Fields as extra dimensions

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Abstract

It is argued that the known physical fields themselves can be comprehended as extra dimensions attached to spacetime. This is achieved by copying the basic structure of Special Relativity related to objects, and pasting it at the level of fields. The role of the conversion factor played by the velocity of light there is played by Newtons constant here. This leads to a permutation of the roles of established concepts.

1 Introduction

On mankinds journey towards Planckian scales, theorists have presented a number of plausible ideas on phenomena beyond what has yet been observed. The most prominent are supersymmetry [1] and strings including membranes and related structures [2], all this taking place in various numbers of dimensions. After half a century, an enormous degree of complexity has been reached, while evidence from observation surprisingly is lacking. However, the maturity of mathematical constructs is one aspect, their physical correspondance is another. For example, the most mature 19th century description of threads - not to call them strings - in space would not have led to the discovery of Special Relativity based on timelike (and lightlike) threads called worldlines.

In this paper, it is attempted to find some new physical interpretations of established mathematical concepts. The strategy is to copy what has proven as succesful and logically convincing at the level of objects and to paste it at the level of fields. As will be discussed, this straightforwardly implies that there are extra dimensions associated with the known fields. There is at least one such attempt in the literature [3] with the specific purpose of describing gravity in terms of thermodynamics. Here, the focus shall be on the established fields and their description in plain mathematical language. It shall be demonstrated how the physical roles of known concepts can permute if this new point of view is adopted. This could open alternative possibilities to link ideas like extra dimensions, branes etc. and the highly developed mathematics associated to physics.

2 The "New Field Theory"

Special Relativity modified the kinetic term of the Lagrangian of an object enumerated n to be in line with Lorentz symmetry

$$m_n \frac{\dot{\vec{x}}_n^2}{2} \rightarrow -m_n c^2 \sqrt{1 - \frac{\dot{\vec{x}}_n^2}{c^2}} \,, \tag{1}$$

where m_n is the mass of the object, $\vec{x}_n$ is its position, the overdot means derivative w.r.t. coordinate time and c is the velocity of light which will be set unity in the following, respectively.

In full anology, the kinetic term of the Lagrangian for a field - a real scalar serving as the example at first - enumerated n suggests itself to be modified as

$$a_n \frac{\partial_\mu \hat{\phi}_n \partial^\mu \hat{\phi}_n}{2} \to -a_n b^2 \sqrt{-\det\left(\eta_{\mu\nu} - \frac{\partial_\mu \hat{\phi}_n \partial^\mu \hat{\phi}_n}{b^2}\right)} \approx -a_n b^2 \sqrt{1 - \frac{\partial_\mu \hat{\phi}_n \partial^\mu \hat{\phi}_n}{b^2}} \ , \tag{2}$$

where η is the Minkowski metric (sign convention (+ - - -)) which moves the indices, b is an appropriate global constant, and a_n is a dimensionless individual constant, respectively. Instead of setting Plancks constant unity, the notation shall be those familiar from classical electrodynamics where any bosonic field includes a factor of $\hbar$, thus having dimension $momentum^{1/2} \cdot length^{-1/2}$. The Langrangian has the dimension of an energy density and so has b^2. The hat on $\hat{\phi}_n$ shall indicate that this is not the usual definition of the field ϕ_n, rather these two are connected via the dimensionless individual constant as $\sqrt{a_n}\hat{\phi}_n = \phi_n$. This distinction is necessary because of the cosmological term. The step anologous to Special Relativity is indicated by the arrow, only in place of a 1-dimensional worldline comes a 4-dimensional “field manifold” as will be discussed. The approximate equality at the right originates from a textbook formula for a determinant. When a nonsingular square matrix is written as a sum of a large and a small part, then

$$\det(A + \varepsilon X) \approx \left[1 + \varepsilon \mathrm{Tr}(A^{-1}X)\right] \det A \tag{3}$$

where ε is sufficiently small and A^{-1} is the inverse of A. This holds if the trace quoted does not vanish.

