Machian MOND: a variable a_0 in galaxy clusters

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ABSTRACT

Modified Newtonian Dynamics (MOND) generally resolves the need for dark matter in galaxy rotation curves introducing a single new constant of acceleration a_0. It is well known that increasing a_0 by a factor of a few can alleviate the residual mass discrepancies that MOND leaves in galaxy clusters. Within a parameter-free Machian interpretation of MOND motivated by $a_0 \sim GM_u/R_u^2$ arising from the scalar sum of inverse-square distance gravitational mass contributions in the universe, we promote a_0 to a variable in spherically symmetric mass distributions, which is sensitive to mass external to the locally enclosed spherical region. Instead of a boost of a_0 in terms of gravitational potentials as in EMOND, we show that a boost in terms of this directionless inverse-square field roughly amounts to the boost needed to accommodate the mass discrepancies of MOND in galaxy clusters, while retaining MOND predictions in galaxies. We begin to generalize this approach by defining a new $a_0' = G\sum_i m_i/|\mathbf{r}-\mathbf{r_i}|^2 - a_N$, where a_N is the magnitude of the usual Newtonian acceleration.

Key words: dark matter; modified gravity; MOND; modified Newtonian Dynamics; EMOND; galaxy clusters; Mach's principle

1 INTRODUCTION

1.1 MOND and galaxy clusters

The observation of rotational velocities in galactic disks higher than those predicted by Newtonian gravity, as shown in galaxy rotation curves by Rubin (1983), has motivated research into modifications of the laws of gravity and mechanics as alternatives to the still elusive physical dark matter (Garrett K., Duda G. 2011). After careful observation of the patterns and regularities found in these systems related to visible mass only, such as the baryonic Tully–Fisher relation (the asymptotic rotational velocity scales as the fourth root of the baryonic mass), Renzo's rule (features in the visible mass distribution strongly correlate with features in the velocity profiles) and the empirical observation that Newton's laws accurately predict dynamics only above a certain acceleration scale, Mordehai Milgrom (1983a,b,c) proposed the simplest modification to Newtonian dynamics consistent with these observational constraints.

The original formulation of Milgrom's Modified Newtonian Dynamics MOND results in a constant asymptotic rotational velocity $v = \sqrt[4]{GMa_0}$ in the low acceleration or deep-MOND regime with the need to introduce an empirically determined constant with dimensions of acceleration a_0. The transition between the classical Newtonian regime where $v = \sqrt{GM/r}$ and the deep-MOND regime is governed by an 'interpolating function' $\mu\,(a/a_0) = 1/(1+(a_0/a)^n)^{(1/n)}$ (with $n = 1$ for the simple and $n = 2$ for the standard interpolating functions) based on $|\mathbf{a}| = a$ as the measured or 'true acceleration' and the constant a_0 (also referred to as the Hubble acceleration or the acceleration scale constant). Such constant sets the characteristic acceleration scale below which MOND effects manifest. The interpolating function, interpreted as modified inertia $\mathbf{F} = m_i\mu(a/a_0)\,\mathbf{a}$, must satisfy the limiting conditions $\mu(x) \to x$ for $x \ll 1$ to reproduce the Tully-Fisher law, and $\mu(x) \to 1$ for $x \gg 1$ to recover the usual Newtonian laws at high accelerations. Switching to an interpolating function $\mathbf{a} = \nu(a_0/a_N)\,\mathbf{a_N}$ in terms of the Newtonian acceleration $|\mathbf{a_N}| = a_N$, $\nu(a_0/a_N) = (1/2)^{1/n}(1+\sqrt{1+4(a_0/a_N)^n})^{1/n}$ (with $n = 1$ for the simple and $n = 2$ for the standard interpolating functions) and

$$\mathbf{a_{MOND}} = \nu\left(\frac{a_0}{a_N}\right)\mathbf{a_N} \sim \mathbf{a_N}\sqrt{1+\frac{a_0}{a_N}}, \tag{1}$$

so that at the deep-MOND regime, $a \sim \sqrt{a_N a_0}$.

