

Study of Double Roman Domination Number in Deg centric graph

Kevin Elijah Valentine

Department of Science
Kristu Jayanti (Deemed to be University)
Bangalore, Karnataka, India
24csmb28@kristujayanti.com

Girish V. R.

Department of Science and Humanities
PES University, Electronic City Campus
Bangalore, Karnataka, India
girishvr1@pes.edu

Abstract

Graph theory plays a vital role in the study of mathematical structures and their applications in communication networks, optimization, and computer science. Among the various domination parameters, the Double Roman Domination Number has attracted significant attention due to its enhanced defensive strategy compared with classical domination concepts. This paper investigates the Double Roman Domination Number of Degree-Centric Graphs obtained from several standard graph families. The degree-centric transformation is first constructed for each graph, and the corresponding Double Roman Domination Numbers are determined. Based on these results, general observations and formulas are established for different classes of graphs, illustrating the relationship between the degree-centric transformation and the domination parameter. The obtained results extend the study of domination in transformed graphs and provide a foundation for future research in degree-centric graph theory.

1 Introduction

Graph theory is a branch of discrete mathematics that studies graphs, which are mathematical structures consisting of vertices and edges used to represent relationships between objects. It has numerous applications in computer science, communication networks, optimization, biology, and social network analysis (west2001,bondy1976).

- **Graph:** A graph $G = (V, E)$ is a mathematical structure consisting of a set of vertices V and a set of edges E joining pairs of vertices (West, 2001).
- **Vertex:** A vertex is a fundamental unit of a graph representing an object or point (West, 2001).
- **Edge:** An edge is a connection between two vertices in a graph (West, 2001).
- **Degree of a Vertex:** The degree of a vertex is the number of edges incident with that vertex (West, 2001).
- **Adjacent Vertices:** Two vertices are said to be adjacent if they are connected by an edge (Bondy & Murty, 1976).

- **Distance:** The distance between two vertices is the length of the shortest path connecting them (West, 2001).
- **Pendant Vertex:** A pendant (or leaf) vertex is a vertex of degree one (West, 2001).
- **Tree:** A tree is a connected graph containing no cycles (West, 2001; Bondy & Murty, 1976).
- **Path Graph (P_n):** A path graph is a connected graph in which each vertex is adjacent to at most two other vertices, forming a single path (West, 2001).
- **Cycle Graph (C_n):** A cycle graph is a connected graph in which every vertex has degree two, forming a closed loop (West, 2001).
- **Complete Graph (K_n):** A complete graph is a graph in which every pair of distinct vertices is connected by an edge (Bondy & Murty, 1976).
- **Star Graph ($K_{1,n-1}$):** A star graph is a tree consisting of one central vertex adjacent to all other vertices (West, 2001).
- **Wheel Graph (W_n):** A wheel graph is obtained by joining a central vertex to every vertex of a cycle graph (West, 2001).
- **Helm Graph (H_n):** A helm graph is obtained from a wheel graph by attaching one pendant vertex to each vertex of the outer cycle (Chartrand & Zhang, 2005).
- **Complete Bipartite Graph ($K_{m,n}$):** A complete bipartite graph is a bipartite graph in which every vertex of one part is adjacent to every vertex of the other part (Bondy & Murty, 1976).
- **Honeycomb Graph:** A honeycomb graph is a planar graph formed by regular hexagons sharing common edges (Chartrand & Zhang, 2005).
- **Domination Number ($\gamma(G)$):** The domination number of a graph is the minimum cardinality of a dominating set (Haynes et al., 1998).
- **Roman Domination Number ($\gamma_R(G)$):** The Roman domination number is the minimum weight of a Roman dominating function on a graph (Cockayne et al., 2004).
- **Double Roman Domination Number ($\gamma_{dR}(G)$):** The double Roman domination number is the minimum weight of a double Roman dominating function on a graph (Beeler et al., 2016).
- **Degree-Centric Graph (G_d):** The degree-centric graph of a graph G has the same vertex set as G , where two vertices are adjacent whenever their distance in G is less than or equal to the degree of one of the vertices (Farahani et al., 2022).