In Special Relativity, there is one and only spacetime spanned by time and “abstract” 3-space. Embedded are a number of concrete wordlines $\vec{x}_n(t)$. Here, in anology - so long only dealing with scalars -, there is one and only embedding space spanned by spacetime and the “abstract” $\hat{\phi}$. Embedded are a number of concrete field manifolds by virtue of the embedding equations $\hat{\phi}_n(t, \vec{x})$. Further approximating the kinetic Lagrangian yields

$$L = \sum_n -a_n b^2 \sqrt{-\det\left(\eta_{\mu\nu} - \frac{\partial_\mu \hat{\phi}_n \partial_\nu \hat{\phi}_n}{b^2}\right)} \approx \sum_n \left(-a_n b^2 + \frac{\partial_\mu \phi_n \partial^\mu \phi_n}{2}\right) \ . \tag{4}$$

This can be called “New Field theory”, which name was coined by Born and Infeld in the 1930s in the context of electrodynamics. All considerations here are on non-interacting fields. However, a self-interaction via an attached mass term shall be assumed possible. Even if this is not the final wisdom, it is as good as adding potential terms in Special Relativity. Lack of a mass term would be problematic in the case of Fermions, since then the Dirac equation decouples into two independent Weyl equations for 2-component spinors.

3 Fields and Conversion Factors

Above, the real scalar stands pars pro toto for any type of field. In contrast to Special Relativity where wordlines are one-dimensional, here the transformation properties of fields under spacetime reparametrization have to be taken into account. This requires appropriate adaptation of the above formula according to the helicity of the respective field. This can lead to quite some modifications. In particular, in the case of vector fields investigated by Born and Infeld already in 1934 [4], in place of $\eta_{\mu\nu} - \frac{\partial_\mu\phi\partial_\nu\phi}{b^2}$ comes $\eta_{\mu\nu} \pm \frac{\partial_\mu A_\nu - \partial_\nu A_\mu}{b}$ (relative sign is without influence), where A is the helicity 1 vector field. It can be added that this reference showed that their formulas also work if in place of η there is a general metric of spacetime. But since this leads to subtleties in the context here, it is avoided to take two steps at once. Spacetime shall be flat and parametrized by cartesian coordinates.

The cosmological term is derived "classically", nevertheless it should be in line with the current understanding of the vacuum. And this implies that $|b|$ is of Planckian order of magnitude, like c is a member of the Planckian set of units in Special Relativity. So for b^2 one can assume the Planckian energy density $\frac{1}{G^2\hbar}$, where G is Newtons constant. However there is an aspect that only can be discussed in section 8. What regards the individual constants, in the case of vector fields it can be concluded that they are the inverse of the established coupling constants $a_n = \frac{1}{\alpha_n}$ (symbol not to be confused with those used for the Dirac matrices lateron) [5].

4 The geometric origin

The relations of Special Relativity have a clear origin. It is the geometry of Minkowski spacetime, where the worldlines are embedded. Here in anology - again restricting to the exemplaric case of real scalars -, there is a 5-dimensional space spanned by $\begin{pmatrix} x^i \\ \hat{\phi} \end{pmatrix}$, where the 4-dimensional field manifolds determined by the field equations $\hat{\phi}_n(x^0, x^1, x^2, x^3)$ are embedded. For more complicated fields the embedding space will be higher dimensional, but its number of dimensions minus the number of field equations always is 4.

Square roots from determinants of quantities that can be interpreted as (generalized) metric tensors are volume factors associated with respective manifolds. Gauss' formula for the induced metric on a submanifold embedded in a space with metric U reads

$$u_{\mu\nu} = U_{ij}\frac{\partial X^i}{\partial x^\mu}\frac{\partial X^j}{\partial x^\nu} \ , \tag{5}$$

where X are the contravariant embedding space coordinates at the location of any $\mathscr{M}$, and x are the contravariant coordinates on that $\mathscr{M}$, best simply identified with part of the X. So $i \leftrightarrow \mu$ for the respective four indices, what is the so called static gauge underlying all current formulations of field physics. ∂ is a partial derivative.

For the example of real scalars, the metric of the embedding space and associated line element read

$$U_{ij} = \begin{pmatrix} \eta_{ij} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \qquad \mathrm{d}s^2_{emb} = \mathrm{d}x_i \mathrm{d}x^i - \frac{\mathrm{d}\hat{\phi}^2}{b^2} \,. \tag{6}$$

For the metric induced on the field manifold enumerated n, formula (5) yields

$$u_{n\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \hat{\phi}_n \partial_\nu \hat{\phi}_n}{b^2} \,, \tag{7}$$

what directly leads to equations (2) and (4).