To satisfy conservation laws and for computation purposes, Lagrangian-based models have been developed, such as AQUAL and QUMOND. Both are special cases of three-potential (TRIMOND) theories (Milgrom 2023) that recover Milgrom's MOND in highly symmetric systems, while showing modest deviations elsewhere. Nevertheless, as non-relativistic models lacking a physical explanation for the origin of the interpolating function and the acceleration scale constant, MOND is best regarded as an effective theory or an approximation to a more fundamental modification of gravity or inertia. Relativistic MOND theories, such as TeVeS (Bekenstein 2004), recover MOND's predictions for galaxies through extra scalar and vector fields and several accompanying parameters and constants.

Milgrom's MOND, while remarkably successful in describing galactic rotation curves, remains insufficient to explain all phenomena attributed to dark matter, such as galaxy cluster dynamics (Sanders 1998, 2003; Pointecouteau & Silk 2005). Roughly 90% of the visible (baryonic) mass in galaxy clusters is in the form of hot and

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diffuse intracluster gas, with the remainder contained in the galaxies themselves. Observations of galaxy motions, X-ray–emitting gas, weak and strong gravitational lensing, and the Sunyaev–Zel'dovich effect all indicate a persistent mass discrepancy between the total (dynamical or lensing) mass and the observed baryonic matter. Under Newtonian dynamics, this discrepancy decreases with increasing distance from the cluster core: in the central regions it is typically of order $10-50$, falling to values of $5-10$ at radii of about $1-2$ Mpc.

There is evidence for an analogous radial acceleration relation in galaxy clusters; an empirical scaling law between the observed acceleration and the expected from visible mass only, with a characteristic acceleration scale of $\sim 10a_0$ (Tian et al. 2020). The typical observed accelerations in the cores of clusters are on the order of, or a few times larger than a_0, and MOND predicts only a small correction in them, underpredicting mass discrepancies by a factor of a few (Milgrom 2008), around $2-6$ (Kelleher & Lelli 2024). At intermediate radii of about 1 Mpc, the measured accelerations are around $0.2\,a_0 - 0.5\,a_0$ (Sanders 1998), and MOND reduces the mass discrepancies at these radii to factors of only $2-3$ (Ettori, S. et al. 2019). At radii above 2 Mpc, the accelerations drop to roughly $0.05\,a_0$, and MOND leaves an even smaller residual mass discrepancy, as low as only around 15%. A recent study suggests that the previously described discrepancies may be significantly alleviated (Zhang et al. 2026).

In analogy with Milgrom's identification of a characteristic acceleration scale as the basis for modifying gravity or inertia in galaxies, Bekenstein identified that a gravitational potential (or velocity) scale could play a similar role for MOND in galaxy clusters. Given that clusters have huge mass distributions that are much more spatially extended than galaxies, their gravitational potentials are correspondingly deeper, and a few times larger a_0 can bridge the mass discrepancy gap in clusters within MOND. In his scale-dependent critical acceleration formulation, Bekenstein introduced the required boost in galaxy clusters of $2-3$ times a_0 while leaving the galactic regime nearly unchanged through $a_0\, e^{-2\Phi/s^2}$ with Φ the negative gravitational potential and s a data-fitted velocity scale on the order of 10^6 m/s (Bekenstein 2011). HongSheng Zhao and Benoit Famaey developed Extended MOND (EMOND) (Zhao & Famaey 2012; Hodson & Zhao 2017; Hodson, Alistair O. & Zhao, Hongsheng 2017) in a modified AQUAL Lagrangian to generalize Bekenstein's idea and address the problematic emergence of unrealistically large effective a_0 values in regions of extremely high gravitational potential $\Phi \sim -c^2$ such as neutron stars and stellar black holes, where any modification to gravity is strongly constrained by General Relativity (GR). But together with an additional interpolating function, they also need to introduce a dimensionless constant parameter $\beta \sim 8\times10^{-3}$ tweaking the gravitational potential scale of the speed of light squared to fit the data. Another extension of MOND addressing the cluster problem is GMOND (Khoury 2015), in which a third regime for clusters is introduced as a stronger inverse-square law by a fine-tuned parameter effectively boosting the gravitational constant. More recently, a p-Laplacian generalization of AQUAL has been suggested to accommodate dwarf spheroidal galaxies and galaxy clusters by allowing the exponent p to depend on the properties of the system through an extra function (Scherer, D. et al. 2025).

