The Aliquot Spectral Basis

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Abstract

The author observed belatedly that $\{9, 4\}$ is the spectral basis of $n = \sigma(9) - 1 = 12$, where $\sigma'(9) = \sigma(9) - 9 = 4$, the aliquot sum of 9. The author provides examples of m such that $\{m, \sigma'(m)\}$ is the spectral basis of $n = \sigma(m) - 1$, called the aliquot spectral basis of n.

Duodecimal declaration: [4] The author uses duodecimal, base 12, throughout this document, except for dates and document counters, with X for 10 and Ɛ for 11.

1 Review

1.1 Number theory

Let n be a positive integer and let d be a divisor of n. Define $\sigma_k(n)$ is the sum of the kth powers of the divisors of n, where k is nonnegative, that is,

$$\sigma_k(n) \triangleq \sum_{d|n} d^k, \quad k \geq 0. \tag{1.1}$$

1 Definition (Number of divisors [9, 13]). Define $\tau(n)$ to be the number of divisors of n, that is,

$$\tau(n) \triangleq \sigma_0(n) = \sum_{d|n} 1. \tag{1.2}$$ ■

2 Theorem (Number of divisors [9]): *Let $\tau(n)$ denote the number of divisors of n. Let p be prime, let $p_1, \ldots, p_j$ be distinct primes, and let $k, k_1, \ldots, k_j$ be positive integers. Then*

$$\tau(p^k) = k + 1. \tag{1.3}$$

Furthermore, τ is multiplicative:

$$\tau(p_1^{k_1} \cdots p_j^{k_j}) = \tau(p_1^{k_1}) \cdots \tau(p_j^{k_j}). \tag{1.4}$$

Proof. An exercise for the reader. □

3 Definition (Sum of divisors [9, 15], aliquot sum [7, 17]). Define $\sigma(n)$ to be the sum of the divisors of n, that is,

$$\sigma(n) \triangleq \sigma_1(n) = \sum_{d|n} d. \tag{1.5}$$

Further, we shall define $\sigma'(n) \triangleq \sigma(n) - n$ to be the aliquot sum of n. ■

4 Theorem (Sum of divisors [9]): *Let $\sigma(n)$ denote the sum of the divisors of n. Let p be prime, let $p_1, \ldots, p_j$ be distinct primes, and let $k, k_1, \ldots, k_j$ be positive integers. Then*

1. *If n is the power of a prime, then*

$$\sigma(2^k) = 2^{k+1} - 1, \tag{1.6}$$

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}. \tag{1.7}$$

2. *Furthermore, σ is multiplicative:*

$$\sigma(p_1^{k_1} \cdots p_j^{k_j}) = \sigma(p_1^{k_1}) \cdots \sigma(p_j^{k_j}). \tag{1.8}$$

3. *The aliquot sums are*

$$\sigma'(2^k) = 2^k - 1, \tag{1.9}$$

$$\sigma'(p^k) = \frac{p^k - 1}{p - 1}. \tag{1.10}$$

The aliquot sum is not multiplicative. ◆

Proof. An exercise for the reader. □

1.2 The spectral basis

Let us consider the isomorphism $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{a_1}} \oplus \mathbb{Z}_{p_2^{a_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{a_k}}$, where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime factorization of n. We shall assume for our purpose that n has at least two prime factors. Otherwise, we would consider a p-adic expansion with respect to $n = p^a$.

5 Definition (Spectral basis [6]). Given $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, define

$$\begin{aligned} h_i &\triangleq n / p_i^{a_i}, \\ b_i &\triangleq h_i^{-1} \mod p_i^{a_i}, \\ s_i &\triangleq b_i h_i, \end{aligned}$$

so that, in $\mathbb{Z}_n$, we have the decomposition of the identity into mutually orthogonal idempotents

$$s_1 + s_2 + \cdots + s_k = 1$$

such that
$$s_i^2 = 0 \quad \text{and} \quad s_i s_j = 0$$
for $1 \le i, j \le k$ with $i \ne j$. Each s_i acts as an orthogonal projections onto its primary part $\mathbb{Z}_{p_i^{a_i}}$ of $\mathbb{Z}_n$. ■

If $x \in \mathbb{Z}_n$, then we have
$$x = x_1 s_1 + x_2 s_2 + \cdots + x_k s_k,$$
where $x \equiv x_i \pmod{p_i^{a_i}}$. Furthermore, if r is nonnegative, we have
$$x^r = x_1^r s_1 + x_2^r s_2 + \cdots + x_k^r s_k.$$
If x is relatively prime to n, then r can be negative.

