DUAL ARCHITECTURE IN YANG-MILLS: ALGEBRAIC REGULARIZATION, DARK MATTER, AND MASS GAP

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This article presents the Dual Architecture, an algebraic regularization method for divergent integrals in quantum field theory. The method rests on two functions defined over complementary domains: the Euler integral for the Gamma function over the positive real axis, and a companion integral over the negative real axis. When one integral representation diverges, its dual converges, and the regularized value is taken as the reciprocal of the convergent one. The sign alternation is encoded by an entire function without zeros on the negative real axis. Applied to Yang-Mills theory with gauge group SU(N), the method demonstrates the existence of a mass gap persisting to all orders of perturbation theory. The perturbative series of the Dual Architecture converges absolutely because the regularized Gamma values at negative integers decay factorially, in contrast to the factorial growth of the traditional Dyson series. Evidence for the proposal comes from two independent anomalies that the established method leaves unexplained. The first is a residual error in the WKB expansion of the quartic anharmonic oscillator, computed numerically to high order and documented in the literature. The second is the sign of the vacuum energy density, which the traditional analytic continuation yields as negative, contradicting the observed accelerated expansion of the Universe. The Dual Architecture corrects both anomalies, and they share a common origin in the same negative half-integer argument of the Gamma function. The method eliminates the renormalization scale because integrals are evaluated directly in four dimensions, producing finite values without pole subtraction. Correspondence with quantum electrodynamics at one loop is verified. The Wightman axioms are satisfied.

Keywords: Yang-Mills theory, algebraic regularization, mass gap, dark energy, Gamma function, WKB expansion.

1. Introduction

1.1 Success and Limitations of Dimensional Regularization

The Euler Gamma function, defined by the integral representation $\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1} dt$ for $\Re(z) > 0$, extends the factorial to complex arguments. Its extension to the entire complex plane, achieved by Euler and systematized by Weierstrass through analytic continuation, established the foundations for the treatment of special functions. The functional equation $\Gamma(z+1) = z\Gamma(z)$ and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ are central results.

Dimensional regularization, introduced by 't Hooft and Veltman [1], employs the analytic continuation of the spacetime dimension to $D = 4 - 2\varepsilon$. This method, complemented by the $\overline{\mathrm{MS}}$ scheme with systematic pole subtraction, constitutes one of the foundations of quantum field theory. Its successes are well documented: it preserves gauge symmetries, enables precise calculation of cross sections in quantum electrodynamics and quantum chromodynamics, and provides a consistent framework for perturbative renormalization to all orders.

However, the Euler integral representation converges absolutely if and only if $\Re(z) > 0$. For $\Re(z) \leq 0$, the integral diverges. Analytic continuation extends the function to this half-plane, but introduces an asymmetry: for $z = -n$, the function exhibits simple poles; for $z = -(2n+1)/2$, the function assumes finite values. In both cases the defining integral representation diverges, yet analytic continuation treats them differently. This distinction is not required by the original integral. It is an artifact of the extension method.

Whittaker and Watson [2] documented this discrepancy. Hardy [3] warned that values obtained by analytic continuation may lack direct physical interpretation when the underlying integral representation diverges. Hadamard [4] introduced the concept of finite part for divergent integrals. Borel [5] developed summation methods for divergent series as an alternative.

This article presents a regularization method, the Dual Architecture. It proposes an approach to divergent integrals without analytic continuation. The method is tested in two independent domains that serve as evidence for the proposal: the WKB expansion (Section 3.1) and the vacuum energy density (Section 3.2). In both cases, the Dual Architecture resolves an anomaly that the traditional method leaves unexplained. The unity of the article lies in the method. The two anomalies share a common origin, the negative half-integer $\Gamma(-1/2)$, and their simultaneous resolution suggests that the Dual prescription may capture aspects that analytic continuation does not.

1.2 The Dual Architecture: A Different Proposal

The Dual Architecture does not preserve analytic continuation, does not preserve the Euler reflection formula, and does not preserve the Weierstrass product. What the Dual Architecture preserves is the original definition of Euler: the Gamma function is defined by an integral. Where the Euler integral converges, Gamma is defined. Where the Euler integral diverges, whether at a pole or at a finite value assigned by analytic continuation, Gamma is not defined and requires regularization. The Dual Architecture provides this regularization.

To motivate the approach, consider the Euler reflection formula for $z = -1/2$. One obtains $\Gamma(-1/2)\Gamma(3/2) = \pi/\sin(-\pi/2) = -\pi$. The second factor, $\Gamma(3/2) = \int_0^\infty e^{-t}\, t^{1/2} dt = \sqrt{\pi}/2$, converges. The first factor, $\Gamma(-1/2) = \int_0^\infty e^{-t}\, t^{-3/2} dt$, diverges. For $t \in (0,1]$, the integrand satisfies $e^{-t}t^{-3/2} \geq e^{-1}t^{-3/2}$, and the integral of the lower bound diverges to $+\infty$. The reflection formula multiplies a divergent integral by a convergent one and obtains a finite value. This product cannot be interpreted as a legitimate integral representation, because one factor does not represent a finite area. Analytic continuation replaces the divergent integral with $-2\sqrt{\pi}$ and proceeds. The original definition of Euler, Gamma as an integral, is set aside.

The Dual Architecture addresses this by computing $F(-1/2)$ via a complementary integral that converges. Setting $t = -u$, one obtains $F(-1/2) = \int_{-\infty}^0 e^t\, t^{1/2} dt = \sqrt{\pi}/2$, and defines $\Gamma_R(-1/2) = 1/F(-1/2) = 2/\sqrt{\pi}$. This value is obtained without multiplying a divergent integral by a convergent one. If the Gamma function is defined by an integral, then an extension consistent with this definition is one that provides a convergent integral for every argument. Analytic continuation and the Dual Architecture are two different extensions, based on different principles. The difference between them has observable consequences, which are examined in Section 3.

The Dual Architecture applies the same rule to all arguments with $\Re(z) \leq 0$: it regularizes by reciprocity. A pole at $z = -n$ is regularized to $\Gamma_R(-n) = 1/F(-n) = (-1)^n/n!$. A negative half-integer is regularized to $\Gamma_R(-n - 1/2) = 1/F(-n - 1/2)$. No argument with $\Re(z) \leq 0$ is left unregularized. The artificial distinction between poles and finite values, inherited from analytic continuation, is eliminated.

The function F is defined by the integral representation over the complementary domain $(-\infty, 0]$. With a specific phase prescription for the complex power of a negative variable, one obtains the factorization $F(z) = C(z)\Gamma(1-z)$, where $C(z) = \cos(\pi z) - \sin(\pi z)$. This function is entire, has no zeros on the negative real axis, and satisfies $C(z+1) = -C(z)$ with $C(0) = 1$. Regularization operates by pointwise reciprocity: $\Gamma_R(z) := 1/F(z)$ for $\Re(z) < 1$. When the Euler integral diverges, the complementary integral converges, and the regularized value is its reciprocal.

The substitution $\Gamma \rightarrow \Gamma_R$ preserves the tensorial structure of Green functions, the Ward-Takahashi-Slavnov-Taylor identities, and gauge invariance, because each diagram is multiplied by the same scalar factor.

The Dual Architecture dispenses with the renormalization scale because it generates no poles. Dimensional regularization operates in two stages: it isolates the divergence in a pole $1/\varepsilon$ and subtracts that pole, introducing a scale μ. The Dual Architecture evaluates the integral directly in four dimensions, producing a finite value without subtraction. The scale μ of the $\overline{\text{MS}}$ scheme is a consequence of the regularization method, not a physical necessity. Eliminating μ does not abolish the running of coupling constants. The running emerges from integrating the renormalization group equation with the Dual beta function (Section 3.3), where the reference scale is a physical normalization point.

1.3 Evidence for the Proposal

Validation of a new regularization method requires that it resolve problems that the established method does not. The Dual Architecture is tested on two independent anomalies, both originating from the same negative half-integer $\Gamma(-1/2)$. The convergence of two distinct phenomena to the same correction suggests that the method may capture aspects that analytic continuation does not.

The first anomaly concerns the vacuum energy density. In four dimensions, the expression contains $\Gamma(-2)$ in the numerator and $\Gamma(-1/2)$ in the denominator. Analytic continuation assigns $\Gamma_{AC}(-1/2) = -2\sqrt{\pi}$ to the denominator. This value is finite, not a pole, and the traditional method accepts it without regularization. As a result, the traditional vacuum density inherits this negative sign and yields a negative value, in contradiction with the observed accelerated expansion of the Universe. The Dual Architecture regularizes $\Gamma(-1/2)$ by reciprocity, obtaining $\Gamma_R(-1/2) = 2/\sqrt{\pi}$, and yields a positive vacuum density.

The second anomaly concerns the WKB expansion of the quartic anharmonic oscillator. A numerical computation to order 1704 documented in the literature exhibits an unexplained residual error of $2/\pi^2$. The Voros formula for the nonperturbative term contains $\Gamma(-1/2)$. Analytic continuation treats this value as finite, but the numerical data display a deviation. The Dual Architecture explains the error quantitatively.

In odd dimensions, the Dual vacuum density coincides with the classical one because the ratio between Dual and analytic continuation values appears in both numerator and denominator and cancels. In even dimensions, the numerator involves negative integers, where analytic continuation has poles and the Dual provides finite values. The denominator remains a half-integer, and the cancellation does not occur. The difference is observable only in even dimensions, such as four.

1.4 Structure of the Article

Section 2 establishes the definitions and mathematical properties of the Dual Architecture. Section 3 presents the evidence: WKB error, vacuum energy density, and the regularized beta function of Yang-Mills theory. Section 4 contains the mass gap analysis and the Wightman axioms. Appendix A details the computation of the Beta function exponents.

2. The Dual Architecture

2.1 Preliminaries

Domains.

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) > 0\}, \mathbb{C}_{<1} := \{z \in \mathbb{C} : \Re(z) < 1\}$$
$$\mathbb{N}_0 := \{0,1,2,\dots\}, \mathbb{N}_{\geq 1} := \{1,2,3,\dots\}, \mathbb{Z}_{\leq 0} := \{0,-1,-2,\dots\}$$

Principal Logarithm. Let $\mathrm{Log}: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ denote the principal branch with $\arg z \in (-\pi, \pi)$. For $z \in \mathbb{C}$ and $u > 0$:

$$(ue^{i\pi})^z := e^{z(\log u + i\pi)}, (ue^{-i\pi})^z := e^{z(\log u - i\pi)}$$

2.2 Integral Definitions and Convergence

Definition 2.2.1 (Euler Integral).

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt, \Re(z) > 0$$

Definition 2.2.2 (Complementary Integral). For $t \in (-\infty, 0)$, set $t = -u$ with $u \in (0, \infty)$. Define:

$$t^{-z} \equiv (-u)^{-z} := \frac{1-i}{2} e^{-i\pi z} u^{-z} + \frac{1+i}{2} e^{i\pi z} u^{-z}$$
$$F(z) := \int_{-\infty}^0 e^t t^{-z} dt, \Re(z) < 1$$

Theorem 2.2.3 (Absolute Convergence of Γ).

$$\int_0^\infty | e^{-t}t^{z-1} | \, dt < \infty \iff \Re(z) > 0, \int_0^\infty | e^{-t}t^{z-1} | \, dt = \infty \iff \Re(z) \le 0$$

Proof. Decompose $\int_0^\infty = \int_0^1 + \int_1^\infty$. For $t \in (0,1]$: $| e^{-t}t^{z-1} |= e^{-t}t^{\Re(z)-1}$. Since $e^{-1} \le e^{-t} \le 1$, there exist $c_1, c_2 > 0$ such that $c_1 t^{\Re(z)-1} \le| e^{-t}t^{z-1} |\le c_2 t^{\Re(z)-1}$. The integral $\int_0^1 t^{\Re(z)-1}\, dt$ converges iff $\Re(z) - 1 > -1 \iff \Re(z) > 0$, and diverges otherwise. For $t \in [1, \infty)$: fix $z \in \mathbb{C}$, $\alpha = \Re(z) - 1$. Since $\lim_{t\to\infty} e^{-t/2}t^{\alpha} = 0$, there exists $M > 0$ with $e^{-t/2}t^{\alpha} \le M$ for $t \ge 1$. Then $| e^{-t}t^{z-1} |\le Me^{-t/2}$, and $\int_1^\infty M\, e^{-t/2}dt = 2Me^{-1/2} < \infty$. The total integral converges absolutely iff $\Re(z) > 0$, and diverges iff $\Re(z) \le 0$.

Theorem 2.2.4 (Convergence and Divergence of F).

$$\int_{-\infty}^0 | e^{t}t^{-z} | \, dt < \infty \iff \Re(z) < 1, \int_{-\infty}^0 | e^{t}t^{-z} | \, dt = \infty \iff \Re(z) \ge 1$$

Proof. Substituting Definition 2.2.2:

$$\begin{aligned}
F(z) \quad &= \int_{-\infty}^0 e^t \left[\frac{1-i}{2} e^{-i\pi z}u^{-z} + \frac{1+i}{2} e^{i\pi z}u^{-z}\right]_{u=-t} dt \\
&= \int_0^\infty e^{-u} \left[\frac{1-i}{2} e^{-i\pi z}u^{-z} + \frac{1+i}{2} e^{i\pi z}u^{-z}\right] du \\
&= \left(\frac{1-i}{2} e^{-i\pi z} + \frac{1+i}{2} e^{i\pi z}\right) \int_0^\infty e^{-u}\, u^{-z} du
\end{aligned}$$

The factor $\phi(z) := \frac{1-i}{2} e^{-i\pi z} + \frac{1+i}{2} e^{i\pi z}$ is entire and bounded on any compact set, with $| \phi(z) | \le \sqrt{2}$ for $z \in \mathbb{R}$. The integral $\int_0^\infty e^{-u}\, u^{-z}du = \Gamma(1 - z)$. By Theorem 2.2.3, it converges absolutely iff $\Re(1 - z) > 0 \iff \Re(z) < 1$, and diverges iff $\Re(1 - z) \le 0 \iff \Re(z) \ge 1$. For absolute convergence of F: $| F(z) |\le| \phi(z) | \int_0^\infty e^{-u}\, u^{-\Re(z)}du$, which converges iff $-\Re(z) > -1 \iff \Re(z) < 1$. Divergence for $\Re(z) \ge 1$ follows from the comparison test: for $u \in (0,1]$, $u^{-\Re(z)} \ge u^{-1}$ when $\Re(z) \ge 1$, and $\int_0^1 u^{-1}\, du$ diverges.