Progress in fundamental physics has often been achieved by reducing the number of fundamental constants[1]. Therefore, given that MOND only introduces a single new constant into Newton's laws (which will be considered a derived parameter in the next section), it is of considerable theoretical interest to explore modifications to MOND without the need for more fundamental constants or free parameters, in the search for the physical origin of MOND.

1.2 Machian MOND

Mordehai Milgrom has insisted on several occasions that MOND might be a consequence of Mach's principle:

> *An attractive possibility is that MOND results as a non-relativistic, small-scale expression of a fundamental theory by which inertia is a vestige of the interaction of a body with 'the rest of the Universe', in the spirit of Mach's principle.* [...] *We thus envisage inertia as resulting from the interaction of the accelerated body with some agent field, perhaps having to do with the vacuum fields, perhaps with an "inertia field" whose source is matter in the "rest of the universe"–in the spirit of Mach's principle.* (Milgrom 1994)

> *It seems to me that in looking for an ultimate theory, the Mach principle may serve as a most useful guide.* [...] *The possible connection between such numerical "coincidences", which relate parameters of "local" physics to cosmological parameters and the Mach principle, is quite obvious.* (Milgrom 1983a)

but has not developed this relationship further. The coincidences $a_0 \sim c^2\sqrt{\Lambda}$ and $a_0 \sim cH_0 \sim c^2/R_u$ have gained more attention as potential evidence that MOND arises from the quantum vacuum or cosmic acceleration (Milgrom 2020). In contrast to the Machian coincidence $a_0 \sim GM_u/R_u^2$, where M_u (the mass within the horizon of radius R_u) influences local dynamics as in most Machian-inspired theories, the cosmological constant Λ remains obscure. Not only must its existence be assumed, since the evidence for it is purely indirect, but one must also assume that it corresponds to the quantum vacuum energy density driving the universe's accelerated expansion. Following Occam's razor, $a_0(M_u, R_u)$ is more plausible, since we are certain about the existence of M_u and R_u by direct observations, but uncertain about Λ and the origin of the accelerated expansion. Other authors have drawn attention to a possible Machian origin of MOND (Licata, I., Corda, C. and Benedetto, E. 2016; Benedetto et al. 2024; Darabi 2010; Gine 2009, 2012).

It is not our purpose to explain here the deep rabbit hole of Mach's principle (Mach 1881) and the relativity of inertia, despite the historical ambiguity[2] surrounding their interpretation, even though such an understanding is required to realize its connection to MOND. An introduction to the Machian reasoning relevant to MOND has already been developed by Uruena, M. (2024), and Mach's principle has been extensively covered by Reinhardt (1973); Barbour & Pfister (1995); Bondi & Samuel (1997). One of the expected consequences[3] of Mach's principle according to Einstein (1922) is that "*The inertial mass of a body should increase with the agglomeration of masses*

[1] Such as the gravitational acceleration at the surface of the Earth and other planets or Kepler's 3rd law constant for a universal gravitational constant, the unification of the speed of light with vacuum permittivity and permeability, and atomic constants, such as Rydberg's constant, for the Planck constant.

[2] One of such misunderstandings is that it necessarily implies the anisotropy of inertia, which is highly constrained by observations and effectively ruled out. Not only it was simply suggested as a possible feature of Mach's principle and not as a consequence by Cocconi and Salpeter (Cocconi & Salpeter 1958), but it was argued by Robert Dicke (Dicke 1961) that such anisotropic effects would be undetectable in a metric theory. The anisotropy is also avoided if inertial mass is a scalar.