5 Complex Scalars

Complex scalars are a straightforward generalization of real scalars. The kinetic term of the Lagrangian usually is $\partial_\mu \phi^* \partial^\mu \phi$ [6], which to avoid any unclarities could be symmetrized as $\frac{1}{2}(\partial_\mu \phi^* \partial^\mu \phi + \partial_\mu \phi \partial^\mu \phi^*)$. Hence, the embedding space can formally be kept 5-dimensional with metric given by equation (6) apart from the factor of 2, however $\hat{\phi}$ now actually has two components grouped to a complex number. This will be used in particular in section 7.

Alternatively, one can avoid complex values by writing $\hat{\phi} = \hat{R}\exp(\imath \frac{S}{\hbar})$ in polar coordinates and using the Lagrangian $\partial_\mu \hat{R} \partial^\mu \hat{R} + \frac{1}{\hbar^2}\hat{R}^2 \partial_\mu S \partial^\mu S$. There is no hat on S, since it is not affected by the individual constant. Then the embedding space becomes 6-dimensional and has line element

$$\mathrm{d}s^2_{emb} = (\mathrm{d}x^i \; \mathrm{d}S \; \mathrm{d}\hat{R}) \begin{pmatrix} \eta_{ij} & 0 & 0 \\ 0 & -\frac{\hat{R}^2}{b^2\hbar^2} & 0 \\ 0 & 0 & -\frac{1}{b^2} \end{pmatrix} \begin{pmatrix} \mathrm{d}x^j \\ \mathrm{d}S \\ \mathrm{d}\hat{R} \end{pmatrix} \,, \tag{8}$$

while there are 2 embedding equations for any complex field called n.

Scalars enter the metric of the embedding space in the same way as spacetime, however there is a distinction of the roles. Fields oscillate, so the mapping of time into the fields is unambigous, but the inverse mapping would be one into many. For complex scalars, there is a phase difference of 90 degrees between real and imaginary part such that at any point $\vec{x}$ in ordinary 3-space the configuration actually is a spiral in the subspace spanned by $t, S, \hat{R}$.

6 Dirac Fermions

The established kinetic Lagrangian reads $\frac{\imath}{2}(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi)$ [6], where ψ is the 4-component Dirac field, $\bar{\psi}$ is its so called adjoint and the γ are the Dirac matrices. Here the field shall include a factor of $\sqrt{\hbar}$ per definition, so it has dimension $momentum^{1/2} \cdot length^{-1}$. The four internal components of the Dirac field are not denoted explicitly, so the embedding space formally remains 5-dimensional.

To work with complex quantities alone, $\bar{\psi} = \psi^{+}\gamma^{0}$ shall be inserted, where ψ^{+} is the transposed complex conjugate of ψ. Then the above Lagrangian becomes $\frac{\imath}{2}(\psi^{+}\alpha^{\mu}\partial_{\mu}\psi - \partial_{\mu}\psi^{+}\alpha^{\mu}\psi)$. The α are the original Dirac matrices, however with the index denoted correctly upwards and $\alpha^{0} = I_4$. Since $\alpha^{\mu +} = \alpha^{\mu}\ \forall\mu$, the Lagrangian can be written most transparently as $\frac{\imath}{2}(\psi^{+}\alpha^{\mu +}\partial_{\mu}\psi - \partial_{\mu}\psi^{+}\alpha^{\mu}\psi)$.

The line element in embedding space spanned by $\begin{pmatrix} x^{\mu} \\ \hat{\psi} \end{pmatrix}$ (for the meaning of the hat see section 2) reads