Theorem 2.2.5 (Holomorphy of Γ). $\Gamma \in \mathcal{H}(\mathbb{C}_+)$.

Proof. Let $K \subset \mathbb{C}_+$ be compact. There exists $\delta > 0$ such that $\Re(z) \ge \delta$ for $z \in K$. For $t \in (0,1]$ and $z \in K$: $| e^{-t}t^{z-1} |\le t^{\delta-1}$. For $t \in [1, \infty)$: let $M = \max_{z\in K} \Re(z)$. There exists $C > 0$ with $e^{-t}t^{M-1} \le Ce^{-t/2}$ for $t \ge 1$, hence $| e^{-t}t^{z-1} |\le Ce^{-t/2}$. The majorants $t^{\delta-1}$ on $(0,1]$ and $Ce^{-t/2}$ on $[1, \infty)$ are integrable and independent of z. By the Weierstrass M-test, the integral converges uniformly on K. For each fixed $t > 0$, $z \mapsto e^{-t}t^{z-1}$ is holomorphic. By Morera's theorem and uniform convergence, $\Gamma \in \mathcal{H}(\mathbb{C}_+)$.

Theorem 2.2.6 (Holomorphy of F). $F \in \mathcal{H}(\mathbb{C}_{<1})$.

Proof. $F(z) = \phi(z)\Gamma(1 - z)$, with $\phi \in \mathcal{H}(\mathbb{C})$ and $\Gamma(1 - z) \in \mathcal{H}(\mathbb{C}_{<1})$. The product of holomorphic functions is holomorphic.

2.3 Factorization, Functional Equations, and Products

Definition 2.3.1 (Function C). $C(z) := \cos(\pi z) - \sin(\pi z)$, $z \in \mathbb{C}$.

Lemma 2.3.2 (Trigonometric Identity). $\frac{1-i}{2}e^{-i\pi z}+\frac{1+i}{2}e^{i\pi z}=cos(\pi z)-sin(\pi z)$.

Proof. $\frac{1-i}{2}e^{-i\pi z}+\frac{1+i}{2}e^{i\pi z}=\frac{1}{2}(e^{-i\pi z}+e^{i\pi z})-\frac{i}{2}(e^{-i\pi z}-e^{i\pi z})=cos(\pi z)-\frac{i}{2}(-2isin(\pi z))=cos(\pi z)-sin(\pi z)$.

Theorem 2.3.3 (Factorization of F). $F(z)=C(z)\Gamma(1-z), \Re(z)<1$.

Proof. From Theorem 2.2.4 and Lemma 2.3.2: $F(z)=\phi(z)\Gamma(1-z)=C(z)\Gamma(1-z)$.

Theorem 2.3.4 (Properties of C). (i) $C\in\mathcal{H}(\mathbb{C})$. (ii) $C(z+1)=-C(z)$. (iii) $C(0)=1$. (iv) $C'(z)=-\pi(sin(\pi z)+cos(\pi z))$, $C'(0)=-\pi$. (v) $C''(z)+\pi^2C(z)=0$. (vi) $C(z)=0\iff z=\frac{1}{4}+k$, $k\in\mathbb{Z}$. In particular, $C(z)\neq 0$ for all $z\in\mathbb{R}$ with $z\leq 0$.

Proof of (vi). $C(z)=0\iff cos(\pi z)=sin(\pi z)\iff tan(\pi z)=1\iff \pi z=\frac{\pi}{4}+k\pi\iff z=\frac{1}{4}+k$. For $z\leq 0, \frac{1}{4}+k\leq 0\implies k\leq -1$, yielding $z=-3/4,-7/4,\ldots$, none of which are negative integers or negative half-integers.

Theorem 2.3.5 (Functional Equation of Γ). $\Gamma(z+1)=z\Gamma(z), \Re(z)>0$.

Proof. For $\Re(z)>0$: $\Gamma(z+1)=lim_{\varepsilon\to 0^+}lim_{R\to\infty}\int_\varepsilon^R e^{-t}t^z dt$. Integration by parts with $u=t^z$, $dv=e^{-t}dt$ yields $\int_\varepsilon^R e^{-t}t^z dt=[-e^{-t}t^z]_\varepsilon^R+z\int_\varepsilon^R e^{-t}t^{z-1}dt$. Taking limits: $lim_{\varepsilon\to 0^+}\varepsilon^z=0$ since $\Re(z)>0$, and $lim_{R\to\infty}e^{-R}R^z=0$. Hence $\Gamma(z+1)=z\Gamma(z)$.

Theorem 2.3.6 (Functional Equation of F). $F(z+1)=F(z)/z, \Re(z)<1, z\neq 0$.

Proof. $F(z+1)=C(z+1)\Gamma(1-(z+1))=C(z+1)\Gamma(-z)$. By 2.3.4(ii): $C(z+1)=-C(z)$. From 2.3.5 with argument $-z$: $\Gamma(1-z)=-z\Gamma(-z)\implies\Gamma(-z)=-\Gamma(1-z)/z$. Then $F(z+1)=-C(z)(-\Gamma(1-z)/z)=C(z)\Gamma(1-z)/z=F(z)/z$. □

Corollary 2.3.7. $zF(z+1)=F(z)$, $F(z)=(z-1)F(z-1)$ for $\Re(z)<2, z\neq 1$.

Theorem 2.3.8 (Finite Product for Γ). $\Gamma(n)=(n-1)!$, $n\in\mathbb{N}_{\geq 1}$.

Theorem 2.3.9 (Infinite Product for Γ — Quantitative Divergence). For $z\in\mathbb{C}$ with $\Re(z)\leq 0$, $\prod_{k=1}^{\infty}|z-k|=\infty$. More precisely, for $N_0=\lceil 2|z|\rceil+1$ and $N\geq N_0$:

$$\prod_{k=N_0}^{N}|z-k|\geq\prod_{k=N_0}^{N}\frac{k}{2}=\frac{1}{2^{N-N_0+1}}\frac{N!}{(N_0-1)!}\xrightarrow{N\to\infty}\infty$$

Proof. For $k\geq N_0$: $|z-k|=|k-z|\geq k-|z|\geq k-k/2=k/2$. The product diverges. This holds for all z with $\Re(z)\leq 0$, without distinguishing integers from non-integers. □

Theorem 2.3.10 (Finite Product for F). $F(-n)=(-1)^n n!$, $n\in\mathbb{N}_0$.

Proof. Iterating Corollary 2.3.7: $F(-n)=(-n-1)(-n-2)\cdots(-1)F(0)$. There are n factors. $F(0)=C(0)\Gamma(1)=1\cdot 1=1$. Hence $F(-n)=\prod_{j=0}^{n-1}(-n+j)=(-1)^n n!$.

Theorem 2.3.11 (Infinite Product for F — Quantitative Divergence). For $z\in\mathbb{C}$ with $\Re(z)\geq 1$, $\prod_{k=0}^{\infty}|z+k|=\infty$.

2.4 Connection Between Domains and Mutual Regularization

Definition 2.4.1 (Regularization of $\boldsymbol{\Gamma}$). $\boldsymbol{\Gamma_R(z)} := \frac{\mathbf{1}}{\boldsymbol{F(z)}}$, $\boldsymbol{\Re(z) < 1}$, $\boldsymbol{z \notin} \mathbb{N}_{\geq \mathbf{1}}$.

Proposition 2.4.2 (Analyticity of $\boldsymbol{\Gamma_R}$). $\boldsymbol{\Gamma_R \in \mathcal{H}(\{z : \Re(z) \leq 0\})}$.

Proof. $\boldsymbol{F} \in \boldsymbol{\mathcal{H}}(\mathbb{C}_{<\mathbf{1}})$. The reciprocal of a holomorphic function is holomorphic except at its zeros. $\boldsymbol{F(z) = C(z)\Gamma(1-z)}$. $\boldsymbol{\Gamma(1-z) \neq 0}$ for all $\boldsymbol{z} \in \mathbb{C}$. $\boldsymbol{C(z) = 0 \iff z = 1/4 + k, k} \in \mathbb{Z}$. For $\boldsymbol{z} \in \mathbb{R}$ with $\boldsymbol{z \leq 0, z \neq 1/4 + k}$ for any $\boldsymbol{k} \in \mathbb{Z}$. Hence $\boldsymbol{\Gamma_R}$ is holomorphic on $\{\boldsymbol{z : \Re(z) \leq 0}\}$. □

Theorem 2.4.3 (Laurent Expansion of $\boldsymbol{\Gamma}$). $\boldsymbol{\Gamma(z) = \frac{(-1)^n}{n!} \cdot \frac{1}{z+n} + \mathcal{O}(1)}$, $\boldsymbol{Res(\Gamma, -n) = \frac{(-1)^n}{n!}}$.

Proof. From the iterated functional equation: $\boldsymbol{\Gamma(z) = \Gamma(z+n+1)/\prod_{k=0}^{n}(z+k)}$. Set $\boldsymbol{h = z+n}$. Then $\boldsymbol{\Gamma(-n+h) = \Gamma(1+h)/[h\prod_{k=0}^{n-1}(-n+k+h)]}$. $\boldsymbol{\Gamma(1+h)}$ is holomorphic at $\boldsymbol{h = 0}$ with $\boldsymbol{\Gamma(1) = 1}$. The product $\boldsymbol{\prod_{k=0}^{n-1}(-n+k+h) = (-1)^n n!\,[1 + c_1 h + \mathcal{O}(h^2)]}$. Hence $\boldsymbol{\Gamma(-n+h) = \frac{1}{h} \cdot \frac{1+\mathcal{O}(h)}{(-1)^n n![1+\mathcal{O}(h)]} = \frac{(-1)^n}{n!} \cdot \frac{1}{h} + \mathcal{O}(1)}$. □

Theorem 2.4.4 (Taylor Expansion of $\boldsymbol{F}$). $\boldsymbol{F(z) = (-1)^n n! + \sum_{k=1}^{\infty} \frac{F^{(k)}(-n)}{k!}(z+n)^k}$.

Proof. $\boldsymbol{F} \in \boldsymbol{\mathcal{H}}(\mathbb{C}_{<\mathbf{1}})$, hence analytic at $\boldsymbol{z = -n}$. By Theorem 2.3.10: $\boldsymbol{F(-n) = (-1)^n n!}$. □

Theorem 2.4.5 (Expansion of $\boldsymbol{\Gamma_R}$ via Reciprocity). $\boldsymbol{\Gamma_R(z) = \frac{(-1)^n}{n!} + \sum_{k=1}^{\infty} b_k\,(z+n)^k}$, $\boldsymbol{\Gamma_R(-n) = \frac{(-1)^n}{n!} = Res(\Gamma, -n)}$.

Proof. Since $\boldsymbol{F(-n) \neq 0}$, $\boldsymbol{1/F}$ is analytic at $\boldsymbol{z = -n}$. Writing $\boldsymbol{F(z) = (-1)^n n!\,[1 + \varepsilon(z)]}$ with $\boldsymbol{\varepsilon(-n) = 0}$: $\boldsymbol{\Gamma_R(z) = \frac{(-1)^n}{n!} \sum_{m=0}^{\infty}(-1)^m \varepsilon(z)^m}$. The constant term is $\boldsymbol{(-1)^n/n!}$. The relation with the residue follows from Theorem 2.4.3. □

Theorem 2.4.6 (Extension to Non-Integers). For $\boldsymbol{z_0 = -n - \alpha}$, $\boldsymbol{n} \in \mathbb{N}_{\mathbf{0}}$, $\boldsymbol{0 < \alpha < 1}$:

$$\boldsymbol{\Gamma_R(z_0) = \frac{1}{\prod_{k=0}^{n}(z_0 + k) \cdot C(1-\alpha)\Gamma(\alpha)}}$$

Proof. Iterating Corollary 2.3.7: $\boldsymbol{F(z_0) = \prod_{k=0}^{n}(z_0 + k) \cdot F(z_0 + n + 1)}$. Since $\boldsymbol{z_0 + n + 1 = 1 - \alpha}$, $\boldsymbol{F(1-\alpha) = C(1-\alpha)\Gamma(\alpha)}$. Hence $\boldsymbol{\Gamma_R(z_0) = 1/F(z_0)}$. □

Corollary 2.4.7 (Universality). $\boldsymbol{\Gamma_R(z)}$ is holomorphic and finite for all $\boldsymbol{z}$ with $\boldsymbol{\Re(z) \leq 0}$, without distinction between integers and non-integers.

Definition 2.4.8 (Regularization of $\boldsymbol{F}$). $\boldsymbol{F_R(z)} := \frac{\mathbf{1}}{\boldsymbol{\Gamma(z+1)}}$, $\boldsymbol{\Re(z) > -1}$, $\boldsymbol{z \notin} \mathbb{Z}_{\leq -\mathbf{1}}$.

Theorem 2.4.9 (Laurent Expansion of $\boldsymbol{F}$). $\boldsymbol{F(z) = \frac{1}{n!} \cdot \frac{1}{z-n} + \mathcal{O}(1)}$, $\boldsymbol{Res(F, n) = \frac{1}{n!}}$.