2 Research Gap

Roman domination, Double Roman domination, and degree-centric graphs have been studied by many researchers. However, there are only a few studies on the Double Roman Domination Number of degree-centric graphs. Most of the existing work focuses on these concepts separately, and very little attention has been given to studying them together.

There are also very few results available for the Double Roman Domination Number of the degree-centric graphs of standard graph families such as path graphs, cycle graphs, complete graphs, star graphs, wheel graphs, helm graphs, complete bipartite graphs, and honeycomb graphs. In addition, no general formulas or observations have been presented for these graph families after applying the degree-centric transformation.

This motivated the present study. In this work, the Double Roman Domination Number is determined for the degree-centric graphs of several standard graph families. Based on the obtained results, general formulas and observations are derived, which help in understanding the behavior of the Double Roman Domination Number in degree-centric graphs.

3 Main Results

Graph theory is the study of the relationship between vertices and edges. This study is mainly based on the Double Roman Domination Number of degree-centric graphs. Initially, each original graph G was transformed into its corresponding degree-centric graph G_d . We first started with standard graph families such as path graphs, cycle graphs, complete graphs, complete bipartite graphs, wheel graphs, honeycomb graphs, and sunlet graphs.

After constructing the degree-centric graph for each graph family, we observed the changes in the edge set and the degree of the vertices. We then determined the Double Roman Domination Number for each transformed graph and compared it with the original graph. From the results, it was observed that the degree-centric transformation adds new edges, which changes the structure of the graph and, in many cases, reduces the number of vertices required to satisfy the Double Roman domination conditions.

After completing the analysis of all the standard graph families, we extended the study to randomly generated graphs to check whether similar patterns were obtained. Based on all the observations and results, a set of generalizations and formulas was developed for the Double Roman Domination Number of degree-centric graphs.

3.1 Degree-Centric Transformation of Path Graphs and Double Roman Domination Number

The Degree-Centric Graph of a path graph exhibits interesting structural changes. For path graphs with fewer than four vertices, the degree-centric transformation does not introduce any additional edges. Hence, for $n \leq 4$, the transformed graph remains identical to the original graph. The transformation begins when $n \geq 4$. In a path graph, the end vertices have degree 1, while all internal vertices have degree 2. According to the definition of a Degree-Centric Graph, each internal vertex becomes adjacent to vertices that are at distance at most 2. Consequently, new edges are created between alternate vertices of the path.

From the given Degree-Centric graph, we can see that the Double Roman Domination number of the original is higher compared to the Degree-Centric graph due to the increase in edges.

The original graph P_4 contains 3 edges, and 2 more edges are added in the degree-centric transformation.

Table 1: Double Roman Domination Numbers for Degree-Centric Path Graphs

Path	$\gamma_{dR}(P_n)$	$\gamma_{dR}(P_n^d)$	Deg-Centric Edges
P_4	6	3	4
P_5	8	3	6
P_6	9	5	8
P_7	9	6	10
P_8	10	6	12
P_9	12	6	14

After studying the degree-centric transformation of the path graph, it was observed that the number of edges increased. However, the Double Roman Domination Number decreased in the degree-centric path graph. Although the number of edges continued to increase as the graph grew, the decrease in the Double Roman Domination Number was not drastic. This shows that adding more edges improves the domination properties of the graph, but the reduction in the Double Roman Domination Number is gradual rather than significant.

3.2 Degree-Centric Transformation of Cycle Graphs and Double Roman Domination Number

From the analysis of the cycle graph, we observed results similar to those of the path graph. The degree-centric transformation increased the number of edges, making the graph partially complete. Due to this increase in connectivity, the Double Roman Domination Number decreased. All the values obtained from the analysis are organized in the table below.