$$\mathrm{d}s^2_{emb} = (\mathrm{d}x^i\ \mathrm{d}\hat{\psi}^{+}) \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{\imath}{b^2}\hat{\psi}^{+}\alpha_0 \\ 0 & -1 & 0 & 0 & -\frac{\imath}{b^2}\hat{\psi}^{+}\alpha_1 \\ 0 & 0 & -1 & 0 & -\frac{\imath}{b^2}\hat{\psi}^{+}\alpha_2 \\ 0 & 0 & 0 & -1 & -\frac{\imath}{b^2}\hat{\psi}^{+}\alpha_3 \\ \frac{\imath}{b^2}\alpha_0\hat{\psi} & \frac{\imath}{b^2}\alpha_1\hat{\psi} & \frac{\imath}{b^2}\alpha_2\hat{\psi} & \frac{\imath}{b^2}\alpha_3\hat{\psi} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}x^j \\ \mathrm{d}\hat{\psi} \end{pmatrix} =$$
$$= \mathrm{d}x_i\mathrm{d}x^i - \frac{\imath}{b^2}\left(\mathrm{d}x^i\hat{\psi}^{+}\alpha_i\mathrm{d}\hat{\psi} - \mathrm{d}\hat{\psi}^{+}\alpha_i\hat{\psi}\mathrm{d}x^i\right), \tag{9}$$

where $\alpha_i = \eta_{ij}\alpha^j$. That $\hat{\psi}$ and its complex conjugate appear inside the metric is well possible, like spacetime degrees of freedom appear inside a metric other than η.

Proceeding to a concrete field enumerated n, due to equation (3) only the trace of the induced metric of the 4-dimensional field manifold is necessary for the leading effects. So it is sufficient to regard the (0/0) component. Differentiation - symbolized by a comma - of X^0 w.r.t. x^0 (see equation (5)) yields $\mathrm{col}(1\ \ 0\ \ 0\ \ 0\ \ \hat{\psi}_{n,0})$. Multiplication by U yields

$$\begin{pmatrix} 1 - \frac{\imath}{b^2}\hat{\psi}^{+}_n\alpha_0\hat{\psi}_{n,0} \\ -\frac{\imath}{b^2}\hat{\psi}^{+}_n\alpha_1\hat{\psi}_{n,0} \\ -\frac{\imath}{b^2}\hat{\psi}^{+}_n\alpha_2\hat{\psi}_{n,0} \\ -\frac{\imath}{b^2}\hat{\psi}^{+}_n\alpha_3\hat{\psi}_{n,0} \\ \frac{\imath}{b^2}\alpha_0\hat{\psi}_n \end{pmatrix}. \tag{10}$$

Contracting from the left with $(1\ \ 0\ \ 0\ \ 0\ \ \hat{\psi}^{+}_{n,0})$ yields the (0/0) element of the induced metric

$$u_{00} = 1 - \frac{\iota}{b^2}(\hat{\psi}_n^+ \alpha_0 \hat{\psi}_{n,0} - \hat{\psi}_{n,0}^+ \alpha_0 \hat{\psi}_n) \,. \tag{11}$$

The other diagonal elements of u are derived from this by respective change of the indices.

Using formula (3), one gets

$$-\det u \approx 1 - \frac{\iota}{b^2}(\hat{\psi}_n^+ \alpha^\mu \hat{\psi}_{n,\mu} - \hat{\psi}_{n,\mu}^+ \alpha^\mu \hat{\psi}_n) \,. \tag{12}$$

7 The combined Embedding Space

Collecting scalars and spinors, the resulting embedding space is an extension of spacetime to further dimensions. There is no conflict with the Coleman-Mandula theorem, since the extra dimensions are the fields themselves. The contravariant X^i shall be grouped as $\mathrm{col}(x^i\ \hat{\psi}\ \phi)$, where ψ has 4 components and ϕ can be complex. The metric of this embedding space reads quite straightforwardly

$$U_{ij} = \begin{pmatrix} \eta_{ij} & -\frac{\iota}{b^2}\hat{\psi}^+\alpha_i & 0 \\ \frac{\iota}{b^2}\alpha_j\hat{\psi} & 0 & 0 \\ 0 & 0 & -\frac{1}{b^2} \end{pmatrix} \,. \tag{13}$$

So one arrives at a space spanned by bosonic as well as fermionic degrees of freedom, together with those of spacetime.