We shall often say "spectral basis of n" instead of "spectral basis of $\mathbb{Z}_n$."

6 Definition (Power spectral basis [3]). The integer n has a *power spectral basis* if and only if the elements of the basis are prime or a (possibly composite) power. ■

The following theorems on power spectral bases are from *Power Spectral Numbers* [3] and are stated without proof.

7 Theorem (Mersenne [3, 11]): *Let M_p be a Mersenne prime with Mersenne exponent p. Then*

1. $2M_p$ *has spectral basis* $\{M_p, 2^p\}$*;*
2. $2^p M_p$ *has spectral basis* $\{M_p^2, 2^p\}$*;*
3. $2^{p+1} M_p$ *has spectral basis* $\{M_p^2, 2^{2p}\}$ *or, equivalently,* $\{M_p^2, (M_p+1)^2\}$*;*
4. $2^{2p+1} M_p^2$ *has spectral basis* $\{M_p^2(M_p+2)^2, (M_p^2-1)^2\}$*.* ◆

Both 1 and 2 of Theorem 7 occur as examples of an aliquot spectral basis. See Examples 12 and 14, respectively.

8 Theorem (Fermat [3, 10]): *If $F_i = 2^{f_i} + 1$ is a Fermat prime with exponent $f_i = 2^i$, $i \ge 0$, then*

1. $2^{f_i} F_i$ *has spectral basis* $\{F_i, 2^{2f_i}\}$*;*
2. $2^{f_i+1} F_i$ *has spectral basis* $\{F_i^2, 2^{2f_i}\}$*;*
3. $2^{2f_i+1} F_i^2$ *has spectral basis* $\{(F_i-2)^2 F_i^2, (F_i^2-1)^2\}$*.* ◆

None of the examples of Theorem 8 are known to occur as an aliquot spectral basis.

9 Theorem (Semiprime [3, 18]): *Given p an odd prime and $k \ge 3$ an integer, the integer pq has power-spectral basis $\{q, p^k\}$ if and only if $q = (p^k - 1)/(p-1)$ is prime.* ◆

See Example 12.

10 Theorem (Cyclotomic [3]): *Let p, q, r be primes and e a nonnegative integer. Then the integer $n = p^{r^e} q$ is power spectral with spectral basis $\{q, p^{r^{e+1}}\}$ if and only if $q = \Phi_{r^{e+1}}(p)$ is prime, where $\Phi_{r^{e+1}}(x)$ is the r^{e+1}-th cyclotomic polynomial [8].* ◆

None of the examples of Theorem 10 are known to occur as an aliquot spectral basis. In fact, Theorem 9 is Theorem 10 with $e = 0$.

2 The aliquot spectral basis

In investigating which numbers n have power spectral basis [3], the author discovered that $\{9, 4\}$ is the spectral basis of $10 = 12|_{12}$, namely Theorem 7.2. However, in thinking about another project, the author noticed that $\sigma(9) = 11$ and that $\sigma'(9) = 4$, so that the pair $\{9, \sigma'(9)\}$ is the spectral basis of $10 = \sigma(9) - 1$. Of course, this gave the author an interesting idea: Find more examples of integers m such that, if $n = \sigma(m) - 1$ and $m' = \sigma'(m)$, then $\{m, m'\}$ is the spectral basis of n. This motivates the following definition.

11 Definition (Aliquot spectral basis). The positive integer n will have an *aliquot spectral basis* if there exists an integer m such that, if $n = \sigma(m) - 1$ and $m' = \sigma'(m)$, then $\{m, m'\}$ is the spectral basis of n. The ordering of the basis will depend on m. See Table 1. ■

The search was under the condition that $\{m, m'\}$ is contained in the spectral basis of n, and not necessarily equal to the spectral basis of n. The examples below will use Table 1 as a guide.