Table 2: Values of $\gamma_{dR}(C_n)$ and $\gamma_{dR}(C_n^d)$

Graph	n	Edges	Degree	$\gamma_{dR}(C_n)$	$\gamma_{dR}(C_n^d)$	Pattern
C_4	4	8	4	6	3	$2n$
C_5	5	10	4	7	3	$2n$
C_6	6	12	4	8	5	$2n$
C_7	7	14	4	9	6	$2n$
C_8	8	16	4	10	6	$2n$

we can see similar results

3.3 Degree-Centric Transformation of Star Graphs and Complete Graphs and Their Double Roman Domination Number

From the analysis of the star graph and the complete graph, we observed that the degree-centric transformation does not introduce any new edges. Hence, the transformed graph is the same as

the original graph, making both graphs isomorphic. Since there is no structural change after the transformation, the Double Roman Domination Number also remains the same as that of the original graph.

3.4 Degree-Centric Transformation of Honeycomb Graphs and Double Roman Domination Number

In a honeycomb graph, most vertices have degree 2 or 3. Vertices lying on the boundary of the graph generally have degree 2, while vertices shared by two adjacent hexagons have degree 3. The structure exhibits a high degree of symmetry and serves as a useful model for studying domination, coloring, distance-based parameters, and graph transformations.

Theorem A. *Let H_n be a honeycomb graph. Then every vertex of H_n has degree either 2 or 3.*

Proof. Vertices on the boundary of the honeycomb structure belong to only one hexagonal cell and hence have degree 2. Vertices shared by two adjacent hexagons are incident with three edges and therefore have degree 3. Since every vertex of a honeycomb graph is either a boundary vertex or a shared vertex, every vertex has degree either 2 or 3. $\square$

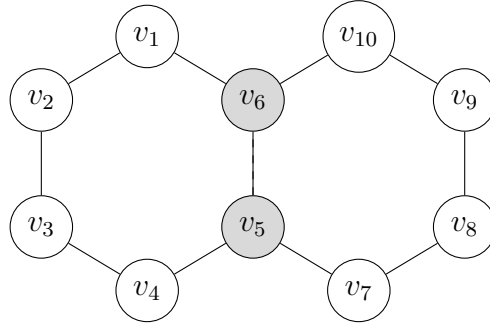


Figure 1: Honeycomb graph G

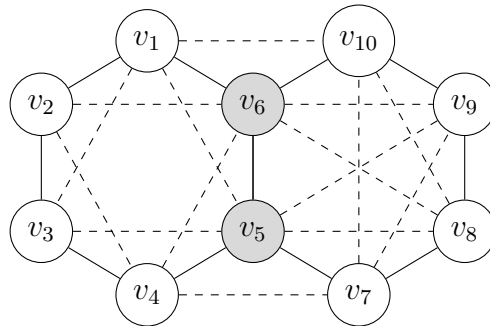


Figure 2: Degree-centric graph G_d

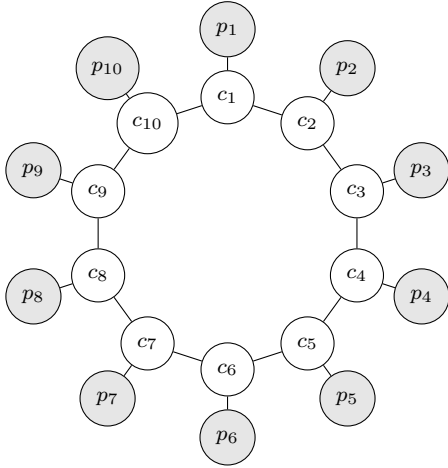
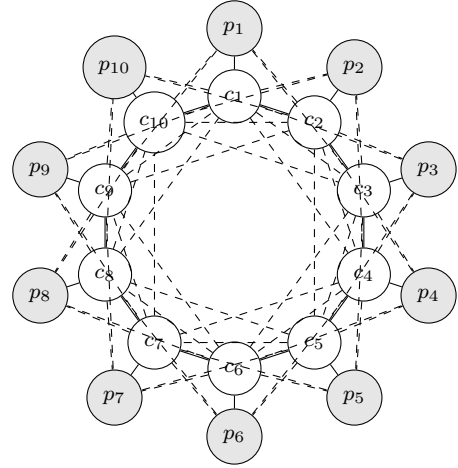
Table 3: Rule Check Table for the Degree Centric Graph

