Self-approximation for Riemann Hypothesis

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Abstract

We present a rigorous formulation of that idea through shifts generated by the continuous Gram function. We study the set of parameters τ for which $\zeta(s+it_\tau)$ uniformly approximates $\zeta(s)$ on compact subsets of D with connected complements. The method is based on Gram-shift universality, weak convergence of probability measures on the space of analytic functions and the support theorem for the random Euler product. The support is precisely the analytic functions that vanish nowhere in D and the identically zero function. This structure leads to two precise equivalences: The Riemann hypothesis is true if and only if the Gram-shift self-approximation holds with positive lower density for each admissible compact set and each approximation radius; and equivalently, when the limiting density exists and it is positive except at most countably many radii.

Keywords: Riemann zeta-function; Gram function; universality; self-approximation; random Euler product

1. Introduction

Let us $s=\sigma+\text{¿}$. For $\sigma>1$, the Riemann zeta-function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \quad \text{¿}\backslash(1\backslash)$$

and it also has the Euler product

$$\zeta(s)=\prod_{p\in P}(1-\text{¿}\,p^{-s})^{-1},\text{¿} \quad \backslash(2\backslash)$$

where P is the set of prime numbers. The product representation is the first clear sign that analytic information about $\zeta(s)$ controls arithmetic information about primes.

Riemann proved that $\zeta(s)$ extends meromorphically to the complex plane with a single simple pole at $s=1$ and satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \quad \text{¿}\backslash(3\backslash)$$

He also observed that the zeros of $\zeta(s)$ are deeply connected with the distribution of primes [1]. Hadamard later proved the prime number theorem by showing that $\zeta(s)$has no zeros on the line $\sigma=1$.

The Riemann hypothesis (RH) states that every nontrivial zero of $\zeta(s)$ lies on the critical line $\sigma=1/2$. Since the functional equation reflects nontrivial zeros across this line, RH is equivalently stated as

$$\zeta(s)\neq 0\left(\Re\ s>\frac{1}{2}\right). \quad \text{¿}\backslash(4\backslash)$$

This zero-free form is the one used in the present paper.

Gram introduced special points related to the argument of the gamma factor in the functional equation [3]. These points now play an important role in numerical and theoretical studies of the zeta-function. Bohr and Jessen then began the probabilistic study of the value distribution of $\zeta(s)$, showing that its values in vertical strips admit a rich statistical structure [4]. Littlewood proved that a positive proportion of nontrivial zeros lie on the critical line [5]. Levinson later proved that more than one third of the zeros lie there [6]. Pratt, Robles, Zaharescu, and Zeindler proved that more than five-twelfths lie on the critical line [7]. A different viewpoint came from Voronin's universality theorem which says, roughly, that vertical shifts of $\zeta(s)$ can approximate broad classes of analytic functions inside the strip $1/2<\sigma<1$ [8]. Bagchi then connected universality and recurrence with RH through weak convergence on spaces of analytic functions [9]. Steuding later presented a broad account of value-distribution theory for zeta and L-functions [10]. Dubickas et al.[11] also studied approximation phenomena involving the Gram function [21].

Considering previous studies, the goal of this paper is to present the Gram-function recurrence criterion in a clean, complete and connected form. The guiding question is simple. Given a compact set K inside D, *how often does the shifted function $\zeta(s+it_\tau)$ return uniformly close to $\zeta(s)$ on K?*

This recurrence has positive density in the correct sense exactly when RH is true. The proof has one central mechanism. Gram-shift universality gives approximation of zero-free analytic targets. Weak convergence identifies the limiting law of the shifted functions. The support of that limiting law consists of zero-free analytic functions on D, together with the identically zero function. Since ζ is not identically zero, it belongs to this support exactly when it has no zeros in D. This is precisely RH.

2. Analytic Framework

To begin, Let us

$$D=\left\{s\in C:\frac{1}{2}<\Re\ s<1\right\}. \quad \text{¿}\backslash(5\backslash)$$

And let K denote the family of compact subsets $K\subset D$ such that $C\setminus K$ is connected. For a continuous function f on K, define

$$\|f\|_K \underset{s\in K}{=} |f(s)|. \quad \text{¿}\backslash(6\backslash)$$

Let $H(D)$ be the space of analytic functions on D, equipped with the topology of uniform convergence on compact subsets. Choose compact sets $K_1, K_2, \ldots \subset D$ satisfying

$$\text{¿}\, j=1\,\text{¿}\,\infty\, K_j=D, \text{¿¿}\backslash(7\backslash)\text{¿}$$

and such that every compact subset of D is contained in some K_j. A standard metric inducing the compact-open topology on $H(D)$is

$$\rho(f,g)=\sum_{j=1}^{\infty} 2^{-j}\frac{\|f-g\|_{K_j}}{1+\|f-g\|_{K_j}}. \quad \text{¿}\backslash(8\backslash)$$

This metric is useful since recurrence can then be expressed as convergence of probability measures on $H(D)$.