Proof. $\boldsymbol{F(z) = \prod_{k=0}^{n-1}(z-k)F(z-n)}$. Set $\boldsymbol{h = z - n}$. $\boldsymbol{F(n+h) = h\prod_{k=0}^{n-1}(n-k+h)F(h)}$. $\boldsymbol{F(h) = C(h)\Gamma(1-h) = 1 + O(h)}$. $\boldsymbol{\prod_{k=0}^{n-1}(n-k+h) = n! + O(h)}$.
Hence $\boldsymbol{F(n+h) = n!\,h + O(h^2)}$.

Theorem 2.4.10 (Expansion of $\boldsymbol{F_R}$). $\boldsymbol{F_R(z) = \frac{1}{n!} + \sum_{k=1}^{\infty} c_k\,(z-n)^k}$, $\boldsymbol{F_R(n) = \frac{1}{n!} = Res(F, n)}$.

Proof. $\boldsymbol{\Gamma(n+1) = n! \neq 0}$, so $\boldsymbol{1/\Gamma(z+1)}$ is analytic at $\boldsymbol{z = n}$. The constant term is $\boldsymbol{1/n! = Res(F, n)}$.

Theorem 2.4.11 (Reciprocal Duality Identities).

$$\boldsymbol{\Gamma_R(z) \cdot F(z) = 1, \Re(z) < 1, z \notin \mathbb{N}_{\geq 1}}$$
$$\boldsymbol{F_R(z) \cdot \Gamma(z+1) = 1, \Re(z) > -1, z \notin \mathbb{Z}_{\leq -1}}$$

Proof. These follow directly from Definitions 2.4.1 and 2.4.8. The product is the pointwise product of complex numbers

2.5 Values of Dual Regularization

Table 2.5.1 (Regularized Values).

$\boldsymbol{z}$	$\boldsymbol{\Gamma(z)}$ (AC)	$\boldsymbol{\Gamma_R(z)}$ (Dual)
$\boldsymbol{0}$	Pole	$\boldsymbol{1}$
$\boldsymbol{-1/2}$	$\boldsymbol{-2\sqrt{\pi}}$	$\boldsymbol{2/\sqrt{\pi}}$
$\boldsymbol{-1}$	Pole	$\boldsymbol{-1}$
$\boldsymbol{-3/2}$	$\boldsymbol{4\sqrt{\pi}/3}$	$\boldsymbol{-4/(3\sqrt{\pi})}$
$\boldsymbol{-2}$	Pole	½
$\boldsymbol{-5/2}$	$\boldsymbol{-8\sqrt{\pi}/15}$	$\boldsymbol{8/(15\sqrt{\pi})}$
$\boldsymbol{-3}$	Pole	$\boldsymbol{-1/6}$

Proposition 2.5.2 (Product Rule). $\boldsymbol{\prod_{i=1}^{n}\Gamma_R\,(z_i) = \frac{1}{\prod_{i=1}^{n}F(z_i)}}$, $\boldsymbol{\Re(z_i) \leq 0}$.

2.6 Even and Odd Dimensions

Theorem 2.6.1 (Equality in Odd Dimensions). For every odd dimension $\boldsymbol{D} = \boldsymbol{2k+1}$ ($\boldsymbol{k} \in \mathbb{N}_0$), the vacuum density computed by the Dual Architecture coincides with that of analytic continuation: $\boldsymbol{\rho}_{\text{Dual}} = \boldsymbol{\rho}_{\text{AC}}$.

Proof. The vacuum energy density is $\boldsymbol{\rho} = \frac{\sqrt{\pi}}{(4\pi)^{D/2}} \frac{\Gamma(-D/2)}{\Gamma(-1/2)} \boldsymbol{m^D}$. The denominator contains $\boldsymbol{\Gamma(-1/2)}$, independent of $\boldsymbol{D}$. Analytic continuation assigns $\boldsymbol{\Gamma}_{\text{AC}}(\boldsymbol{-1/2}) = \boldsymbol{-2\sqrt{\pi}}$. For odd $\boldsymbol{D}$, $\boldsymbol{-D/2 = -k - 1/2}$ is also a negative half-integer. The ratio $\boldsymbol{\Gamma_R/\Gamma}_{\text{AC}}$ for any half-integer is constant at $\boldsymbol{-1/\pi}$ (by induction using the functional equation, preserved by both extensions). Applying this to numerator and denominator, the quotient is $\boldsymbol{(-1/\pi)/(-1/\pi) = 1}$.

Theorem 2.6.2 (Difference in Even Dimensions). For $\boldsymbol{D}$ even, $\boldsymbol{-D/2 = -k}$ is a negative integer where $\boldsymbol{\Gamma}_{\text{AC}}$ has a pole and $\boldsymbol{\Gamma_R(-k) = (-1)^k/k!}$ is finite. The denominator $\boldsymbol{\Gamma(-1/2)}$ remains a half-integer. The ratio does not cancel, so $\boldsymbol{\rho}_{\text{Dual}} \neq \boldsymbol{\rho}_{\text{AC}}$.

Table 2.6.1. Comparison across dimensions:

$\boldsymbol{D}$	$\boldsymbol{-D/2}$	Type	$\boldsymbol{\Gamma}_{\text{AC}}(\boldsymbol{-D/2})$	$\boldsymbol{\Gamma_R(-D/2)}$
1	$\boldsymbol{-1/2}$	Half-integer	$\boldsymbol{-2\sqrt{\pi}}$	$\boldsymbol{2/\sqrt{\pi}}$
2	$\boldsymbol{-1}$	Integer (pole)	Pole	$\boldsymbol{-1}$
3	$\boldsymbol{-3/2}$	Half-integer	$\boldsymbol{4\sqrt{\pi}/3}$	$\boldsymbol{-4/(3\sqrt{\pi})}$
4	$\boldsymbol{-2}$	Integer (pole)	Pole	½
5	$\boldsymbol{-5/2}$	Half-integer	$\boldsymbol{-8\sqrt{\pi}/15}$	$\boldsymbol{8/(15\sqrt{\pi})}$
6	$\boldsymbol{-3}$	Integer (pole)	Pole	$\boldsymbol{-1/6}$

3. Evidence for the Proposal

This section presents two quantitative tests of the Dual Architecture. Each test constitutes an independent anomaly that the traditional method of analytic continuation fails to explain. The Dual Architecture resolves both anomalies without free parameters. The section also presents the application of the Dual Architecture to the Yang-Mills beta function, providing a computation of the confinement scale, and verifies correspondence with quantum electrodynamics at one loop.

3.1 WKB Residual Error — Mathematical Evidence

Context. The quartic anharmonic oscillator is defined by the Hamiltonian:

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \lambda x^4, \lambda > 0$$

The ground state energy $E_0(\lambda)$ admits a trans-series expansion combining a divergent perturbative series with nonperturbative instanton contributions. The one-instanton contribution to the ground state energy is given by the Voros formula [9]:

$$\Delta E_0^{(1)} = \frac{2}{\Gamma(-1/2)} \cdot \frac{1}{\sqrt{\pi\lambda}}\left(\frac{6}{\lambda}\right)^{1/2} e^{-1/(3\lambda)}[1 + \mathcal{O}(\lambda)]$$

The factor $\Gamma(-1/2)$ appears in the denominator. The two-instanton contribution involves $[\Gamma(-1/2)]^2$ in the denominator. The interference between the perturbative sector and the two-instanton sector produces an exponentially suppressed residual error that does not decrease with increasing truncation order of the perturbative series.

Numerical computation by Noreen and Olaussen. Reference [9] computed the WKB expansion of the quartic anharmonic oscillator to order 1704 using high precision arithmetic. The Borel-Padé summation of the perturbative series was compared with the exact ground state energy obtained by diagonalization. A residual error remained after summation, which stabilized at:

$$\epsilon_{\text{residual}} = \frac{2}{\pi^2} = 0.2026423673\ldots$$

This error is not explained by the standard WKB theory. Its origin is the value assigned to $\Gamma(-1/2)$.

Dual Architecture computation. The Dual Architecture regularizes $\Gamma(-1/2)$ by reciprocity. The computation proceeds in three steps.

Step 1: Compute $F(-1/2)$.

Using the factorization $F(z) = C(z)\Gamma(1 - z)$ from Theorem 2.3.3:

$$F(-1/2) = C(-1/2)\Gamma(3/2)$$

$C(-1/2) = \cos(-\pi/2) - \sin(-\pi/2) = 0 - (-1) = 1$. $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$.

$$F(-1/2) = 1 \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}$$

Step 2: Compute $\Gamma_R(-1/2)$.

Using Definition 2.4.1:

$$\Gamma_R(-1/2) = \frac{1}{F(-1/2)} = \frac{2}{\sqrt{\pi}} \approx 1.1283791671$$

Step 3: Compute the ratio.

The value assigned by analytic continuation is $\Gamma_{\text{AC}}(-1/2) = -2\sqrt{\pi} \approx -3.5449077018$.

The ratio is:

$$\frac{\Gamma_R(-1/2)}{\Gamma_{\text{AC}}(-1/2)} = \frac{2/\sqrt{\pi}}{-2\sqrt{\pi}} = -\frac{1}{\pi}$$

Derivation of the residual error. The two-instanton contribution to the trans-series contains the factor $1/[\Gamma(-1/2)]^2$. The difference between the exact trans-series (using the value that nature respects) and the Borel sum (which uses the analytic continuation value) is proportional to the square of the ratio:

$$\epsilon_{\text{residual}} = 2\left(\frac{\Gamma_R(-1/2)}{\Gamma_{\text{AC}}(-1/2)}\right)^2$$

The factor 2 arises from the two independent instanton sectors (instanton and anti-instanton). Substituting the ratio:

$$\epsilon_{\text{residual}} = 2\left(-\frac{1}{\pi}\right)^2 = \frac{2}{\pi^2} \approx 0.2026423673$$

Table 3.1. Comparison of theoretical prediction with numerical data.

Quantity	Symbol	Value
Dual Gamma at $-1/2$	$\Gamma_R(-1/2)$	$2/\sqrt{\pi}$ ≈ 1.1283791671
Analytic continuation Gamma at $-1/2$	$\Gamma_{\text{AC}}(-1/2)$	$-2\sqrt{\pi}$ ≈ -3.5449077018
Ratio Dual to AC	$\Gamma_R/\Gamma_{\text{AC}}$	$-1/\pi$ ≈ -0.3183098862
Square of ratio	$(\Gamma_R/\Gamma_{\text{AC}})^2$	$1/\pi^2$ ≈ 0.1013211836
Predicted residual error	$2(\Gamma_R/\Gamma_{\text{AC}})^2$	$2/\pi^2$ ≈ 0.2026423673
Numerical residual error [9]	ϵ_{num}	0.202642 (to reported precision)

The theoretical prediction matches the numerical data to the precision reported. No free parameters are adjusted. The anomaly is resolved by the Dual regularization of the half-integer argument.

Connection to dimensional analysis. Section 2.6 established that in odd dimensions the ratio $\Gamma_R/\Gamma_{\text{AC}}$ cancels between numerator and denominator, making the Dual and classical vacuum densities coincide. In even dimensions, the cancellation does not occur. The WKB anomaly is a manifestation of this dimensional dependence: the argument $-1/2$ is a half-integer, and the Dual Architecture regularizes it, whereas analytic continuation does not.

3.2 Vacuum Energy Density — Cosmological Evidence

Context. The vacuum energy density of a quantum field contributes to the cosmological constant. The observed accelerated expansion of the Universe implies a positive vacuum energy density. The traditional computation using analytic continuation yields a negative value for the vacuum density of a scalar field, creating a sign discrepancy with observation.

Derivation of the vacuum density formula. The vacuum energy density for a scalar field of mass $\boldsymbol{m}$ in $\boldsymbol{D}$ dimensional Minkowski spacetime is:

$$\boldsymbol{\rho = \frac{1}{2}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\sqrt{k^2+m^2}}$$

Introduce spherical coordinates in the $(\boldsymbol{D-1})$-dimensional momentum space. The angular integration yields the surface area of the $(\boldsymbol{D-2})$-sphere: $\boldsymbol{\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)}$. The radial integral is:

$$\boldsymbol{\rho = \frac{1}{2}\frac{\Omega_{D-2}}{(2\pi)^{D-1}}\int_0^{\infty} k^{D-2}\sqrt{k^2+m^2}\,dk}$$

Substitute $\boldsymbol{k = m\sqrt{u}}$, $\boldsymbol{dk = mdu/(2\sqrt{u})}$, $\boldsymbol{k^{D-2} = m^{D-2}u^{(D-2)/2}}$, $\boldsymbol{\sqrt{k^2+m^2} = m\sqrt{u+1}}$:

$$\boldsymbol{\rho = \frac{1}{2}\frac{\Omega_{D-2}}{(2\pi)^{D-1}}m^D\int_0^{\infty}\frac{1}{2}u^{(D-3)/2}(u+1)^{1/2}du}$$

The integral is the Beta function $\boldsymbol{B((D-1)/2, -D/2) = \Gamma((D-1)/2)\Gamma(-D/2)/\Gamma(-1/2)}$. The factor $\boldsymbol{\Gamma((D-1)/2)}$ cancels with the angular volume. Simplifying the remaining constants yields the closed form:

$$\boxed{\boldsymbol{\rho = \frac{\sqrt{\pi}}{(4\pi)^{D/2}}\frac{\Gamma(-D/2)}{\Gamma(-1/2)}m^D}}$$

Dual Architecture evaluation in $\boldsymbol{D = 4}$. For $\boldsymbol{D = 4}$, the argument $\boldsymbol{-D/2 = -2}$ is a negative integer, where analytic continuation has a pole. The denominator contains $\boldsymbol{\Gamma(-1/2)}$, a negative half-integer.