Vertex	$\deg(v)$	$d \leq \deg \rightarrow$ New Neighbours	Edges Added
v_1	2	v_3, v_5, v_{10} ($d = 2$)	3
v_2	2	v_4, v_6 ($d = 2$)	2
v_3	2	v_1, v_5 ($d = 2$)	2 (v_1 shared)
v_4	2	v_2, v_6, v_7 ($d = 2$)	3
v_5	3	v_1, v_3, v_{10}, v_8 ($d = 2$); v_2, v_9 ($d = 3$)	6
v_6	3	v_2, v_4, v_7, v_9 ($d = 2$); v_3, v_8 ($d = 3$)	6

3.5 Degree-Centric Transformation of Sunlet Graphs and Double Roman Domination Number

The sunlet graph also showed results similar to the cycle graph. Since both graphs share similar structural properties, the degree-centric transformation was almost the same. The transformed graph became partially complete due to the addition of new edges, and this resulted in a decrease in the Double Roman Domination Number.

Therefore, the Sunlet graph S_n contains $2n$ vertices and $2n$ edges.


 Figure 3: Sunlet Graph S_{10}

 Figure 4: Degree-Centric Graph $(S_{10})_d$

3.6 Degree-Centric Transformation of Wheel Graphs and Double Roman Domination Number

From the analysis of the wheel graph, we observed that all the outer vertices are already connected to the centre vertex. Because of this, the Double Roman Domination Number of the original graph is already very low, with a value less than or equal to 3. After the degree-centric transformation, more edges were added to the graph. However, when comparing the original and the transformed graph, we found that both graphs could still be dominated using a single dominating set. Hence, there was no significant change in the Double Roman Domination Number.

Hence the result follows.

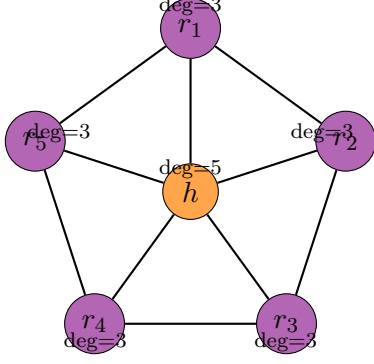


Figure 5: Wheel Graph W_6

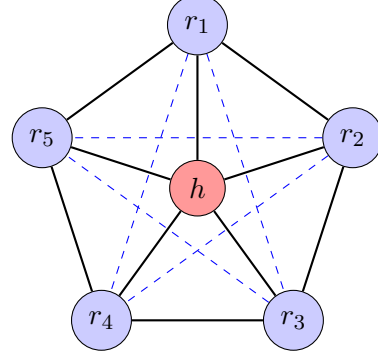


Figure 6: Degree-Centric Graph $(W_6)_d$

4 Generalization of Results

After analyzing the Double Roman Domination Number of the degree-centric graphs for different graph families, several common patterns were observed. These observations made it possible to generalize the obtained results and derive formulas that relate the structure of the degree-centric graph to its Double Roman Domination Number. The following theorems summarize the general results obtained from the analysis of both standard and randomly generated graphs.

Theorem 1.

$$3 \leq \gamma_{dR}(G_d) \leq n$$

Notation. Let $G = (V, E)$ be a graph, where

- $n = |V(G)|$ denotes the number of vertices of G ,
- $m = |E(G)|$ denotes the number of edges of G ,
- $\gamma_t(G)$ denotes the total domination number of G ,
- $\gamma_{dR}(G)$ denotes the Double Roman domination number of G .