For $K\in K$, let $H_0(K)$ be the class of functions that are continuous and nonzero on K, and analytic in the interior of K. The nonvanishing condition is essential. The limiting random Euler product used below has a support that contains zero-free analytic functions and the zero function only. Thus zero-free targets are the natural functions in the approximation theorem.

3. The Continuous Gram Function

The Gram function is built from the gamma factor in the functional equation. We define

$$\theta(t)=\Im\ \log\ \Gamma\ \left(\frac{1}{4}+\frac{\text{¿}}{2}\right)-\frac{t}{2}\log\ \pi. \quad \text{¿}\backslash(9\backslash)$$

For sufficiently large t, $\theta(t)$ is strictly increasing. Hence the equation

$$\theta(t_\tau)=\pi(\tau-1) \quad \text{¿}\backslash(10\backslash)$$

defines a unique real number t_τ in that range. The function $\tau\mapsto t_\tau$ is called the continuous Gram function. For integer τ, it gives the classical Gram points up to the usual indexing convention.

The large-τ size of t_τ is

$$t_\tau=\frac{2\pi\tau}{\log\tau}\left(1+\frac{\log\ \log\ \tau}{\log\tau}\right)(1+o(1)), \tau\to\infty. \quad \text{¿}\backslash(11\backslash)$$

Its derivative satisfies

$$t'_\tau = \frac{2\pi}{\log\tau}\left(1+\frac{\log\log\tau}{\log\tau}\right)(1+o(1)). \quad ¿\(12\)$$

These two estimates show that the Gram shift is unbounded but slower than the ordinary linear height. The shift $s \mapsto s+it_\tau$ therefore samples the vertical behavior of $\zeta(s)$through a nonlinear clock tied directly to the functional equation.

For $K \in K$, $\varepsilon > 0$, and $T > 0$, define the recurrence set

$$R_T(K,\varepsilon) = \{\tau \in [0,T] : \| \zeta(\cdot + it_\tau) - \zeta \|_K < \varepsilon\}. \quad ¿\(13\)$$

The central object of the paper is the asymptotic size of this set as $T \to \infty$.

Figure 1 shows the scale on which the nonlinear shift $s \mapsto s+it_\tau$moves. The two-term approximation lies above the first-order term because of the correction factor $\log\log\tau/\log\tau$.