3 Examples of the aliquot spectral basis

The following example is Theorem 9 established as an aliquot spectral basis.

12 Example (Semiprime [18]): If $q = (p^k - 1)/(p - 1)$ is prime, $k \geq 1$, then $\{q, p^k\}$ is the aliquot spectral basis of $n = pq$.

13 Remarks: See entries 1, 2, 4, 5 of Table 1 as illustrations of Example 12. Theorem 7.1 is a consequence of Example 12.

Proof. Observe that, if $m = p^k$, then $\sigma(m) = (p^{k+1} - 1)/(p - 1)$ and $n = \sigma(m) - 1 = pq$, where $q = (p^k - 1)/(p - 1)$. If q is prime, then $\{q, p^k\}$ is the spectral basis of $n = pq$. □

The following example is Theorem 7.2 established as an aliquot spectral basis.

14 Example (Mersenne): Let M_p be a Mersenne prime with Mersenne exponent p. Then $2^p M_p$ has aliquot spectral basis $\{M_p^2, 2^p\}$.

15 Remark: See entries 3, 6, 8, 15 of Table 1 as illustrations of Example 14.

Proof. Let $m = M_p^2$. Then $\sigma(m) = 2^{2p} - 2^p + 1$, $n = 2^{2p} - 2^p = 2^p M_p$, and $\sigma'(m) = 2^p$. Verification of spectral basis is left as an exercise for the reader. □

The following example is new to the author.

16 Example (Perfect aliquot basis with $P = 6$): If $m = 6\cdot\mathcal{E}^k$, $k \geq 1$, and $q' = 2(m-1)/(\mathcal{E}-1)$ is prime, then $\{m', m\} = \{6q', 6\cdot\mathcal{E}^k\}$ is the ordered spectral basis of $n = \mathcal{E}q'$. See Theorem 18 for a generalization.

17 Remark: See entries 7, 13, 19, 24, 30 of Table 1 as illustrations of Example 16. See also Table 2.

Proof. Observe that if we assume

$$\sigma(6\cdot\mathcal{E}^k) = \mathcal{E}q' + 1,$$

then

$$\begin{aligned} q' &= \frac{10\cdot\dfrac{\mathcal{E}^{k+1}-1}{\mathcal{E}-1}-1}{\mathcal{E}} \\ &= \frac{10\cdot\mathcal{E}^{k+1}-10-\mathcal{E}+1}{\mathcal{E}(\mathcal{E}-1)} \\ &= \frac{10\cdot\mathcal{E}^{k+1}-2\mathcal{E}}{\mathcal{E}(\mathcal{E}-1)} \\ &= \frac{10\cdot\mathcal{E}^{k}-2}{\mathcal{E}-1} \\ q' &= 2\cdot\frac{6\cdot\mathcal{E}^{k}-1}{\mathcal{E}-1}. \end{aligned}$$

It is an exercise for the reader to verify that, if q' is prime, then $\{m', m\} = \{6q', 6\cdot\mathcal{E}^k\}$ is the ordered spectral basis of $n = \mathcal{E}q'$. □

Suppose P is a perfect number such that $q = 2P - 1$ is prime. Consider

$$\begin{aligned} \sigma(Pq^k) &= qq' + 1 \\ q' &= \frac{\sigma(Pq^k)-1}{q} \\ &= \frac{2P\cdot\dfrac{q^{k+1}-1}{q-1}-1}{q} \\ &= \frac{2P(q^{k+1}-1)-q+1}{q(q-1)} \\ &= \frac{2Pq^{k+1}-2P-q+1}{q(q-1)} \\ &= \frac{2Pq^{k+1}-2q}{q(q-1)} \\ q' &= 2\cdot\frac{Pq^{k}-1}{q-1}. \end{aligned}$$

An alternate form is

$$q' = \frac{q^k P - 1}{P - 1} = q^k + \frac{q^k - 1}{P - 1}.$$

Observe that k must be even. The change of variable $u = P - 1$, $q = 2u + 1$ shows that q' is always an integer. Thus, we have the following theorem:

18 Theorem (Perfect aliquot basis [12, 16]): *Suppose $m = Pq^k$, where P is a perfect number such that $q = 2P - 1$ is prime, and further suppose that $q' = 2(m-1)/(q-1) = (m-1)/(P-1)$ is prime. Then $\{m', m\} = \{Pq', Pq^k\}$ is the ordered aliquot spectral basis of $n = qq'$.* ◆

19 Remark: The only known perfect numbers P such that $2P - 1$ is prime are $P = 2(2^2 - 1) = 6$, $q = \mathcal{E}$, and $P = 2^4(2^5 - 1) = 354$, $q = 6X7$. The author found additional solutions for Theorem 18 with $P = 354$, namely, $k \in \{12, 98, 1732\}$, $k \le 10^4$.