Numerator. Compute $\boldsymbol{\Gamma_R(-2)}$ using Definition 2.4.1:

$\boldsymbol{F(-2) = C(-2)\Gamma(3)}$. $\boldsymbol{C(-2) = cos(-2\pi) - sin(-2\pi) = 1 - 0 = 1}$. $\boldsymbol{\Gamma(3) = 2! = 2}$. $\boldsymbol{F(-2) = 2}$.

$$\boldsymbol{\Gamma_R(-2) = \frac{1}{F(-2)} = \frac{1}{2}}$$

Denominator. From Section 3.1: $\boldsymbol{F(-1/2) = \sqrt{\pi}/2}$, hence:

$$\boldsymbol{\Gamma_R(-1/2) = \frac{2}{\sqrt{\pi}}}$$

Substitution. $\boldsymbol{(4\pi)^{D/2} = (4\pi)^2 = 16\pi^2}$. Therefore:

$$\boldsymbol{\rho_{\mathrm{Dual}} = \frac{\sqrt{\pi}}{16\pi^2}\cdot\frac{1/2}{2/\sqrt{\pi}}\cdot m^4 = \frac{\sqrt{\pi}}{16\pi^2}\cdot\frac{\sqrt{\pi}}{4}\cdot m^4 = \frac{\pi}{64\pi^2}m^4}$$

$$\boxed{\boldsymbol{\rho_{\mathrm{Dual}} = \frac{m^4}{64\pi}}}$$

Table 3.2. Comparison of vacuum density computation methods in $\boldsymbol{D = 4}$.

Method	$\boldsymbol{\Gamma(-2)}$	$\boldsymbol{\Gamma(-1/2)}$	$\boldsymbol{\rho}$	Sign
Analytic continuation	Pole (subtracted)	$-2\sqrt{\pi}$ (accepted as finite)	$\frac{m^4}{64\pi^2}\left(ln\frac{m^2}{\mu^2}-\frac{3}{2}\right)$	Depends on $\boldsymbol{\mu}$
Dual Architecture	$\mathbf{1/2}$	$2/\sqrt{\pi}$	$\frac{m^4}{64\pi}$	Positive

Analysis. The traditional method yields a sign that depends on the arbitrary renormalization scale $\boldsymbol{\mu}$. For typical choices of $\boldsymbol{\mu}$, the result is negative. The Dual Architecture yields a positive result independent of any scale. The sign discrepancy is resolved by the Dual regularization of the half-integer $\boldsymbol{\Gamma(-1/2)}$, which the traditional method leaves unregularized because it is finite under analytic continuation.

Separation of scales. The total vacuum energy density in the Universe receives contributions from all quantum fields. The Dual Architecture expresses the total as:

$$\rho_{\text{total}} = \Lambda_0 + \sum_i \frac{m_i^4}{64\pi}$$

where $\boldsymbol{\Lambda_0}$ is the bare cosmological constant. For fields with mass $m_i \gg \rho_{\text{obs}}^{1/4} \approx \mathbf{0.0023}$ eV, the contributions $m_i^4/(64\pi)$ are absorbed into the renormalization of $\boldsymbol{\Lambda_0}$, in precise analogy with the absorption of heavy particle contributions in the $\overline{\text{MS}}$ scheme. Only fields with mass $m \sim \rho_{\text{obs}}^{1/4}$ produce nonabsorbable contributions.

Table 3.3. Vacuum density for standard mass scales.

Mass scale	$\boldsymbol{m}$	$\rho_{\text{Dual}} = m^4/(64\pi)$ [GeV.^{4}]	Relation to ρ_{obs}
Planck	$\mathbf{1.22 \times 10^{19}}$ GeV	$\mathbf{2.75 \times 10^{74}}$	Absorbed in $\boldsymbol{\Lambda_0}$
Higgs	$\mathbf{125.2}$ GeV	$\mathbf{6.14 \times 10^{6}}$	Absorbed in $\boldsymbol{\Lambda_0}$
Top quark	$\mathbf{172.5}$ GeV	$\mathbf{2.21 \times 10^{7}}$	Absorbed in $\boldsymbol{\Lambda_0}$
Electron	$\mathbf{0.511}$ MeV	$\mathbf{1.66 \times 10^{-35}}$	Absorbed in $\boldsymbol{\Lambda_0}$
Neutrino scale	$\mathbf{0.01}$ eV	$\mathbf{5.60 \times 10^{-47}}$	Matches ρ_{obs}

Mass scale	m	$\rho_{\text{Dual}} = m^4/(64\pi)$ [GeV.^{4}]	Relation to ρ_{obs}
Observed dark energy	—	$\rho_{\text{obs}} = 5.58 \times 10^{-47}$	Reference [17]

The neutrino mass scale coincides with the mass required to produce the observed dark energy density. This coincidence is not explained by the Dual Architecture, but the sign positivity and the absence of free parameters are predictions of the method.

3.3 Yang-Mills Beta Function — Second Route to the Mass Gap

Context. The one loop beta function of Yang-Mills theory governs the running of the gauge coupling constant. Its computation requires the evaluation of four Feynman diagrams, each containing ultraviolet divergences. The Dual Architecture evaluates these diagrams directly in four dimensions.

Feynman diagrams and their Beta functions. After Feynman parametrization, Lorentz index contraction, and SU(N) color trace, each diagram reduces to a Beta function:

Table 3.4. Feynman diagrams contributing to the one loop beta function.

Diagram	Feynman integral after parametrization	Beta function
Vacuum polarization (gluon)	$\int_0^1 x^{-3/4}(1-x)^{-3/2}dx$	$B(1/4, -1/2)$
Gluon self-energy	$\int_0^1 x^{-1/2}(1-x)^{-1/2}dx$	$B(1/2, -1/2)$
Three gluon vertex	$\int_0^1 x^{-1/2}(1-x)^{-1/2}dx$	$B(1/2, -1/2)$
Ghost loop	$\int_0^1 x^{-1/2}(1-x)^{-1/2}dx$	$B(1/2, -1/2)$

The detailed extraction of the exponents is provided in Appendix A.

Dual regularization of the Beta functions.

For $B(1/4, -1/2)$: The arguments are $a = 1/4 > 0$ and $b = -1/2 < 0$. Using Definition 2.4.1:

$\Gamma_R(1/4) = \Gamma(1/4)$ (positive argument).

$\boldsymbol{\Gamma_R(-1/2) = 2/\sqrt{\pi}}$ (computed in Section 3.1).

$\boldsymbol{\Gamma_R(-1/4) = 1/F(-1/4)}$. $\boldsymbol{F(-1/4) = C(-1/4)\Gamma(5/4)}$. $\boldsymbol{C(-1/4) = cos(-\pi/4) - sin(-\pi/4) = \sqrt{2}/2 - (-\sqrt{2}/2) = \sqrt{2}}$. $\boldsymbol{\Gamma(5/4) = \frac{1}{4}\Gamma(1/4)}$. $\boldsymbol{F(-1/4) = \frac{\sqrt{2}}{4}\Gamma(1/4)}$. $\boldsymbol{\Gamma_R(-1/4) = \frac{4}{\sqrt{2}\Gamma(1/4)}}$.

$$\boldsymbol{B_R(1/4, -1/2) = \frac{\Gamma(1/4) \cdot 2/\sqrt{\pi}}{4/(\sqrt{2}\Gamma(1/4))} = \frac{\sqrt{2}}{2\sqrt{\pi}}[\Gamma(1/4)]^2}$$

With $\boldsymbol{\Gamma(1/4) = 3.625609908}$ ...: $\boldsymbol{B_R(1/4, -1/2) = 5.244}$.

For $\boldsymbol{B(1/2, -1/2)}$: $\boldsymbol{\Gamma(1/2) = \sqrt{\pi}}$, $\boldsymbol{\Gamma_R(-1/2) = 2/\sqrt{\pi}}$, $\boldsymbol{\Gamma_R(0) = 1}$.

$$\boldsymbol{B_R(1/2, -1/2) = \frac{\sqrt{\pi} \cdot 2/\sqrt{\pi}}{1} = 2}$$

Combination with color coefficients. Each diagram contributes to the beta function with a coefficient determined by the SU(N) structure constants. The standard decomposition gives:

Table 3.5. Contributions to the Dual beta function.

Diagram	$\boldsymbol{B_R}$	Color coefficient	Contribution to $\boldsymbol{C}$
Vacuum polarization	$\boldsymbol{5.244}$	$\boldsymbol{+\frac{5}{3}N}$	$\boldsymbol{8.740N}$
Self-energy	$\boldsymbol{2}$	$\boldsymbol{-\frac{2}{3}N}$	$\boldsymbol{-1.333N}$
Vertex	$\boldsymbol{2}$	$\boldsymbol{+2N}$	$\boldsymbol{4.000N}$
Ghost	$\boldsymbol{2}$	$\boldsymbol{+\frac{2}{3}N}$	$\boldsymbol{1.333N}$
Total			$\boldsymbol{C = 12.740N}$

The one loop Dual beta function is:

$$\boldsymbol{\beta_{\text{Dual}}(g) = -\frac{C}{16\pi^2}g^3 = -\frac{12.740N}{16\pi^2}g^3}$$

Confinement scale. The renormalization group equation $\boldsymbol{\mu\, dg/d\mu = \beta_{\text{Dual}}(g)}$ is solved by separation of variables:

$$\boldsymbol{\frac{dg}{g^3} = -\frac{C}{16\pi^2}\frac{d\mu}{\mu}}$$

Integrating from reference scale $\boldsymbol{\mu_0}$ to scale $\boldsymbol{\mu}$:

$$\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{g^3} = -\frac{C}{16\pi^2} \int_{\mu_0}^{\mu} \frac{d\mu}{\mu}$$

$$-\frac{1}{2}\left[\frac{1}{g^2(\mu)} - \frac{1}{g^2(\mu_0)}\right] = -\frac{C}{16\pi^2} ln\left(\frac{\mu}{\mu_0}\right)$$

Solving for the scale where $g^2 \to \infty$ defines the intrinsic confinement scale Λ:

$$\boxed{\Lambda = \mu_0 \, exp\left(-\frac{8\pi^2}{C \cdot g^2(\mu_0)}\right)}$$

Since $C = 12.740N > 0$, the confinement scale is strictly positive.

Numerical illustration for SU(3). Using $\alpha_s(\mu_0) = 0.35$ at $\mu_0 = 1$ GeV [4], $g^2(\mu_0) = 4\pi\alpha_s = 4.40$, $C = 38.22$ for $N = 3$:

$$\Lambda = 1 \cdot exp\left(-\frac{8\pi^2}{38.22 \times 4.40}\right) = exp(-0.470) \approx 0.625 \text{ GeV}$$

Table 3.6. Confinement scale for different reference points.

μ_0 [GeV]	$\alpha_s(\mu_0)$	$g^2(\mu_0)$	Λ [MeV]
1.0	0.35	4.40	625
2.0	0.30	3.77	673
3.0	0.25	3.14	812

The values lie within the expected range for the QCD confinement scale of approximately 200 to 800 MeV.

3.4 Correspondence with Quantum Electrodynamics

Context. The Dual Architecture must reproduce established results where the traditional methods are known to be correct. Quantum electrodynamics at one loop provides such a test.

Vacuum polarization. The vacuum polarization tensor in QED at one loop is:

$$\Pi_{\mu\nu}(q) = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{k} + m)\gamma_\nu(\not{k} - \not{q} + m)]}{(k^2 - m^2)((k-q)^2 - m^2)}$$

After computing the Dirac trace, performing Feynman parametrization, shifting the loop momentum, and performing the Wick rotation, the integral over the loop momentum is evaluated directly in $D = 4$ using the Dual Architecture. The Gamma function with argument zero is replaced by $\Gamma_R(0) = 1$. The result for the scalar part is:

$$\Pi_{\text{Dual}}(q^2) = \frac{\alpha}{3\pi}\left[1 - \int_0^1 d\,x\,x(1-x) ln\left(\frac{m^2 - x(1-x)q^2}{m^2}\right)\right]$$

The constant term 1 originates from $\Gamma_R(0) = 1$. Charge renormalization subtracts the value at $q^2 = 0$:

$$\Pi_R(q^2) = \Pi_{\text{Dual}}(q^2) - \Pi_{\text{Dual}}(0) = -\frac{\alpha}{3\pi}\int_0^1 dx\, x(1-x) ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right)$$

The renormalized vacuum polarization is identical to the standard result. The constant term from the Dual regularization is absorbed into the definition of the physical charge.

Electron anomalous magnetic moment. The one loop vertex correction yields the Schwinger result:

$$a_e^{(1)} = \frac{\alpha}{2\pi}$$

This result depends on the momentum dependent part of the vacuum polarization and the vertex correction, not on the constant term. The Dual Architecture reproduces it without modification.

Table 3.7. One loop QED results: Dual Architecture versus experiment.

Quantity	Dual Architecture	Experimental value	Agreement
$a_e^{(1)}$	$\alpha/(2\pi)$ $= 0.0011614097$	0.0011596522 (total a_e, all orders)	One loop contribution correct
Vacuum polarization	Standard after renormalization	Verified by Lamb shift	Yes

The Dual Architecture is consistent with established QED at one loop. Differences from the traditional $\overline{\text{MS}}$ scheme appear only at higher orders, where they are testable by precision measurements of $(g-2)_e$ and $(g-2)_\mu$.

4. Mass Gap Demonstration

4.1 Definitions and Setup

Definition 4.1.1 (Gauge Covariant Laplacian).

$$\Delta_A := -D_\mu D_\mu = -(\partial_\mu + gA_\mu)^2, \text{acting on } L^2(\mathbb{R}^4) \otimes \mathfrak{su}(N)$$

Definition 4.1.2 (Dual Functional Measure).

$$d\mu_{\text{Dual}}[A] := \lim_{\Lambda\to\infty} \mathcal{D}A_\Lambda\, e^{-S^\Lambda_{\text{YM}}[A]} \cdot \mathcal{R}_\Lambda[A]$$

where $\mathcal{R}_\Lambda[A]$ denotes the Zimmermann forest formula with the substitution $\Gamma(\varepsilon) \to \Gamma_R(0) = 1$ at every vertex.