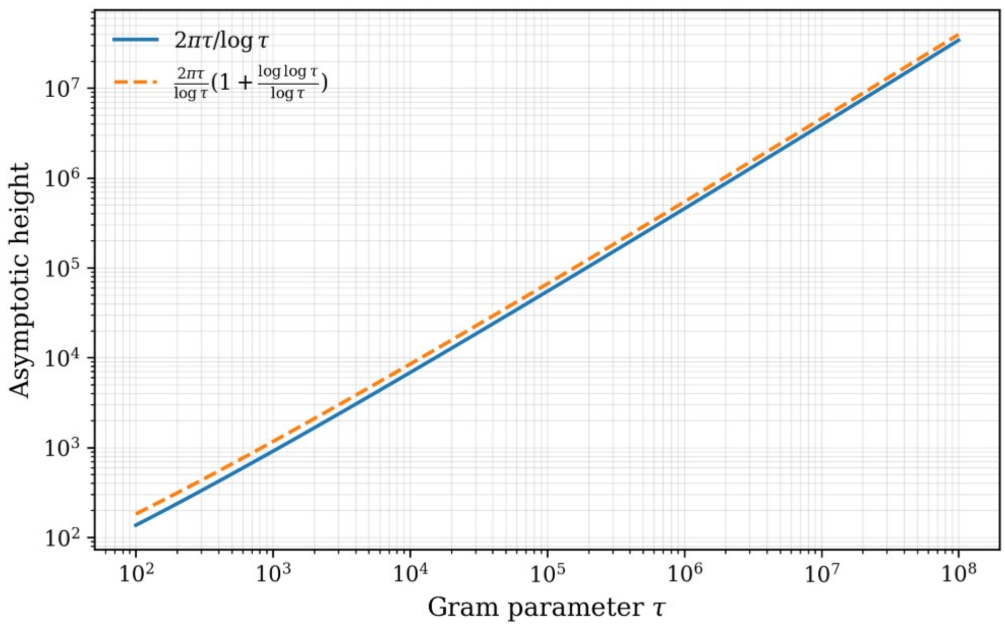

Figure 1. First-order and two-term asymptotic growth laws.

The graph makes clear that the Gram shift is not a linear vertical translation. It grows to infinity, but with logarithmic compression.

4. Limit Law and Support

Our proof uses a probabilistic model for the values of $\zeta(s)$. We set

$$\Omega = \prod_{p \in P} \{z \in C : |z| = 1\}, \quad ¿\(14\)$$

equipped with product topology and normalized Haar probability measure m_H. For $\omega = (\omega(p))_{p \in P} \in \Omega$, define the random Euler product

$$\zeta(s,\omega) = \prod_{p \in P} (1-\omega(p)p^{-s})^{-1}. \quad ¿\(15\)$$

For m_H-almost every ω, this product defines an analytic function on D. Let P_ζ be its distribution on $H(D)$.

Now we can define probability measures $P_T$on $H(D)$ by

$$P_T(A) = \frac{1}{T} meas\{\tau \in [0,T] : \zeta(\cdot + it_\tau) \in A\}, A \in B(H(D)). \quad ¿\(16\)$$

The Gram-shift limit theorem states that

$$P_T \Rightarrow P_\zeta, T \to \infty, \quad ¿\(17\)$$

where $\Rightarrow$ denotes weak convergence of probability measures.
The standard portmanteau theorem gives two consequences. If $G \subset H(D)$ is open, then

$$\liminf_{T \to \infty} P_T(G) \geq P_\zeta(G). \tag{18}$$

If $A \subset H(D)$ is a P_ζ-continuity set, then

$$\lim_{T \to \infty} P_T(A) = P_\zeta(A). \tag{19}$$

The decisive structural fact is the support identity

$$supp\ P_\zeta = \{g \in H(D) : g(s) \neq 0 \text{ for every } s \in D\} \cup \{0\}. \tag{20}$$

This formula is the bridge from probability to RH. Since ζ is not identically zero, the statement $\zeta \in supp\ P_\zeta$ is equivalent to $\zeta(s) \neq 0$ throughout D. By (4), which is exactly RH.

5. Gram-Shift Universality

The approximation theorem needed in the proof is the following.
Proposition 1. Let $K \in K$, $f \in H_0(K)$, and $\varepsilon > 0$. Then

$$\liminf_{T \to \infty} \frac{1}{T} meas\ \{\tau \in [0, T] : \| \zeta(\cdot + i t_\tau) - f \|_K < \varepsilon\} > 0. \tag{21}$$

Moreover, for fixed K and f, the corresponding limit exists and is positive for all but at most countably many values of $\varepsilon > 0$.
This proposition says that the Gram-shift orbit of $\zeta(s)$ approximates every admissible zero-free analytic target with positive frequency. The theorem is a Gram-scale analogue of Voronin-type universality. The result immediately explains why RH should imply self-approximation. If RH holds, then $\zeta(s)$ has no zeros in D. Hence the restriction of ζ to any $K \in K$ belongs to $H_0(K)$. Taking $f = \zeta$ in Proposition 1 then gives positive-frequency approximation of $\zeta(s)$ by its Gram shifts. The converse needs the support theorem. If RH fails, then ζ has a zero in D. The function ζ is not the zero function, so it lies outside the support in (20). A small enough neighborhood of ζ then has limiting probability zero. Weak convergence converts that support exclusion into failure of positive-density recurrence.