Proof. Consider a connected graph of order n , where $n \geq 3$. When any connected graph G is transformed using the degree-centric graph transformation,

$$E(G_d) = \{v_i v_j : d(v_i, v_j) \leq \deg(v_i)\}$$

and $V(G_d) = V(G)$, let the degree-centric graph be denoted by G_d . Then the graph obtained is either complete or partially complete. Since G is connected, the maximum degree of a vertex satisfies

$$\Delta(G) \geq 2,$$

where it forms the maximum number of edges. After the degree-centric graph transformation, that vertex is assigned the maximum Double Roman value, namely 3.

Hence, $\gamma_{dR}(G_d)$ will always be less than or equal to n , since the transformed graph covers most of the vertices. If a vertex is extended to an extent, then it would be assigned in the following sequence:

$$(2, 1, 0)$$

or

$$(3, 0, 0),$$

depending on the minimization of the Double Roman domination number. Therefore,

$$\gamma_{dR}(G_d) \leq n.$$

Hence,

$$3 \leq \gamma_{dR}(G_d) \leq n.$$

□

Theorem 2. *Let G be a connected graph and let G_d be the deg-centric graph of G . Then*

$$V_{DR}(G_d) \leq 3V_R(G).$$

Proof. From Theorem 1, we know that

$$3 \leq V_{DR}(G_d) \leq n.$$

Also,

$$V(G) = V(G_d) = n.$$

By the Handshaking Lemma,

$$2m = \sum_{v \in V(G)} \deg(v).$$

By the Handshaking Lemma,

$$2|E(G)| = \sum_{v \in V(G)} \deg_G(v).$$

Since the deg-centric transformation is based on the degrees of the vertices, the graph G_d contains at least as many edges as G , that is,

$$|E(G_d)| \geq |E(G)|.$$

Hence, G_d is a spanning supergraph of G .

Lemma 3. *If H is a spanning supergraph of G , then*

$$V_R(H) \leq V_R(G).$$

By the above lemma,

$$V_R(G_d) \leq V_R(G).$$

Also,

$$V_{DR}(G_d) \leq 3V_R(G_d).$$

Therefore,

$$V_{DR}(G_d) \leq 3V_R(G_d) \leq 3V_R(G).$$

This completes the proof. □

Theorem 4. *Let G be a graph of order n and maximum degree $\Delta(G)$. Then*

$$3 \leq \gamma_{dR}(G_d) \leq 3 \left\lceil \frac{n}{1 + \Delta(G)} \right\rceil.$$

Proof. Consider a graph G with maximum degree $\Delta(G)$. Since the maximum degree of a graph of order n cannot exceed $n - 1$, we have

$$\Delta(G) \leq n - 1.$$

From Theorem 1, we have

$$\gamma_{dR}(G_d) \leq 3 \left\lceil \frac{n^2}{n + 2m} \right\rceil.$$

Also, by the Handshaking Lemma,

$$2m = \sum_{v \in V(G)} \deg(v).$$

Since every vertex has degree at most $\Delta(G)$,

$$2m \leq n\Delta(G).$$

Adding n to both sides gives

$$n + 2m \leq n(\Delta(G) + 1).$$

Hence,

$$\frac{n^2}{n + 2m} \geq \frac{n^2}{n(\Delta(G) + 1)} = \frac{n}{\Delta(G) + 1}.$$

Therefore,

$$\left\lceil \frac{n^2}{n + 2m} \right\rceil \geq \left\lceil \frac{n}{\Delta(G) + 1} \right\rceil.$$

Substituting into Theorem 1, we obtain

$$\gamma_{dR}(G_d) \leq 3 \left\lceil \frac{n}{1 + \Delta(G)} \right\rceil.$$

Hence,

$$3 \leq \gamma_{dR}(G_d) \leq 3 \left\lceil \frac{n}{1 + \Delta(G)} \right\rceil.$$