Proof. The proof of Theorem 18 is similar to that of Example 16 and is left as an interesting exercise for the reader. □

20 Example (Nagell–Ljunggren [1, 2, 5]): Entry X of Table 1, $n = 3 \cdot \mathcal{E}^2$, with ordered spectral basis $\{m, m'\} = \{3^5, \mathcal{E}^2\}$, is a consequence of the fact that the only two solutions to the equation

$$\frac{x^m - 1}{x - 1} = y^2$$

are $m = 4$, $x = 7$, $y = 18$, and $m = 5$, $x = 3$, $y = \mathcal{E}$. The only known solutions to

$$\frac{x^m - 1}{x - 1} = y^p, \quad (m, p > 2)$$

are $m = p = 3$, $x = 16$, $y = 7$, and $m = p = 3$, $x = -17$, $y = 7$, and it is conjectured there are no others.

4 Table of aliquot spectral bases

i	$n = \sigma(m) - 1$	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
1.	$(2)(3)$	$(2)^2$	(3)	[1,1]	1
2.	$(2)(7)$	$(2)^3$	(7)	[1,1]	1
3.	$(2)^2(3)$	$(3)^2$	$(2)^2$	[1,1]	1
4.	$(3)(11)$	$(3)^3$	(11)	[1,1]	1
5.	$(2)(27)$	$(2)^5$	(27)	[1,1]	1
6.	$(2)^3(7)$	$(7)^2$	$(2)^3$	[1,1]	1
7.	$(\mathcal{E})(11)$	$(2)(3)(\mathcal{E})$	$(2)(3)(11)$	[3,3]	6
8.	$(5)(27)$	$(5)^3$	(27)	[1,1]	1

▽

i	$n = \sigma(m) - 1$	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
9.	(2)(X7)	$(2)^7$	(X7)	[1,1]	1
X.	(3)(Ɛ)^2	$(3)^5$	(Ɛ)^2	[1,1]	1
Ɛ.	$(2)^5(27)$	$(27)^2$	$(2)^5$	[1,1]	1
10.	(3)(771)	$(3)^7$	(771)	[1,1]	1
11.	(7)(775)	(3)(7)(Ɛ)(17)	(3)(775)	[4,2]	3
12.	(15)(217)	$(15)^3$	(217)	[1,1]	1
13.	(Ɛ)(Ɛ11)	(2)(3)(Ɛ)^3	(2)(3)(Ɛ11)	[3,3]	6
14.	(2)(48X7)	$(2)^{11}$	(48X7)	[1,1]	1
15.	$(2)^7$(X7)	(X7)^2	$(2)^7$	[1,1]	1
16.	(7)(1755)	$(7)^5$	(1755)	[1,1]	1
17.	(35)(ƐƐ7)	$(35)^3$	(ƐƐ7)	[1,1]	1
18.	(5)(Ɛ377)	$(5)^7$	(Ɛ377)	[1,1]	1
19.	(Ɛ)(X201)	(2)(3)(Ɛ)^4	(2)(3)(X201)	[3,3]	6
1X.	(2)(63X27)	$(2)^{15}$	(63X27)	[1,1]	1
1Ɛ.	(4Ɛ)(2071)	(4Ɛ)^3	(2071)	[1,1]	1
20.	(5Ɛ)(2Ɛ61)	(5Ɛ)^3	(2Ɛ61)	[1,1]	1
21.	(11)(15XX5)	$(11)^5$	(15XX5)	[1,1]	1
22.	(2)(2134X7)	$(2)^{17}$	(2134X7)	[1,1]	1
23.	(75)(4777)	$(75)^3$	(4777)	[1,1]	1
24.	(Ɛ)(93X11)	(2)(3)(Ɛ)^5	(2)(3)(93X11)	[3,3]	6
25.	(85)(5Ɛ67)	$(85)^3$	(5Ɛ67)	[1,1]	1
26.	(15)(43431)	$(15)^5$	(43431)	[1,1]	1
27.	(3)(3253X1)	$(3)^{11}$	(3253X1)	[1,1]	1
28.	(XƐ)(X011)	(XƐ)^3	(X011)	[1,1]	1
29.	(11Ɛ)(142X1)	(11Ɛ)^3	(142X1)	[1,1]	1
2X.	(125)(15507)	$(125)^3$	(15507)	[1,1]	1
2Ɛ.	(1Ɛ)(121381)	(1Ɛ)^5	(121381)	[1,1]	1
30.	(Ɛ)(866301)	(2)(3)(Ɛ)^6	(2)(3)(866301)	[3,3]	6
31.	(25)(2Ɛ3Ɛ11)	$(25)^5$	(2Ɛ3Ɛ11)	[1,1]	1
32.	(205)(41X27)	$(205)^3$	(41X27)	[1,1]	1
33.	(5)(4108307)	(5)^{Ɛ}	(4108307)	[1,1]	1
34.	(27Ɛ)(71141)	(27Ɛ)^3	(71141)	[1,1]	1
35.	(11)(1902097)	$(11)^7$	(1902097)	[1,1]	1
36.	$(2)^{11}$(48X7)	(48X7)^2	$(2)^{11}$	[1,1]	1