5.1 Main Equivalence Theorems

Theorem 1. RH holds if and only if, for every $K \in K$ and every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{meas\ R_T(K, \varepsilon)}{T} > 0. \tag{22}$$

Proof. Assume RH. Then $\zeta(s) \neq 0$ for all $s \in D$. Therefore $\zeta \in H_0(K)$ for every $K \in K$. Proposition 1 with $f = \zeta$ gives (22).
Conversely, assume that (22) holds for every K and every ε. Suppose that RH is false. Then ζ has a zero in D. Since $\zeta \not\equiv 0$, the support identity (20) implies that $\zeta \notin supp\ P_\zeta$.
Hence there is an open neighborhood U of ζ in $H(D)$ such that

$$P_\zeta(U) = 0. \tag{23}$$

Choose $\eta > 0$ with

$$B_\eta(\zeta) = \{g \in H(D) : \rho(g, \zeta) < \eta\} \subset U. \tag{24}$$

Pick m so large that

$$\sum_{j > m} 2^{-j} < \frac{\eta}{2}. \tag{25}$$

Then choose $r > 0$ satisfying

$$\frac{r}{1+r}<\frac{\eta}{2}. \quad ¿\backslash(26\backslash)$$

If $\| g-\zeta \|_{K_m}<r$, then $K_1,\ldots,K_m \subset K_m$, and (8) gives

$$\rho(g,\zeta)\le\sum_{j=1}^{m}2^{-j}\frac{r}{1+r}+\sum_{j>m}2^{-j}<\eta. \quad ¿\backslash(27\backslash)$$

Therefore the set

$$G_r=\{g\in H(D):\| g-\zeta \|_{K_m}<r\} \quad ¿\backslash(28\backslash)$$

is contained in U. Thus $P_\zeta(G_r)=0$.

For distinct positive radii, the level sets $\| g-\zeta \|_{K_m}=r$ are disjoint. A probability measure can charge at most countably many such level sets. Hence one may choose r so that $G_r$is a P_ζ-continuity set. By (19),

$$\lim_{T\to\infty} P_T(G_r)=0. \quad ¿\backslash(29\backslash)$$

Using the definition of P_T, this becomes

$$\lim_{T\to\infty}\frac{meas\ R_T(K_m,r)}{T}=0. \quad ¿\backslash(30\backslash)$$

This contradicts (22). Hence RH is true.

Theorem 2. RH holds if and only if, for every $K\in K$, the limit

$$d(K,\varepsilon)=\lim_{T\to\infty}\frac{meas\ R_T(K,\varepsilon)}{T} \quad ¿\backslash(31\backslash)$$

exists and satisfies $d(K,\varepsilon)>0$for all but at most countably many $\varepsilon>0$.

Proof. Assume RH. Proposition 1 applied to $f=\zeta$gives the existence and positivity of $d(K,\varepsilon)$for all but at most countably many radii.

Conversely, suppose the density statement holds for every K. If RH were false, the proof of Theorem 1 would provide a compact set $K_m$and an interval of radii $0<r<r_0$such that

$$P_\zeta\{g\in H(D):\| g-\zeta \|_{K_m}<r\}=0. \quad ¿\backslash(32\backslash)$$

All but at most countably many of these balls are P_ζ-continuity sets. For each continuity radius, weak convergence gives

$$\lim_{T\to\infty}\frac{meas\ R_T(K_m,r)}{T}=0. \quad ¿\backslash(33\backslash)$$

This contradicts the assumed positivity of $d(K,\varepsilon)$ outside at most countably many radii. Therefore RH holds. The countable exceptional set is natural. Weak convergence gives convergence of probabilities on continuity sets, and a nested family of balls can fail to be continuous for only countably many radii.

6. Conclusion

This paper establishes two exact Gram-function recurrence criteria for RH. The first criterion states that RH is equivalent to positive lower density of the parameter set on which $\zeta(s+it_\tau)$ uniformly approximates $\zeta(s)$on every admissible compact subset of the strip D. The second criterion refines this to the existence and positivity of the limiting density, with at most countably many exceptional approximation radii. The proof relies on a single structural principle. The support of the random Euler-product limit law is the analytic zero-free functions on D, plus the zero function. Since ζ is not identically zero, its membership in that support is exactly the zero-free condition in D. That zero-free condition is the R.H. The Gram function is not an arbitrary parametrization, it is tied to the argument of the gamma factor in the functional equation and its asymptotic behavior shows that it gives a genuine nonlinear sampling of the vertical orbit of $\zeta(s)$. The recurrence result therefore translates RH into a dynamical statement along a natural arithmetic-analytic scale. This formulation does not prove RH independently. It gives an equivalent condition and its value is that it connects three major structures: zeros of $\zeta(s)$, universality of zeta shifts and weak convergence of random Euler products.

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