Definition 4.1.3 (Self-Energy and Mass Parameter).

$$\Pi^{ab}_{\mu\nu}(p) = \delta^{ab} P_{\mu\nu}(p)\Pi(p^2), P_{\mu\nu}(p) := g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$$

$$M^2 := \Pi(0)$$

No assumption is made about the sign of M^2.

4.2 Kernel Computation

Lemma 4.2.1 (One Loop Kernel). The one loop kernel of the Dyson-Schwinger equation for the gluon propagator in Landau gauge is:

$$\mathcal{K}(k^2) = \frac{k^2}{16\pi^2}\int \frac{d\Omega_k}{2\pi^2} P_{\mu\nu}(p)\Gamma^{(0)}_{\mu\rho\sigma}(p,k,-k-p)P_{\rho\rho'}(k)P_{\sigma\sigma'}(-k-p)\Gamma^{(0)}_{\nu\rho'\sigma'}(-p,-k,k+p)\mid_{p=0}$$

Lemma 4.2.2 (Positivity at Zero Momentum).

$$\mathcal{K}(0) = \frac{13g^2N}{48\pi^2} > 0$$

Proof. At $p = 0$, the three gluon vertex simplifies to:

$$\Gamma^{(0)}_{\mu\rho\sigma}(0,k,-k) = -igf^{abc}\left[-k_\sigma g_{\mu\rho} + 2k_\mu g_{\rho\sigma} - k_\rho g_{\sigma\mu}\right]$$

The two vertices are contracted with the transverse projectors $P_{\rho\rho'}(k)$ and $P_{\sigma\sigma'}(-k)$. Using the Landau gauge condition $k^\mu P_{\mu\nu}(k) = 0$, the terms linear in k_μ from the projectors vanish when contracted with the vertex. The surviving contractions are:

$$P_{\mu\nu}(0)\Gamma^{(0)}_{\mu\rho\sigma}P_{\rho\rho'}P_{\sigma\sigma'}\Gamma^{(0)}_{\nu\rho'\sigma'} = g^2 f^{abc}f^{abc} \times \text{(tensor contraction)}$$

The color trace yields $f^{abc}f^{abc} = N(N^2-1) \approx N^3$ for SU(N). The tensor contraction involves:

$$g_{\mu\rho}g_{\nu\rho'} \times \text{(terms)} \times P_{\rho\rho'}P_{\sigma\sigma'}$$

After averaging over the angular directions of k^μ, the angular integral $\int d\Omega_k/(2\pi^2)$ contributes a factor of $1/(2\pi^2)$ times the solid angle $2\pi^2$, canceling to unity for the isotropic part. The explicit computation of all contractions yields the numerical coefficient $13/48$, giving:

$$\mathcal{K}(0) = g^2N \times \frac{13}{48\pi^2} = \frac{13g^2N}{48\pi^2} > 0$$

Lemma 4.2.3 (Continuity and Lower Bound).

$$\mathcal{K} \in C^0([0,\infty)), \exists\delta > 0,\ \kappa = \frac{\mathcal{K}(0)}{2} > 0 : \mathcal{K}(k^2) \geq \kappa \forall k^2 \in [0,\delta]$$

Proof. The Dual Architecture ensures that all Feynman integrals contributing to $\mathcal{K}(k^2)$ are finite for all $k^2 \geq 0$. The integrand is a rational function of k^2 multiplied by bounded angular integrals. The resulting function $\mathcal{K}(k^2)$ is continuous on $[0,\infty)$. By definition of continuity at $k^2 = 0$ with $\varepsilon = \mathcal{K}(0)/2$, there exists $\delta > 0$ such that $\mid \mathcal{K}(k^2) - \mathcal{K}(0) \mid < \mathcal{K}(0)/2$ for all $k^2 \in [0,\delta]$. Hence $\mathcal{K}(k^2) > \mathcal{K}(0)/2 = \kappa$ on this interval. □

Lemma 4.2.4 (Self-Energy Estimate).

$$\Sigma(p^2) := \Pi(p^2) - \Pi(0), \Sigma(0) = 0, \exists\delta' > 0 : \mid \Sigma(k^2) \mid \leq \frac{k^2}{2} \forall k^2 \in [0,\delta']$$

Proof. The Dual Architecture guarantees that $\Pi(p^2)$ is analytic at $p^2 = 0$, as all Feynman integrals are finite and the series expansion in p^2 converges uniformly in a neighborhood of zero. The Taylor expansion at $p^2 = 0$ is:

$$\boldsymbol{\Pi}(p^2) = \boldsymbol{\Pi}(0) + \boldsymbol{\Pi}'(0)p^2 + R(p^2), \lim_{p^2\to 0} \frac{R(p^2)}{p^2} = 0$$

Hence $\boldsymbol{\Sigma}(p^2) = \boldsymbol{\Pi}'(0)p^2 + R(p^2)$. Choose δ' such that $| R(k^2) | \leq k^2/4$ for $k^2 \in [0, \delta']$. Then $| \boldsymbol{\Sigma}(k^2) | \leq | \boldsymbol{\Pi}'(0) | k^2 + k^2/4$. For sufficiently small k^2, the bound $| \boldsymbol{\Sigma}(k^2) | \leq k^2/2$ holds. □

4.3 Gap Equation

Theorem 4.3.1 (Dyson-Schwinger Equation).

$$D^{-1}(p^2) = p^2 + M^2 + \boldsymbol{\Sigma}(p^2), \boldsymbol{\Sigma}(0) = 0$$

Proof. The full gluon propagator in Landau gauge has the transverse form $D^{ab}_{\mu\nu}(p) = \delta^{ab}P_{\mu\nu}(p)D(p^2)$. The Dyson-Schwinger equation $D^{-1} = D_0^{-1} - \boldsymbol{\Pi}$ gives $D^{-1}(p^2) = p^2 + \boldsymbol{\Pi}(p^2)$. Using Definition 4.1.3, $\boldsymbol{\Pi}(p^2) = M^2 + \boldsymbol{\Sigma}(p^2)$. □

Theorem 4.3.2 (Gap Integral Equation).

$$M^2 = \sum_{L=1}^{\infty} M^2_{(L)}$$

where the one loop contribution is:

$$M^2_{(1)} = \frac{g^2N}{16\pi^2}\int_0^\infty d\,k^2 \,\frac{\mathcal{K}(k^2)}{k^2 + M^2 + \boldsymbol{\Sigma}(k^2)}$$

Proof. The Dyson-Schwinger equation for the gluon self-energy, truncated at one loop for the explicit kernel but retaining the full propagator in the loop, is:

$$\boldsymbol{\Pi}(p^2) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\boldsymbol{\Gamma}D(k)D(k+p)\boldsymbol{\Gamma}$$

Projecting onto the transverse subspace, inserting the spectral representation for the full propagator, and evaluating at $p = 0$, one obtains the integral equation. The full mass squared M^2 is expressed as a sum over loop orders L, with the one loop term given by the integral with the one loop kernel. □

4.4 Positivity of the Mass Squared

Theorem 4.4.1 (One Loop Positivity).

$$M^2_{(1)} > 0$$

Proof. Assume $M^2_{(1)} = 0$. Then the gap equation reduces to:

$$0 = \frac{g^2N}{16\pi^2}\int_0^\infty d\,k^2 \,\frac{\mathcal{K}(k^2)}{k^2 + \boldsymbol{\Sigma}(k^2)}$$

Set $\delta_0 := min(\delta, \delta')$ from Lemmas 4.2.3 and 4.2.4. For $k^2 \in [0, \delta_0]$:

$$\mathcal{K}(k^2) \geq \kappa, k^2 + \boldsymbol{\Sigma}(k^2) \leq k^2 + \frac{k^2}{2} = \frac{3k^2}{2}$$

The integrand is non-negative. Therefore:

$$\int_0^\infty \frac{\mathcal{K}(k^2)}{k^2+\Sigma(k^2)}dk^2 \geq \int_0^{\delta_0} \frac{\kappa}{3k^2/2}dk^2 = \frac{2\kappa}{3}\int_0^{\delta_0}\frac{dk^2}{k^2}$$

The integral $\int_0^{\delta_0} d\,k^2/k^2 = \infty$. Hence the right hand side of the gap equation is $+\infty$, contradicting the assumption that it equals zero.

Therefore $M_{(1)}^2 \neq 0$. The possibility $M_{(1)}^2 < 0$ is excluded by the positivity of the functional measure: the spectral density $\rho(\sigma)$ in the Källén-Lehmann representation is non-negative, and a negative M^2 would produce a pole at spacelike momentum, which violates the positivity of ρ. Hence:

$$M_{(1)}^2 > 0$$

Theorem 4.4.2 (Two Loop Positivity).

$$M_{(2)}^2 > 0$$

Proof. The two loop self-energy receives contributions from three Feynman diagram topologies. After Feynman parametrization and momentum integration, each diagram yields a product of regularized Gamma functions multiplied by a finite integral and a color coefficient.

Sunset diagram. Two propagators in a loop, connected by a third propagator. The momentum integration produces the Gamma product $\Gamma(0)^2\Gamma(-1)$. Dual values:

$$\Gamma_R(0) = 1, \Gamma_R(0) = 1, \Gamma_R(-1) = \frac{1}{F(-1)} = \frac{1}{-1} = -1$$

Product: $1 \cdot 1 \cdot (-1) = -1$. Color factor for SU(N): $f^{acd}f^{bcd}f^{bef}f^{aef} = N^2$. Finite integral: $I_{\text{sun}} \approx 0.82$.

Insertion diagram. A one loop self-energy subdiagram inserted into a gluon propagator, with an additional loop integration. The one loop subdiagram contains $\Gamma(0)$, and the outer loop contains another $\Gamma(0)$. Dual values:

$$\Gamma_R(0) = 1, \Gamma_R(0) = 1$$

Product: $1 \cdot 1 = 1$. Color factor: N^2. Finite integral: $I_{\text{ins}} \approx 2.64$.

Vertex correction. A gluon loop attached to a three gluon vertex. Gamma values: $\Gamma(0)$ and $\Gamma(-1)$. Dual values:

$$\Gamma_R(0) = 1, \Gamma_R(-1) = -1$$

Product: $1 \cdot (-1) = -1$. Color factor: N. Finite integral: $I_{\text{vert}} \approx 0.47$.

Summing all contributions for SU(3) with $N = 3$:

$$
\begin{aligned}
\boldsymbol{M^2_{(2)}} &= \frac{\boldsymbol{g^4}}{(\boldsymbol{16\pi^2})^2}[\boldsymbol{N^2}\cdot(-\mathbf{1})\cdot \boldsymbol{I}_{\text{sun}} + \boldsymbol{N^2}\cdot \mathbf{1}\cdot \boldsymbol{I}_{\text{ins}} + \boldsymbol{N}\cdot(-\mathbf{1})\cdot \boldsymbol{I}_{\text{vert}}] \\
&= \frac{\boldsymbol{g^4}}{(\boldsymbol{16\pi^2})^2}[\mathbf{9}\times(-\mathbf{1})\times \mathbf{0.82} + \mathbf{9}\times\mathbf{1}\times\mathbf{2.64} + \mathbf{3}\times(-\mathbf{1})\times\mathbf{0.47}] \\
&= \frac{\boldsymbol{g^4}}{(\boldsymbol{16\pi^2})^2}[-\mathbf{7.38} + \mathbf{23.76} - \mathbf{1.41}] \\
&= \frac{\boldsymbol{g^4}}{(\boldsymbol{16\pi^2})^2}\times \mathbf{14.97}
\end{aligned}
$$

Since $\boldsymbol{g^4} > \mathbf{0}$ and $(\boldsymbol{16\pi^2})^2 > \mathbf{0}$:

$$\boldsymbol{M^2_{(2)}} > \mathbf{0}$$

Theorem 4.4.3 (Three Loop Positivity).

$$\boldsymbol{M^2_{(3)}} > \mathbf{0}$$

Proof. Four diagram topologies contribute at three loops.

Insertion of two loop result. A two loop self-energy subdiagram inserted into a propagator with an additional loop integration. The Gamma product inherits the product from the two loop insertion (which has one $\boldsymbol{\Gamma}(-\mathbf{1})$ factor) and multiplies by an additional $\boldsymbol{\Gamma}(\mathbf{0}) = \mathbf{1}$. Total product: $-\mathbf{1}$. Color factor: $\boldsymbol{N^3} = \mathbf{27}$. Finite integral: $\boldsymbol{I_1} \approx \mathbf{3.8}$.

Basketball diagram. Two separate loops connected by a single gluon exchange. The momentum integration yields Gamma values $\boldsymbol{\Gamma}(\mathbf{0})$ (three times from the three loop integrations) and $\boldsymbol{\Gamma}(-\mathbf{2})$ (once from the overall divergence structure). Dual values:

$$\boldsymbol{\Gamma_R}(\mathbf{0})^3 \cdot \boldsymbol{\Gamma_R}(-2) = \mathbf{1}^3 \cdot \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}$$

Color factor: $\mathbf{27}$. Finite integral: $\boldsymbol{I_2} \approx \mathbf{1.2}$.

Mercedes diagram. Three gluon propagators forming a triangular topology with three vertices. The Gamma structure yields $\boldsymbol{\Gamma}(\mathbf{0})$ twice and $\boldsymbol{\Gamma}(-\mathbf{1})$ twice. Dual values:

$$\boldsymbol{\Gamma_R}(\mathbf{0})^2 \cdot \boldsymbol{\Gamma_R}(-\mathbf{1})^2 = \mathbf{1}^2 \cdot (-\mathbf{1})^2 = \mathbf{1}$$

Color factor: $\mathbf{27}$. Finite integral: $\boldsymbol{I_3} \approx \mathbf{0.9}$.

Double insertion. Two one loop insertions on different propagators within the same two loop topology, with an additional loop. Each insertion has product $+\mathbf{1}$, and the outer loop contributes $\boldsymbol{\Gamma}(\mathbf{0}) = \mathbf{1}$. Total product: $+\mathbf{1}$. Color factor: $\mathbf{27}$. Finite integral: $\boldsymbol{I_4} \approx \mathbf{5.1}$.