$\triangledown$

i	$n = \sigma(m) - 1$	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
37.	(17)(14479Ɛ1)	(17)(271)(5ƐƐ1)	(14479Ɛ1)	[3,1]	1
38.	(37)(12096Ɛ5)	$(37)^5$	(12096Ɛ5)	[1,1]	1
39.	(320Ɛ)(35311)	(5)(15)(46Ɛ)(320Ɛ)	$(5)^2$(3Ɛ)(35311)	[4,3]	5
3X.	(485)(1X1767)	$(485)^3$	(1X1767)	[1,1]	1
3Ɛ.	(4X5)(1Ɛ8947)	$(4X5)^3$	(1Ɛ8947)	[1,1]	1
40.	(51Ɛ)(227XX1)	$(51Ɛ)^3$	(227XX1)	[1,1]	1
41.	(15)(8709647)	$(15)^7$	(8709647)	[1,1]	1
42.	(535)(23Ɛ6Ɛ7)	$(535)^3$	(23Ɛ6Ɛ7)	[1,1]	1
43.	(545)(24X2X7)	$(545)^3$	(24X2X7)	[1,1]	1
44.	(58Ɛ)(290331)	$(58Ɛ)^3$	(290331)	[1,1]	1
45.	(59Ɛ)(29ƐX21)	$(59Ɛ)^3$	(29ƐX21)	[1,1]	1
46.	(5Ɛ5)(2Ɛ5637)	$(5Ɛ5)^3$	(2Ɛ5637)	[1,1]	1
47.	(63Ɛ)(340981)	$(63Ɛ)^3$	(340981)	[1,1]	1
48.	(5)(86252431)	$(5)^{11}$	(86252431)	[1,1]	1
49.	(76Ɛ)(495551)	$(76Ɛ)^3$	(495551)	[1,1]	1
4X.	(775)(4X1077)	$(775)^3$	(4X1077)	[1,1]	1
4Ɛ.	(80Ɛ)(5534Ɛ1)	$(80Ɛ)^3$	(5534Ɛ1)	[1,1]	1
50.	(825)(573X07)	$(825)^3$	(573X07)	[1,1]	1
51.	(835)(5883Ɛ7)	$(835)^3$	(5883Ɛ7)	[1,1]	1
52.	(855)(5Ɛ5997)	$(855)^3$	(5Ɛ5997)	[1,1]	1
53.	(61)(9786505)	$(61)^5$	(9786505)	[1,1]	1
54.	(2)(4ƐƐ2308X7)	$(2)^{27}$	(4ƐƐ2308X7)	[1,1]	1
55.	(965)(76Ɛ887)	$(965)^3$	(76Ɛ887)	[1,1]	1
56.	(9XƐ)(823311)	$(9XƐ)^3$	(823311)	[1,1]	1
57.	(67)(11265655)	$(67)^5$	(11265655)	[1,1]	1
58.	(X3Ɛ)(8X8581)	$(X3Ɛ)^3$	(8X8581)	[1,1]	1
59.	(X9Ɛ)(993521)	$(X9Ɛ)^3$	(993521)	[1,1]	1
5X.	(6Ɛ)(141072X1)	$(6Ɛ)^5$	(141072X1)	[1,1]	1
5Ɛ.	(XƐƐ)(X0Ɛ101)	$(XƐƐ)^3$	(X0Ɛ101)	[1,1]	1
60.	(ƐX5)(Ɛ8Ɛ247)	$(ƐX5)^3$	(Ɛ8Ɛ247)	[1,1]	1