□

Theorem 5. *Let G be a connected graph and let G_d be its deg-centric graph. Then*

$$V_{DR}(G_d) \leq 3V_D(G_d) \leq 3V_D(G).$$

Proof. Let

$$D_G = \{v_1, v_2, \dots, v_k\}$$

be a minimum dominating set of the graph G , where

$$|D_G| = V_D(G).$$

Let

$$D_{G_d} = \{u_1, u_2, \dots, u_t\}$$

be a minimum dominating set of the deg-centric graph G_d , where

$$|D_{G_d}| = V_D(G_d).$$

After applying the deg-centric graph transformation, the number of vertices remains the same, whereas the number of edges increases. Due to the increase in the number of edges, the graph becomes more connected and every dominating vertex can dominate more adjacent vertices. Hence, fewer vertices are required to dominate the transformed graph. Therefore,

$$|D_{G_d}| \leq |D_G|.$$

Also,

$$V_{DR}(G_d) \leq 3|D_{G_d}|.$$

Hence,

$$V_{DR}(G_d) \leq 3|D_{G_d}| \leq 3|D_G| \leq 3V_D(G).$$

Therefore,

$$V_{DR}(G_d) \leq 3V_D(G).$$

Hence the proof. □

Theorem 6. *Let G be a connected graph of order n and let G_d be its deg-centric graph. Then*

$$V_{DR}(G_d) \leq 3(n - \beta(G)),$$

where $\beta(G)$ denotes the vertex cover number of G .

Proof. From Theorem 2, we have

$$V_{DR}(G_d) \leq 3V_D(G).$$

Let $\beta(G)$ denote the minimum vertex cover of G . Since G is a connected graph, every minimum vertex cover is also a dominating set. The vertices in a minimum vertex cover cover all the edges of the graph, while a dominating set is a minimum set of vertices that dominates all the vertices of the graph. Hence, every minimum vertex cover is a dominating set, although the converse need not be true.

Also, by Gallai's Theorem,

$$\alpha(G) + \beta(G) = n,$$

where $\alpha(G)$ is the independence number of G . Since every maximal independent set is a dominating set, it follows that

$$V_D(G) \leq \alpha(G) = n - \beta(G).$$

Therefore,

$$V_{DR}(G_d) \leq 3V_D(G) \leq 3(n - \beta(G)).$$

Hence,

$$V_{DR}(G_d) \leq 3(n - \beta(G)).$$

□

Theorem 7. *Let G be a graph of order n with no isolated vertices, and let $\alpha'(G)$ denote the edge independence (matching) number of G . Then the Double Roman Domination Number of the degree-centric graph G_d satisfies*

$$\gamma_{dR}(G_d) \leq 3(n - \alpha'(G)).$$

Proof. From Theorem 1, for any dominating set D of the degree-centric graph G_d , we have

$$\gamma_{dR}(G_d) \leq 3|D|.$$

Since every vertex cover is also a dominating set in a graph with no isolated vertices, it follows that

$$|D| \leq \beta(G),$$

where $\beta(G)$ denotes the vertex cover number of G .

By Gallai's Theorem,

$$\beta(G) + \alpha'(G) = n,$$

where $\alpha'(G)$ is the edge independence (maximum matching) number of G .

Hence,

$$\beta(G) = n - \alpha'(G).$$

Substituting this into the previous inequality gives

$$|D| \leq n - \alpha'(G).$$

Therefore,

$$\gamma_{dR}(G_d) \leq 3|D| \leq 3(n - \alpha'(G)).$$

Thus,

$$\gamma_{dR}(G_d) \leq 3(n - \alpha'(G)).$$

This completes the proof. □

Theorem 8. *Let G be a connected graph of order $n \geq 3$ with m edges. Then,*

$$\gamma_{dR}(G_d) \leq 3 \left(\frac{n^2}{n + 2m} \right).$$

Proof. Let G be a connected graph with n vertices and m edges. Since the maximum possible number of edges in a simple graph of order n is