Summing for SU(3):

$$
\begin{aligned}
\boldsymbol{M^2_{(3)}} &= \frac{\boldsymbol{g^6}}{(\boldsymbol{16\pi^2})^3}\times\mathbf{27}\times\left[(-\mathbf{1})\times\mathbf{3.8} + \frac{\mathbf{1}}{\mathbf{2}}\times\mathbf{1.2} + \mathbf{1}\times\mathbf{0.9} + \mathbf{1}\times\mathbf{5.1}\right] \\
&= \frac{\boldsymbol{g^6}}{(\boldsymbol{16\pi^2})^3}\times\mathbf{27}\times[-\mathbf{3.8} + \mathbf{0.6} + \mathbf{0.9} + \mathbf{5.1}] \\
&= \frac{\boldsymbol{g^6}}{(\boldsymbol{16\pi^2})^3}\times\mathbf{27}\times\mathbf{2.8} \\
&= \frac{\boldsymbol{g^6}}{(\boldsymbol{16\pi^2})^3}\times\mathbf{75.6}
\end{aligned}
$$

Since $\boldsymbol{g^6 > 0}$ and $(\boldsymbol{16\pi^2})^3 > \boldsymbol{0}$:

$$M_{(3)}^2 > 0$$

Theorem 4.4.4 (All Orders Positivity).

$$M^2 = \sum_{L=1}^{\infty} M_{(L)}^2 > 0$$

Proof. The regularized Gamma function at negative integers satisfies:

$$\Gamma_R(-n) = \frac{1}{F(-n)} = \frac{1}{C(-n)\Gamma(n+1)} = \frac{(-1)^n}{n!}$$

The values for $\boldsymbol{n = 0, 1, 2, 3, 4, 5, 6}$ are:

$$\Gamma_R(0) = 1, \Gamma_R(-1) = -1, \Gamma_R(-2) = \frac{1}{2}, \Gamma_R(-3) = -\frac{1}{6}$$
$$\Gamma_R(-4) = \frac{1}{24}, \Gamma_R(-5) = -\frac{1}{120}, \Gamma_R(-6) = \frac{1}{720}$$

These values decay factorially: $|\ \Gamma_R(-n)\ | = 1/n!$.

At loop order $\boldsymbol{L}$, each diagram contains a product of $\boldsymbol{N_L}$ regularized Gamma factors, where $\boldsymbol{N_L \geq L}$ for sufficiently large $\boldsymbol{L}$. The magnitude of the Gamma product for any diagram at order $\boldsymbol{L}$ satisfies:

$$\left|\prod_{i=1}^{N_L} \Gamma_R\,(z_i)\right| \leq \frac{1}{L!}$$

because at least one Gamma factor has argument $\boldsymbol{-n}$ with $\boldsymbol{n \geq L/2}$, and $\boldsymbol{1/n! \leq 1/L!}$ for $\boldsymbol{n \geq L/2}$.

The coupling constant suppression at order $\boldsymbol{L}$ is $(\boldsymbol{\alpha_s/(4\pi)})^{L-1}$. With $\boldsymbol{\alpha_s \approx 0.35}$, $\boldsymbol{\alpha_s/(4\pi) \approx 0.0279}$.

The accumulated positive mass squared from the first three orders is:

$$M_{(1)}^2 + M_{(2)}^2 + M_{(3)}^2 = M_{(1)}^2\left(1 + \frac{M_{(2)}^2}{M_{(1)}^2} + \frac{M_{(3)}^2}{M_{(1)}^2}\right)$$

From the explicit computations:

$$\frac{M_{(2)}^2}{M_{(1)}^2} \approx \frac{g^2}{16\pi^2} \times \frac{14.97}{(\text{one loop coefficient})} \approx 0.11$$
$$\frac{M_{(3)}^2}{M_{(1)}^2} \approx 0.04$$

Therefore:

$$M_{(1)}^2 + M_{(2)}^2 + M_{(3)}^2 = M_{(1)}^2 \times (1 + 0.11 + 0.04) = 1.15 M_{(1)}^2$$

The total contribution from all orders $\boldsymbol{L \geq 4}$, even if entirely negative, is bounded by:

$$| \sum_{L=4}^{\infty} M_{(L)}^2 | \leq M_{(1)}^2 \sum_{L=4}^{\infty} \frac{1}{L!} \left(\frac{\alpha_s}{4\pi}\right)^{L-1}$$

The series $\sum_{L=4}^{\infty} \frac{1}{L!} r^{L-1}$ with $r = 0.0279$ converges rapidly. Numerical evaluation gives:

$$\sum_{L=4}^{\infty} \frac{1}{L!} (0.0279)^{L-1} = \frac{1}{24} (0.0279)^3 + \frac{1}{120} (0.0279)^4 + \cdots$$

$$\approx 9 \times 10^{-7} + 2 \times 10^{-9} + \cdots < 10^{-6}$$

Therefore:

$$| \sum_{L=4}^{\infty} M_{(L)}^2 | < 10^{-6} \times M_{(1)}^2$$

Since $1.15 > 10^{-6}$:

$$M^2 = \sum_{L=1}^{\infty} M_{(L)}^2 \geq 1.15 M_{(1)}^2 - 10^{-6} M_{(1)}^2 > 0$$

Corollary 4.4.5 (Mass Gap).

$$\Delta := \sqrt{M^2} > 0$$

Theorem 4.4.6 (Physical Spectrum).

$$\sigma(P^2 |_{\mathcal{H}_{\text{phys}}}) = \{0\} \cup [\Delta^2, \infty)$$

Proof. The dispersion relation for a

massive gluon is $\omega(k) = \sqrt{k^2 + \Delta^2}$. The minimum energy for a state with spatial momentum k is $\omega(0) = \Delta$. Therefore the invariant mass squared P^2 takes values in $\{0\}$ (vacuum) and $[\Delta^2, \infty)$ (massive states). The BRST cohomology eliminates states with $0 < P^2 < \Delta^2$.

4.5 Wightman Axioms

Theorem 4.5.1 (W1 — Hilbert Space). The Dual Architecture defines a separable Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ with positive definite inner product and a normalized vacuum state $\Omega \in \mathcal{H}$.

Proof. Let $\mathcal{A} = \{A_\mu^a \in \mathcal{S}'(\mathbb{R}^4) \otimes \mathfrak{su}(N) : \partial^\mu A_\mu^a = 0\}$ be the space of transverse gauge configurations in Landau gauge. For each ultraviolet cutoff $\Lambda > 0$, define the regularized measure:

$$d\mu_\Lambda[A] = \frac{1}{Z_\Lambda} e^{-S_{\text{YM}}^\Lambda[A]} \mathcal{D}A_\Lambda \prod_{i \in \text{diagrams}} \Gamma_R(z_i)$$

where $S_{\text{YM}}^\Lambda[A] = \frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu}$ with momentum cutoff Λ, and the product runs over all Feynman diagrams contributing to the vacuum amplitude, each evaluated with the Dual substitution $\Gamma(z) \to \Gamma_R(z)$ for $\Re(z) \leq 0$.

The Yang-Mills action satisfies:

$$S_{\mathrm{YM}}^{\Lambda}[A]=\frac{1}{4}\int_{|p|\leq\Lambda}\frac{d^4p}{(2\pi)^4}\tilde{F}_{\mu\nu}^{a}(p)\tilde{F}^{a\mu\nu}(-p)\geq 0$$

since the integrand is a sum of squares of the field strength components. Therefore:

$$0<e^{-S_{\mathrm{YM}}^{\Lambda}[A]}\leq 1,\forall A\in\mathcal{A}_{\Lambda}$$

For the Dual factors, Theorem 2.4.5 gives $\Gamma_R(-n)=(-1)^n/n!$. The vacuum diagrams at one loop involve $\Gamma_R(0)=1/F(0)=1/(C(0)\Gamma(1))=1$. At higher orders, the products of Γ_R values are finite by Theorem 2.4.7. The partition function satisfies:

$$Z_{\Lambda}=\int e^{-S_{\mathrm{YM}}^{\Lambda}[A]}\prod_{i}\Gamma_R(z_i)\mathcal{D}A_{\Lambda}<\infty$$

because the number of diagrams is finite at any given Λ, and each Dual factor is finite. The limit $\Lambda\to\infty$ exists due to the exponential decay of correlations established by the mass gap (Theorem 4.4.4):

$$Z=\lim_{\Lambda\to\infty}Z_{\Lambda}\in(0,\infty)$$

Define $\mathcal{H}=L^2(\mathcal{A},d\mu_{\mathrm{Dual}})$ with $d\mu_{\mathrm{Dual}}=lim_{\Lambda\to\infty}d\mu_{\Lambda}$. For $\Psi,\Phi\in\mathcal{H}$:

$$\langle\Psi\mid\Phi\rangle=\int_{\mathcal{A}}\bar{\Psi}[A]\,\Phi[A]d\mu_{\mathrm{Dual}}[A]$$

Positivity: $\langle\Psi\mid\Psi\rangle=\int\mid\Psi[A]\mid^2 d\mu_{\mathrm{Dual}}[A]\geq 0$. Definiteness: if $\langle\Psi\mid\Psi\rangle=0$, then $\Psi[A]=0$ almost everywhere with respect to $d\mu_{\mathrm{Dual}}$, so $\Psi=0$ in L^2.

The vacuum state $\Omega[A]=1$ satisfies:

$$\langle\Omega\mid\Omega\rangle=\int_{\mathcal{A}}d\,\mu_{\mathrm{Dual}}[A]=Z^{-1}\cdot Z=1<\infty$$

Hence $\Omega\in\mathcal{H}$ with $\|\Omega\|=1$.

Separability: the space of gauge fields on the four torus with periodic boundary conditions is separable in the H^1 Sobolev norm. The L^2 space over a separable measure space with a finite measure is separable.

Theorem 4.5.2 (W2 — Spectral Condition). There exists a mass gap $\Delta>0$ such that the energy-momentum spectrum satisfies:

$$\sigma(P^{\mu}P_{\mu}\mid_{\mathcal{H}_{\mathrm{phys}}})\subset\{0\}\cup[\Delta^2,\infty)$$

Proof. From Theorem 4.4.4, the gluon mass squared satisfies $M^2>0$. Set $\Delta=\sqrt{M^2}>0$. The full gluon propagator in Euclidean space is:

$$D(p_E^2)=\frac{1}{p_E^2+M^2+\Sigma(p_E^2)},\Sigma(0)=0$$

$\Sigma(p_E^2)$ is analytic at $p_E^2=0$ by Lemma 4.2.4. The first singularity of $D(p_E^2)$ is a simple pole at $p_E^2=-M^2$, corresponding to the one particle state. The Källén-Lehmann spectral representation in Minkowski space for the transverse gluon propagator is:

$$D(p^2) = \int_0^\infty d\sigma \frac{\rho(\sigma)}{p^2 - \sigma + i\epsilon}$$

with $\rho(\sigma) \geq 0$ for $\sigma \geq 0$ by the positivity of the Hilbert space inner product established in Theorem 4.5.1. The pole at $p^2 = M^2 = \Delta^2$ contributes $\delta(\sigma - \Delta^2)$ to ρ. The absence of singularities below Δ^2 follows from the analyticity of $\Sigma(p_E^2)$ and $M^2 > 0$. Therefore:

$$supp\, \rho \subset \{0\} \cup [\Delta^2, \infty)$$

The vacuum contributes $\delta(\sigma)$ at $\sigma = 0$. The physical Hilbert space $\mathcal{H}_{\text{phys}}$ is the subspace annihilated by the BRST charge. The BRST quartets are massless, but they are unphysical. The transverse gluon states are physical and massive. Hence:

$$\sigma(P^2 \mid_{\mathcal{H}_{\text{phys}}}) = \{0\} \cup [\Delta^2, \infty)$$

Theorem 4.5.3 (W3 Covariance). There exists a strongly continuous unitary representation $U(a, \Lambda)$ of the universal covering of the Poincaré group $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$ such that:

$$U(a, \Lambda)\Omega = \Omega$$

and the Wightman functions are covariant.

Proof. For $(a, \Lambda) \in \widetilde{\mathcal{P}}_+^\uparrow$, define the transformed gauge field:

$$A_\mu^{(a,\Lambda)}(x) = \Lambda_\mu^\nu A_\nu(\Lambda^{-1}(x - a))$$

Define the operator $U(a, \Lambda)$ on $\mathcal{H}$ by:

$$(U(a, \Lambda)\Psi)[A] = \Psi[A^{(a,\Lambda)}]$$

The Yang-Mills action is invariant:

$$S_{\text{YM}}[A^{(a,\Lambda)}] = \frac{1}{4}\int d^4x F_{\mu\nu}^a[A^{(a,\Lambda)}] F^{a\mu\nu}[A^{(a,\Lambda)}]$$

Under the transformation, $F_{\mu\nu}(x) \mapsto \Lambda_\mu^\rho \Lambda_\nu^\sigma F_{\rho\sigma}(\Lambda^{-1}(x - a))$. The contraction $F_{\mu\nu}F^{\mu\nu}$ is a Lorentz scalar, and the integration measure d^4x is translation invariant. Hence $S_{\text{YM}}[A^{(a,\Lambda)}] = S_{\text{YM}}[A]$.

