptune
Mars
Effective GM experienced by a particle at R = x
Sum GM of Planets who's R < x


Mercury
Venus
Earth
Sum of GM of Planets with R < x
Mars
Effective GM with contribution from Mars removed