Table 1: Integers $m \leq 10^9$ such that $n = \sigma(m) - 1$ has aliquot spectral basis $\{m, m'\}$. See Table 3 for the anomalous examples.

5 Table of m

k	$m = 6 \cdot Ɛ^k$	q
1	(2)(3)(Ɛ)	(11)
3	(2)(3)(Ɛ)^{3}	(Ɛ11)
4	(2)(3)(Ɛ)^{4}	(X201)
5	(2)(3)(Ɛ)^{5}	(93X11)
6	(2)(3)(Ɛ)^{6}	(866301)
13	(2)(3)(Ɛ)^{13}	(3XX25X500050511)
25	(2)(3)(Ɛ)^{25}	(11X352X8321X90042572101832X11)
56	(2)(3)(Ɛ)^{56}	(679355094090998412112Ɛ677784Ɛ389925Ɛ70487352XƐ187X394702X7Ɛ42901)
6Ɛ	(2)(3)(Ɛ)^{6Ɛ}	(162110650XX99165X4420Ɛ80182Ɛ240X722Ɛ1114521157ƐX49175803741X439184X2XX179Ɛ8139711)
Ɛ6	(2)(3)(Ɛ)^{Ɛ6}	(19X2453724ƐƐ19203XX01Ɛ19921Ɛ111X634X8X23368X03758725848658Ɛ087150Ɛ1X843451841749X6Ɛ76539794829X3Ɛ65202593658532ƐX25X15X89Ɛ373384375901)

Table 2: Primes q of the form $q = 2(6 \cdot Ɛ^k - 1)/(Ɛ - 1)$ for $k \le 100$. See Example 16, Theorem 18, and Remark 19.

6 Anomalous examples

i	n	m	$m' = \sigma'(m)$	#PF	gcd(m, m')
11.	(7)(775)	(3)(7)(Ɛ)(17)	(3)(775)	[4,2]	3
37.	(17)(14479Ɛ1)	(17)(271)(5ƐƐ1)	(14479Ɛ1)	[3,1]	1
39.	(320Ɛ)(35311)	(5)(15)(46Ɛ)(320Ɛ)	(5)^{2}(3Ɛ)(35311)	[4,3]	5

Table 3: Anomalous examples of aliquot spectral basis from Table 1: they work because they work. The author was trying to find another anomalous example when he searched out to $m \le 10^9$.

7 Sequences

The definitions of the sequences have been modified to suit the needs of this document.

7.1 Sequence (A000005). $\tau(n)$ is the number of divisors of n. Also called $\sigma_0(n)$, where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. ■

7.2 Sequence (A000043). Mersenne exponents: primes p such that $M_p = 2^p - 1$ is prime, and M_p is called a Mersenne prime. ■

7.3 Sequence (A000203). $\sigma(n)$ is the sum of the divisors of n. Also called $\sigma_1(n)$, where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. ■

7.4 Sequence (A000396). Perfect numbers n: n is equal to its sum of its proper divisors or, equivalently, $\sigma(n) = 2n$. ■

7.5 Sequence (A001065). $\sigma'(n) = n$, that is, n is the sum of its proper divisors, or aliquot parts. ■

7.6 Sequence (A023194). Numbers n whose sum of divisors $\sigma(n)$ is prime. ■

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