$$\frac{n(n-1)}{2},$$

we have

$$m \leq \frac{n(n-1)}{2}.$$

Therefore,

$$n + 2m \leq n + 2 \left(\frac{n(n-1)}{2} \right) = n + n(n-1) = n^2.$$

Hence,

$$n + 2m \leq n^2.$$

Dividing both sides by the positive quantity $n + 2m$ gives

$$1 \leq \frac{n^2}{n + 2m},$$

which implies

$$3 \leq 3 \left(\frac{n^2}{n + 2m} \right).$$

From Theorem 1,

$$\gamma_{dR}(G_d) \leq 3.$$

Combining the above inequalities, we obtain

$$\gamma_{dR}(G_d) \leq 3 \leq 3 \left(\frac{n^2}{n + 2m} \right).$$

Therefore,

$$\boxed{\gamma_{dR}(G_d) \leq 3 \left(\frac{n^2}{n + 2m} \right)}.$$

This completes the proof. □

5 Conclusion

This paper presented a detailed study of the Double Roman Domination Number of Degree-Centric Graphs obtained from various standard graph families. The degree-centric transformation was applied to path graphs, cycle graphs, complete graphs, star graphs, wheel graphs, honeycomb graphs, and sunlet graphs to investigate the effect of increased connectivity on the Double Roman Domination Number.

The results show that the degree-centric transformation generally increases the number of edges while preserving the vertex set. For several graph families, this increase in connectivity reduces the Double Roman Domination Number, indicating that additional edges improve the defensive capability of the graph. However, for graph families such as complete graphs and star graphs, where no structural changes occur after transformation, the Double Roman Domination Number

remains unchanged.

Based on the obtained results, several general theorems and inequalities were established, providing bounds and relationships between the Double Roman Domination Number and other graph parameters. These results contribute to the growing literature on domination in transformed graphs and provide a theoretical foundation for further investigations in Degree-Centric Graph Theory.

Future research may focus on determining the exact Double Roman Domination Numbers for other graph transformations, studying algorithmic approaches for larger graph classes, and exploring applications of degree-centric graphs in communication networks, social networks, optimization problems, and network security.

Declaration

We hereby declare that this research work entitled “*Study of Double Roman Domination Number in Degree-Centric Graph*” is our original work carried out under the guidance of our respective institutions. The work presented in this paper has not been submitted, either in whole or in part, to any other journal, conference, or institution for the award of any degree or diploma.

All sources of information used in this work have been properly acknowledged through appropriate citations and references. We take full responsibility for the originality and authenticity of the work presented.

Future Scope

The present work opens several directions for future research. The study may be extended to other graph transformations such as subdivision graphs, central graphs, total graphs, line graphs, middle graphs, and shadow graphs. Determining the Double Roman Domination Number for these graph classes may lead to new theoretical results and generalized formulas.

Further research may also focus on developing efficient algorithms for computing the Double Roman Domination Number of large graphs, investigating computational complexity, and exploring applications in communication networks, wireless sensor networks, cyber security, social network analysis, and optimization problems. These directions can further strengthen the importance of Degree-Centric Graphs in modern graph theory.

References

1. Douglas B. West, *Introduction to Graph Theory*, 2nd Edition, Prentice Hall, Upper Saddle River, NJ, 2001.
2. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1976.
3. Gary Chartrand and Ping Zhang, *Introduction to Graph Theory*, McGraw-Hill Higher Education, New York, 2005.
4. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

5. E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi, “Roman Domination in Graphs,” *Discrete Mathematics*, Vol. 278, No. 1–3, pp. 11–22, 2004.
6. Richard A. Beeler, Teresa W. Haynes, and Stephen T. Hedetniemi, “Double Roman Domination,” *Discrete Applied Mathematics*, Vol. 211, pp. 23–29, 2016.
7. M. R. Farahani, et al., “Degree-Centric Graphs and Their Properties,” 2022.