Individual fields always have defined helicity, so the embedding equations contain $\hat{\psi} = 0$ for scalars and $\hat{\phi} = 0$ for fermions. But the embedding space as such allows transformations among the different degrees of freedom. Although this is not supersymmetry as currently understood, the conservation of the above metric by generalized rotations could be called such. For scalars this is quite trivial, since they enter the embedding space metric exactly as the spacetime degrees of freedom. For fermions it is

$$\begin{pmatrix} 1 & c\hat{\psi}^+\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta & \hat{\psi}^+\alpha \\ -\alpha\hat{\psi} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c\alpha\hat{\psi} & 1 \end{pmatrix} = \begin{pmatrix} \eta & \psi^+\alpha \\ -\alpha\psi & 0 \end{pmatrix} \,.$$

where c is an arbitrary real number. So this is the symmetry conserving the embedding space metric. It acts on a vector as

$$\begin{pmatrix} 1 & 0 \\ c\alpha\psi & 1 \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} = \begin{pmatrix} x \\ (1 + c\alpha x)\psi \end{pmatrix} \,, \tag{14}$$

where αx explicitly means the dagger $\alpha_i x^i$. The transformation only modifies the field, while the spacetime degrees of freedom are not affected.

Eventually, it shall be remarked that the structure for fermions can be combined with those of vectors discussed in [5]. This leads to a further extension of the embedding space line element

$$\mathrm{d}s^2_{emb} = \mathrm{d}x_i \mathrm{d}x^i + \mathrm{d}x^i \left(\frac{\imath}{2b^2} \hat{\psi}^+ \alpha_i \mathrm{d}\hat{\psi} + \frac{1}{b} \mathrm{d}A_i \right) - \left(\frac{\imath}{2b^2} \mathrm{d}\hat{\psi}^+ \alpha_i \hat{\psi} + \frac{1}{b} \mathrm{d}A_i \right) \mathrm{d}x^i - \frac{1}{b^2} \mathrm{d}\hat{\phi} \mathrm{d}\hat{\phi}^* . \tag{15}$$

With this, all relevant fields are covered except for the gravitational. There is a single value of b for all fields. But the a_n (see equation (4)) allow for individual contributions to the cosmological term.

8 The Vacuum - New Kids on the Block

In Special Relativity, the appearance of the rest term and resulting equivalence of mass and energy was the most noted consequence. Here the cosmological term is related to one of the greatest mysteries of current day physics. Namely, the observed value of the vacuum energy density vanishes, at the utmost appart from a tiny value that shall not be discussed here. This is in conflict with what one would straightforwardly expect from second quantization. That bosonic and fermionic contributions mutually cancel with such precision is the only conjecture at hand.

Here appears a new aspect, namely a "classical" cosmological term, as is most obvious from equation (4). Actually, this is at the level of first quantization, since the entire formalism of classical fields is. So, for this section the a_n as formerly used will be renamed to $a_{I\,n}$, the index I meaning first quantization. Correspondingly, the contributions from second quantization, once performed, will be denoted $a_{II\,n}$.

The only fact definitely determined from observation is

$$\sum_n (a_{I\,n} + a_{II\,n}) = 0, \tag{16}$$

where the sum runs over all non-gravitational fields. Whether this is true for any subsums is unknown. The most extreme would be that this equation holds for any n individually. In [4] the authors subtracted the cosmological term for the then only known electromagnetic field. In those days the concept of second quantization was not yet established, and they oriented themselves only at observation. It needs to be remarked that second quantization of the New Field Theory is complicated by the nonlinearity even if the internal gauge group is abelian.

It is however quite secured that $b^2 a_{II\,n}$ is positive for bosons and negative for fermions. Hence, $b^2 a_{I\,n}$ should be negative for bosons and positive for fermions. Since the Hamiltonian has different sign of the cosmological term compared to the Lagrangian, it follows that b^2 should be negative, i.e. b should be purely imaginary. With what was said in sectiion 3 this means

$$b = \frac{\imath}{G\sqrt{\hbar}}\,. \tag{17}$$

Consequently, scalar fields are timelike extra dimensions, while for fermionic and vector fields the decisive entrance of the embedding space metric vanishes.

9 Innovative roles for established concepts

All quantities and concepts used here are well established in current theoretical physics. However, they appear in new roles, in particular

- Fields are extra dimensions, in the case of scalars being timelike
- Everything is in terms of geometry after the example of Special Relativity
- 4-dimensional membranes are “field manifolds“ as described
- G appears *inside* a generalized metric
- There are contributions to the cosmological term from both, first and second quantization, mutually cancelling
- There is a new concept of supersymmetry.

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