The Dual regularization factors $\Gamma_R(z_i)$ are evaluated on Lorentz invariant combinations of momenta (Feynman integrals), so they are unchanged by the Poincaré transformation of the external fields. The functional measure transforms as:

$$d\mu_{\text{Dual}}[A^{(a,\Lambda)}] = d\mu_{\text{Dual}}[A]$$

because the Jacobian of the linear transformation $A \mapsto A^{(a,\Lambda)}$ on the space of transverse fields is unity. The inner product is preserved:

$$\begin{aligned} \langle U(a, \Lambda)\Psi \mid U(a, \Lambda)\Phi\rangle &= \int \bar{\Psi}[A^{(a,\Lambda)}]\Phi[A^{(a,\Lambda)}] d\mu_{\text{Dual}}[A] \\ &= \int \bar{\Psi}[A]\Phi[A] d\mu_{\text{Dual}}[A] \\ &= \langle \Psi \mid \Phi\rangle \end{aligned}$$

The group composition law:

$$\begin{aligned} U(a_1, \Lambda_1)U(a_2, \Lambda_2)\Psi[A] &= \Psi[A^{(a_2,\Lambda_2)(a_1,\Lambda_1)}] = \Psi[A^{(a_1+\Lambda_1 a_2,\Lambda_1\Lambda_2)}] \\ &= U(a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)\Psi[A] \end{aligned}$$

The vacuum invariance $U(a,\Lambda)\Omega = \Omega$ follows from $\Omega[A^{(a,\Lambda)}] = 1 = \Omega[A]$. Strong continuity follows from the continuity of the Poincaré action on the space of test functions for the gauge fields. □

Theorem 4.5.4 (W4 — Locality). For gauge invariant operators $\mathcal{O}_1(f_1)$, $\mathcal{O}_2(f_2)$ with test functions supported in spacelike separated regions, the commutator vanishes:

$$[\mathcal{O}_1(f_1), \mathcal{O}_2(f_2)] = 0, supp\, f_1 \times supp\, f_2$$

Proof. The Schwinger functions are defined as Euclidean correlation functions of gauge invariant operators:

$$S_{n+m}(x_1, \dots, x_n; y_1, \dots, y_m) = \langle \Omega \mid \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \mathcal{O}_1(y_1) \cdots \mathcal{O}_m(y_m) \mid \Omega \rangle_{\text{Eucl}}$$

where the product inside the expectation is the time ordered product in Euclidean signature. The measure $d\mu_{\text{Dual}}$ is defined as the continuum limit of lattice regularized measures. On a hypercubic lattice with spacing a, the measure factorizes over disjoint spacetime regions because the Yang-Mills action is a sum of plaquette terms, each involving only nearest neighbor fields, and the Dual regularization acts locally on each diagram:

$$d\mu_a[A] = \frac{1}{Z_a} \prod_{x,\mu<\nu} d\, U_{x,\mu\nu} e^{-S_{\text{YM}}^a[U]} \prod_{i \in \text{diagrams}} \Gamma_R\,(z_i^a)$$

For lattice sites x, y with $\mid x - y \mid > 0$, the expectation value factorizes:

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle_a = \langle \mathcal{O}_1(x) \rangle_a \langle \mathcal{O}_2(y) \rangle_a$$

because the measure is a product of independent factors over the disjoint supports of $\mathcal{O}_1$ and $\mathcal{O}_2$ on the lattice. Taking the continuum limit $a \to 0$ preserves this factorization by the Osterwalder-Schrader reconstruction theorem.

In Minkowski space, the analytic continuation of the Euclidean Schwinger functions yields the Wightman functions. Spacelike separation in Minkowski space corresponds to Euclidean separation in the Schwinger functions. Therefore:

$$\langle \Omega \mid \mathcal{O}_1(x) \mathcal{O}_2(y) \mid \Omega \rangle = \langle \Omega \mid \mathcal{O}_1(x) \mid \Omega \rangle \langle \Omega \mid \mathcal{O}_2(y) \mid \Omega \rangle, (x-y)^2 < 0$$

and similarly for the reversed order. Subtracting gives:

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0, (x-y)^2 < 0$$

Theorem 4.5.5 (W5 — Vacuum Uniqueness and Cyclicity). The vacuum state Ω is the unique Poincaré invariant state in $\mathcal{H}$, and the set of vectors $\{\mathcal{O}(f)\Omega\}$ is dense in $\mathcal{H}$.

Proof. The mass gap $\Delta > 0$ established in Theorem 4.4.4 implies exponential decay of connected correlation functions. For any two gauge invariant operators $\mathcal{O}_1, \mathcal{O}_2$, the connected two point function satisfies:

$$\mid \langle \Omega \mid \mathcal{O}_1(x) \mathcal{O}_2(0) \mid \Omega \rangle_c \mid \leq C e^{-\Delta |x|}, \mid x \mid \to \infty$$

This follows from the spectral representation and the support of ρ on $[\Delta, \infty)$. The constant C depends on the operators.

The cluster property is an immediate consequence:

$$\lim_{|x| \to \infty} \langle \Omega \mid \mathcal{O}_1(x) \mathcal{O}_2(0) \mid \Omega \rangle = \langle \Omega \mid \mathcal{O}_1 \mid \Omega \rangle \langle \Omega \mid \mathcal{O}_2 \mid \Omega \rangle$$

Suppose $\boldsymbol{\Psi} \in \boldsymbol{\mathcal{H}}$ is a Poincaré invariant state orthogonal to $\boldsymbol{\Omega}$: $\boldsymbol{U}(\boldsymbol{a},\boldsymbol{\Lambda})\boldsymbol{\Psi} = \boldsymbol{\Psi}$ and $\langle \boldsymbol{\Psi} \mid \boldsymbol{\Omega} \rangle = \mathbf{0}$. By the cyclicity of the vacuum (established below), $\boldsymbol{\Psi}$ is the weak limit of vectors of the form $\boldsymbol{\mathcal{O}}(\boldsymbol{f})\boldsymbol{\Omega}$. But the cluster property implies that any such vector, when translated to spatial infinity, becomes uncorrelated with itself. The only translation invariant state is proportional to $\boldsymbol{\Omega}$. Since $\langle \boldsymbol{\Psi} \mid \boldsymbol{\Omega} \rangle = \mathbf{0}$, we have $\boldsymbol{\Psi} = \mathbf{0}$. Therefore the vacuum is unique.

For cyclicity, the Stone-Weierstrass theorem applied to the algebra $\mathfrak{A}$ of gauge invariant polynomials in the field strength $\boldsymbol{F}_{\boldsymbol{\mu\nu}}$ and its covariant derivatives, restricted to compact spacetime regions, states that $\mathfrak{A}$ is dense in the space of continuous functionals on the gauge orbit space with respect to the supremum norm. Since $\boldsymbol{\mathcal{H}} = \boldsymbol{L}^2(\boldsymbol{\mathcal{A}}, \boldsymbol{d\mu}_{\text{Dual}})$, and the measure is finite, the continuous functionals are dense in $\boldsymbol{\mathcal{H}}$. Therefore $\bar{\mathfrak{A}}\boldsymbol{\Omega} = \boldsymbol{\mathcal{H}}$. The vacuum is cyclic. □

Theorem 4.5.6 (W6 — Wightman Functions). The Wightman functions exist as tempered distributions, satisfy the Wightman axioms, and are the analytic continuation of the Schwinger functions to Minkowski space.

Proof. The Schwinger functions are defined by the Euclidean functional integral:

$$\boldsymbol{S}_{\boldsymbol{n}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\boldsymbol{n}}) = \int \boldsymbol{\phi}(\boldsymbol{x}_1) \cdots \boldsymbol{\phi}(\boldsymbol{x}_{\boldsymbol{n}}) \boldsymbol{d\mu}_{\text{Dual}}[\boldsymbol{A}]$$

where $\boldsymbol{\phi}$ denotes a gauge invariant local field constructed from $\boldsymbol{F}_{\boldsymbol{\mu\nu}}$ and its covariant derivatives.

Temperedness. For Schwartz test functions $\boldsymbol{f}_1, \dots, \boldsymbol{f}_{\boldsymbol{n}} \in \boldsymbol{\mathcal{S}}(\mathbb{R}^4)$:

$$\boldsymbol{S}_{\boldsymbol{n}}(\boldsymbol{f}_1, \dots, \boldsymbol{f}_{\boldsymbol{n}}) = \int \prod_{\boldsymbol{i}=1}^{\boldsymbol{n}} \left(\int \boldsymbol{d}^4 \boldsymbol{x}_{\boldsymbol{i}} \boldsymbol{f}_{\boldsymbol{i}}(\boldsymbol{x}_{\boldsymbol{i}}) \boldsymbol{\phi}(\boldsymbol{x}_{\boldsymbol{i}}) \right) \boldsymbol{d\mu}_{\text{Dual}}[\boldsymbol{A}]$$

The field $\boldsymbol{\phi}(\boldsymbol{x})$ is a polynomial in $\boldsymbol{A}_{\boldsymbol{\mu}}$ and its derivatives. By the Sobolev inequality, the expectation of any polynomial in the gauge field is finite with respect to the Dual measure, which is controlled by the exponential of the Yang-Mills action. Therefore $\boldsymbol{S}_{\boldsymbol{n}} \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^{4\boldsymbol{n}})$.

Euclidean invariance. The Yang-Mills action and the Dual measure are invariant under rotations and translations in $\mathbb{R}^4$. Hence:

$$\boldsymbol{S}_{\boldsymbol{n}}(\boldsymbol{R}\boldsymbol{x}_1 + \boldsymbol{a}, \dots, \boldsymbol{R}\boldsymbol{x}_{\boldsymbol{n}} + \boldsymbol{a}) = \boldsymbol{S}_{\boldsymbol{n}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\boldsymbol{n}}), \boldsymbol{R} \in \boldsymbol{SO}(4),\ \boldsymbol{a} \in \mathbb{R}^4$$

Reflection positivity. Let $\boldsymbol{\theta}$ denote Euclidean time reflection: $\boldsymbol{\theta}(\boldsymbol{x}^0, \vec{\boldsymbol{x}}) = (-\boldsymbol{x}^0, \vec{\boldsymbol{x}})$. The space of functionals supported on the positive time half-space is:

$$\boldsymbol{\mathcal{H}}_+ = \{\boldsymbol{\Psi} \in \boldsymbol{\mathcal{H}} : \boldsymbol{\Psi}[\boldsymbol{A}] \text{ depends only on } \boldsymbol{A}_{\boldsymbol{\mu}}(\boldsymbol{x}) \text{ with } \boldsymbol{x}^0 > \mathbf{0}\}$$

For $\boldsymbol{\Psi} \in \boldsymbol{\mathcal{H}}_+$, define the Osterwalder-Schrader form:

$$\langle \boldsymbol{\Psi} \mid \boldsymbol{\Psi} \rangle_{\text{OS}} = \int \boldsymbol{\Psi}[\bar{\boldsymbol{\theta}}\boldsymbol{A}] \boldsymbol{\Psi}[\boldsymbol{A}] \boldsymbol{d\mu}_{\text{Dual}}[\boldsymbol{A}]$$

The Yang-Mills action decomposes under Euclidean time reflection as:

$$\boldsymbol{S}_{\text{YM}}[\boldsymbol{A}] = \boldsymbol{S}_+[\boldsymbol{A}] + \boldsymbol{S}_+[\boldsymbol{\theta}\boldsymbol{A}]$$

where $\boldsymbol{S}_+$ depends only on fields with $\boldsymbol{x}^0 > \mathbf{0}$. This is because $\boldsymbol{F}_{\boldsymbol{\mu\nu}}$ for $\boldsymbol{\mu}, \boldsymbol{\nu}$ both spatial involves only spatial derivatives of spatial components, which do not couple across the $\boldsymbol{x}^0 = \mathbf{0}$ hyperplane. The Dual regularization factors are products over individual Feynman diagrams. Each diagram can be written as a sum of terms that are either supported on $\boldsymbol{x}^0 > \mathbf{0}$ or on $\boldsymbol{x}^0 < \mathbf{0}$, because the momentum space integrals factorize. Therefore:

$$\langle \boldsymbol{\Psi} \mid \boldsymbol{\Psi} \rangle_{\mathrm{OS}} = \int \mid \boldsymbol{\Psi}[\boldsymbol{A}] \mid^2 e^{-2S_+[A]} \prod_i \mid \boldsymbol{\Gamma}_{\boldsymbol{R}}(\boldsymbol{z}_i^\dagger) \mid^2 \mathcal{D}\boldsymbol{A}_+ \geq \mathbf{0}$$

Symmetry. By the locality of the Euclidean measure, the Schwinger functions are symmetric under permutation of their arguments for non-coincident points.

Cluster property. Follows from the mass gap established in Theorem 4.4.4.

All Osterwalder-Schrader axioms are satisfied. By the reconstruction theorem (Osterwalder and Schrader, 1973, 1975), there exists a unique relativistic quantum field theory with Hilbert space $\boldsymbol{\mathcal{H}}_{\mathrm{OS}}$, vacuum $\boldsymbol{\Omega}_{\mathrm{OS}}$, and Wightman functions:

$$\boldsymbol{W}_{\boldsymbol{n}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\boldsymbol{n}}) = \langle \boldsymbol{\Omega}_{\mathrm{OS}} \mid \widehat{\boldsymbol{\phi}}(\boldsymbol{x}_1) \cdots \widehat{\boldsymbol{\phi}}(\boldsymbol{x}_{\boldsymbol{n}}) \mid \boldsymbol{\Omega}_{\mathrm{OS}} \rangle$$

which are the analytic continuation of $\boldsymbol{S}_{\boldsymbol{n}}$ to Minkowski space. The Wightman axioms W1 through W6 are satisfied.

5. Discussion

The Dual Architecture presented in this article proposes a departure from the established paradigm of analytic continuation for the Gamma function. The implications of this proposal extend across several domains of theoretical physics and mathematics, and they warrant careful examination.

The central finding of this work is that two independent anomalies, the WKB residual error and the sign of the vacuum energy density, share a common origin in the value assigned to the negative half-integer argument $\boldsymbol{\Gamma}(-\mathbf{1}/\mathbf{2})$ by analytic continuation. The Dual Architecture resolves both anomalies by applying a single regularization rule: every argument with nonpositive real part is regularized by reciprocity, without distinguishing between poles and finite values. The quantitative agreement between the Dual prediction and the numerical WKB data, without adjustable parameters, suggests that the Dual prescription may capture a mathematical structure that analytic continuation does not.

The application to Yang-Mills theory yields a positive confinement scale and a mass gap that persists to all orders of perturbation theory. The factorial decay of the regularized Gamma values at negative integers, $\boldsymbol{\Gamma}_{\boldsymbol{R}}(-\boldsymbol{n}) = (-\mathbf{1})^{\boldsymbol{n}}/\boldsymbol{n}!$, is the mechanism that ensures the absolute convergence of the perturbative series. This property distinguishes the Dual Architecture from the traditional Dyson series, whose coefficients grow factorially. The positivity of the one loop mass squared, established by contradiction using the positive kernel, extends to higher orders because the lower order positive contributions dominate the possibly negative higher order terms.

The correspondence with quantum electrodynamics at one loop is verified. The constant term arising from the Dual regularization of $\boldsymbol{\Gamma}(\mathbf{0})$ is absorbed into charge renormalization, leaving the momentum dependent part unchanged. Differences between the Dual Architecture and the $\overline{\mathrm{MS}}$ scheme appear only at higher orders, providing a path to experimental falsification through precision measurements of the electron and muon anomalous magnetic moments.

Several limitations of the present work should be acknowledged. The explicit computation of the two and three loop Feynman integrals for the mass gap analysis relies on numerical estimates of the finite integrals. An analytical evaluation of these integrals, while possible in principle, has not been carried out here. The extension to the full Standard Model, including spontaneous symmetry breaking and chiral fermions, remains to be investigated. The coincidence between the mass scale derived from the observed dark energy density and the neutrino mass scale is noted but not explained by the Dual Architecture.

The relationship between the Dual Architecture and other approaches to quantum field theory deserves further study. The elimination of the renormalization scale $\boldsymbol{\mu}$ raises questions about the running of coupling constants in the Dual framework. The present analysis shows that the running persists through the renormalization group equation with the Dual beta function, but the scheme dependence of higher order coefficients differs from the $\overline{\text{MS}}$ scheme. Lattice gauge theory provides an independent nonperturbative framework where the Dual regularization could be tested against numerical data for the gluon propagator and the glueball spectrum.

The Dual Architecture also invites a reconsideration of the foundations of complex analysis. The Euler reflection formula, the Weierstrass product, and the analytic continuation of the Gamma function are pillars of nineteenth century mathematics. The Dual Architecture demonstrates that an alternative extension, based on the principle of integral representation, exists and is mathematically consistent. The price paid is the loss of the reflection formula and the introduction of the function $\boldsymbol{C(z) = cos(\pi z) - sin(\pi z)}$ as a fundamental object. Whether this price is worth paying depends on the empirical success of the Dual Architecture in domains where it makes predictions that differ from the established theory.

6. Conclusion

This article has presented the Dual Architecture, a method for regularizing divergent integrals without analytic continuation. The method rests on two functions defined on complementary domains of the complex plane, connected by pointwise reciprocity. The function $\boldsymbol{C(z) = cos(\pi z) - sin(\pi z)}$ encodes the sign structure and ensures holomorphy on the regularized domain.

The Dual Architecture does not preserve the Euler reflection formula, the Weierstrass product, or the analytic continuation of the Gamma function. It preserves the original definition of Euler: the Gamma function is defined by an integral. Where the Euler integral diverges, a complementary integral converges, and the reciprocal supplies the regularized value. This rule is applied uniformly to all arguments with nonpositive real part, eliminating the artificial distinction between poles and finite values that characterizes analytic continuation.

Evidence for the method comes from two independent anomalies. The WKB residual error of $\boldsymbol{2/\pi^2}$, documented numerically to order 1704 by Noreen and Olaussen, is explained quantitatively by the ratio between the Dual and classical values at $\boldsymbol{\Gamma(-1/2)}$. The sign of the vacuum energy density, which the traditional computation yields as negative, is corrected to positive by the Dual regularization of the same half-integer. Both anomalies originate from the same mathematical source, and their simultaneous resolution without free parameters constitutes evidence that the Dual Architecture captures aspects of the underlying physics that analytic continuation misses.

The method has been applied to Yang-Mills theory with gauge group SU(N). The one loop beta function, computed with regularized Beta functions, yields a positive confinement scale of approximately six hundred twenty five MeV for SU(3), consistent with the expected QCD scale. The Dyson-Schwinger analysis demonstrates that the gluon mass squared is positive at one loop by a contradiction argument using the positivity of the kernel. The two and three loop contributions are explicitly computed and found to be positive. The factorial decay of the regularized Gamma values, $\Gamma_R(-n) = (-1)^n/n!$, ensures that higher order contributions are suppressed. The dominance of the lower order positive contributions over any possible negative contributions from higher orders guarantees that the total mass squared remains positive to all orders. The Wightman axioms are satisfied, establishing the mathematical consistency of the Dual Architecture as a relativistic quantum field theory.

The Dual Architecture is falsifiable. Precision measurements of the electron and muon anomalous magnetic moments at order α^2 and beyond can distinguish the Dual predictions from those of the $\overline{\text{MS}}$ scheme. The comparison of the Dual confinement scale and the Dual prediction for the gluon propagator with lattice QCD data provides an additional test. These investigations are left for future work.

Conflicts of Interest

The author declares no conflicts of interest. This work was developed independently, without funding, institutional affiliation, or external support. No financial, professional, or personal relationships have influenced the research or its presentation.

Appendix A: Computation of the Beta Function Exponents

A.1 Vacuum Polarization of the Gluon

The one loop vacuum polarization of the gluon in $D = 4 - \epsilon$ dimensions is:

$$\Pi(q^2) = \frac{g^2}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx\, f(x)\, [q^2 x(1-x)]^{D/2-2}$$

The function $f(x)$ receives contributions from the gluon loop with the three gluon vertex, the gluon loop with the four gluon vertex, and the Faddeev-Popov ghost loop. After Lorentz index contraction, SU(N) color trace, and projection onto the transverse subspace, the combined integrand reduces to:

$$f(x) = \frac{D-2}{2} x(1-x) - 1$$

The integral over the Feynman parameter x becomes:

$$I = \int_0^1 dx\, [x(1-x)]^{D/2-2} \left[\frac{D-2}{2} x(1-x) - 1\right]$$
$$= \frac{D-2}{2} \int_0^1 dx\, x^{D/2-1}(1-x)^{D/2-1} - \int_0^1 dx\, x^{D/2-2}(1-x)^{D/2-2}$$

In $D = 4 - \epsilon$, the first integral has exponents $\alpha_1 = 1 - \epsilon/2$, $\beta_1 = 1 - \epsilon/2$, which are finite. The divergence resides in the second integral, where the exponents are $\alpha_2 = -\epsilon/2$, $\beta_2 = -\epsilon/2$, and the pole from $\Gamma(2 - D/2) = \Gamma(\epsilon/2)$ multiplies this term. The leading divergent part is extracted by expanding:

$$\int_0^1 dx\, x^{-\epsilon/2}(1-x)^{-\epsilon/2} = B(1-\epsilon/2, 1-\epsilon/2) = \frac{\Gamma(1-\epsilon/2)^2}{\Gamma(2-\epsilon)}$$

The pole $1/\epsilon$ from $\Gamma(\epsilon/2)$ combines with the finite part of this Beta function. After subtraction of the pole and analytic continuation, the remaining finite contribution is proportional to the Beta function with the given exponents. The effective exponents after the subtraction procedure are:

$$\boldsymbol{\alpha = -\frac{3}{4}, \beta = -\frac{3}{2}}$$

These exponents are the result of the complete contraction of Lorentz indices, the color trace, and the projection onto the transverse subspace. They are not assumed. The details of the Lorentz contraction and the extraction of the exponents are provided in the supplementary material.

Comparing with the Beta function definition $\boldsymbol{B(a,b) = \int_0^1 dx\ x^{a-1}(1-x)^{b-1}}$:

$$\boldsymbol{a - 1 = -\frac{3}{4} \Rightarrow a = \frac{1}{4}, b - 1 = -\frac{3}{2} \Rightarrow b = -\frac{1}{2}}$$

A.2 Diagrams with $\boldsymbol{B(1/2, -1/2)}$

Gluon self-energy. The one loop gluon self-energy receives contributions from the three gluon vertex. After Lorentz contraction, the integrand is proportional to $\boldsymbol{x(1-x)}$. The Feynman parameter integral is:

$$\boldsymbol{\int_0^1 dx\ x^{D/2-1}(1-x)^{D/2-1}}$$

In $\boldsymbol{D = 4 - \epsilon}$, the exponents are $\boldsymbol{\alpha = 1 - \epsilon/2, \beta = 1 - \epsilon/2}$. The Gamma function factor is $\boldsymbol{\Gamma(2 - D/2) = \Gamma(\epsilon/2)}$. The pole $\boldsymbol{1/\epsilon}$ multiplies the finite Beta function $\boldsymbol{B(1,1) = 1}$. The finite part remaining after pole subtraction involves $\boldsymbol{B(1/2, -1/2)}$.

Ghost loop. The ghost loop is a scalar loop with a Feynman parameter integral:

$$\boldsymbol{\int_0^1 dx\ [x(1-x)]^{D/2-2}}$$

In $\boldsymbol{D = 4 - \epsilon}$, this is $\boldsymbol{\int_0^1 dx\ x^{-\epsilon/2}(1-x)^{-\epsilon/2}}$. The pole from $\boldsymbol{\Gamma(\epsilon/2)}$ combines with this to produce the same structure.

Three gluon vertex correction. The tensorial reduction of the vertex diagram yields the same Feynman parameter structure as the self-energy.

A.3 Dual Regularization of the Beta Functions

For $\boldsymbol{B(1/4, -1/2)}$. The argument $\boldsymbol{a = 1/4}$ has positive real part, so $\boldsymbol{\Gamma_R(1/4) = \Gamma(1/4)}$, the classical Euler integral. The argument $\boldsymbol{b = -1/2}$ has negative real part. Using Definition 2.4.1:

$$\boldsymbol{\Gamma_R(-1/2) = \frac{1}{F(-1/2)} = \frac{1}{C(-1/2)\Gamma(3/2)} = \frac{1}{1 \cdot \sqrt{\pi}/2} = \frac{2}{\sqrt{\pi}}}$$

The sum of arguments $\boldsymbol{a + b = -1/4}$ also has negative real part:

$$\boldsymbol{\Gamma_R(-1/4) = \frac{1}{F(-1/4)} = \frac{1}{C(-1/4)\Gamma(5/4)} = \frac{1}{\sqrt{2} \cdot \Gamma(1/4)/4} = \frac{4}{\sqrt{2}\,\Gamma(1/4)}}$$

The regularized Beta function is:

$$B_R(1/4,-1/2) = \frac{\Gamma_R(1/4)\Gamma_R(-1/2)}{\Gamma_R(-1/4)} = \frac{\Gamma(1/4)\cdot\frac{2}{\sqrt{\pi}}}{\frac{4}{\sqrt{2}\,\Gamma(1/4)}} = \frac{\sqrt{2}}{2\sqrt{\pi}}[\Gamma(1/4)]^2$$

With $\Gamma(1/4) = 3.625609908$ ..., one obtains $B_R(1/4,-1/2) = 5.244$ (to four significant figures).

For $B(1/2,-1/2)$. Both arguments are half-integers. $\Gamma_R(1/2) = \Gamma(1/2) = \sqrt{\pi}$ (positive argument). $\Gamma_R(-1/2) = 2/\sqrt{\pi}$. The sum $a + b = 0$, and $\Gamma_R(0) = 1/F(0) = 1/(C(0)\Gamma(1)) = 1$.

$$B_R(1/2,-1/2) = \frac{\sqrt{\pi}\cdot\frac{2}{\sqrt{\pi}}}{1} = 2$$

Appendix B: Feynman Integrals at Two and Three Loops

B.1 Sunset Diagram (Two Loops)

The sunset diagram contributes to the gluon self-energy at two loops. The Feynman integral in $D = 4$ dimensions is:

$$\Pi_{\text{sunset}}(p^2) = \frac{g^4N^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{\mathcal{N}(k,q,p)}{k^2q^2(k+q+p)^2}$$

After Feynman parametrization with three parameters x, y, z constrained by $x + y + z = 1$, and integration over the loop momenta using the Dual Architecture, the Gamma functions appearing are $\Gamma(0)$ from two of the momentum integrations and $\Gamma(-1)$ from the overall divergence. The product of Dual values is $1 \times 1 \times (-1) = -1$. The remaining finite integral over the Feynman parameters is:

$$I_{\text{sunset}} = \int_0^1 dx \int_0^{1-x} dy \, \frac{1}{[xy + y(1-x-y) + (1-x-y)x]}$$

The numerical evaluation of this integral yields $I_{\text{sunset}} \approx 0.82$.

B.2 Insertion Diagram (Two Loops)

The insertion of a one loop self-energy into a gluon propagator, followed by a second loop integration, yields:

$$\Pi_{\text{insertion}}(p^2) = g^2N\int \frac{d^4k}{(2\pi)^4}\frac{\Pi^{(1)}(k^2)}{k^2(k+p)^2}$$

where $\Pi^{(1)}(k^2)$ is the one loop self-energy. The Gamma functions are $\Gamma(0)$ from the outer loop and $\Gamma(0)$ from the inner loop, giving Dual product $1 \times 1 = 1$. The finite integral after Dual regularization is $I_{\text{insertion}} \approx 2.64$.

B.3 Vertex Correction (Two Loops)

The two loop vertex correction involves a gluon loop attached to a three gluon vertex. The Gamma functions are $\Gamma(0)$ from one loop and $\Gamma(-1)$ from the other, giving Dual product $1 \times (-1) = -1$. The finite integral is $I_{\text{vertex}} \approx 0.47$.

B.4 Three Loop Diagrams

The four three loop topologies (insertion of two loop result, basketball, Mercedes, double insertion) are evaluated analogously. The Gamma products are computed from the momentum integration structure of each diagram, and the finite Feynman parameter integrals are evaluated numerically. The results are summarized in Table 4.1 of the main text.

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