[3] The other two expected consequences correspond to the Lense-Thirring effect and to the fact that a body should experience an acceleration if nearby masses are accelerated, with the same direction as their acceleration. Both are featured in GR by frame-dragging.

in its neighborhood", or equivalently, *the inertia of a body should decrease when masses are removed from its neighborhood*. Such consequence, unlike the other two Einstein mentioned, is not satisfied by GR, in which inertia is still absolute[4]. In MOND, this isolation of a body from other masses trivially corresponds to the deep-MOND regime, where the system is subjected to very low gravitational field intensities.

Given that the gravitational potential of the universe $GM_u/R_u \sim c^2$ does not naturally correspond to the potential scale featured in EMOND removing MOND's problem in galaxy clusters, and that a huge dimensionless correction factor (as a dimensionless parameter) must be introduced to adjust this scale, we instead turn to local field intensities as the ones relevant for modifying a_0 in an attempt to address the cluster problem. We are motivated by the notion that since accelerations in clusters are of the order of a_0, if these sum to a_0 through some as yet unknown mechanism in clusters, the required boost to a_0 could be approximately achieved without introducing new constants.

2 VARIABLE a_0 IN MACHIAN MOND

2.1 Mach against Newton's shell and Birkhoff's theorems

A brief exception to omitting the history of Mach's principle is made here as a justification for the physical foundations of the following section.

At the beginning of the 'golden age' of GR, Brans (1962) presented a thought experiment considering the ratio between the inertial and gravitational masses of a body within a Machian framework. He idealized a static universe consisting solely of a massive hollow shell of radius R_u and inertial mass M_u (serving as a simplification to the large-scale mass and size of the universe, with approximately equivalent potential at its center), together with a relatively small body of inertial mass M at the center and a test particle (Figure 1).

He expected the acceleration of the test particle to be independent of its own mass (as required by the equivalence principle), to depend on the mass M and its distance r from M, and because of Machian arguments about the inertial properties of the test body being determined by the distribution of matter in the universe and not being intrinsic of the body, conceivably also to depend on M_u and R_u. Thus, $a = f(M_u, R_u)M/r^2$ and he related f to the gravitational constant by $f = G \sim c^2/(M_u/R_u)$ recovering the Newtonian gravitational acceleration. This relationship for the gravitational constant[5], which inspired the Brans-Dicke theory, dates back to Sciama (1953) with its roots in the works of Schrödinger (1925) and Reissner (1915) on Mach's principle. The difference in predictions between the Brans-Dicke theory and GR has been locally tested in the Solar System near the Sun through the Shapiro time delay effect measured by the Cassini experiment, showing that the Brans-Dicke dimensionless coupling factor of the scalar field to gravity, which differentiates between the theory and GR, must be so high that Brans-Dicke theory is almost indistinguishable from GR. All of these tests have been performed in the high-acceleration regime of the Solar System.

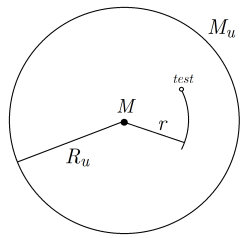

Figure 1. Brans' shell model of the universe: the total mass M_u within the observable universe is represented by an equivalent static mass shell of radius R_u.

Milgrom's MOND can be formulated very similarly to Brans' Machian idea with $a = f(M_u, R_u, M, r)M/r^2$ by introducing[6] $a_0 \sim GM_u/R_u^2$ as

$$a_{\text{Machian MOND}} \sim c^2 \frac{M/r^2}{M_u/R_u}\sqrt{1 + \frac{M_u/R_u{}^2}{M/r^2}}. \tag{2}$$

With the interpolating function acting on Newton's second law rather than modifying gravity, the resulting inertial law depends on the inertial mass of the test particle, the gravitationally active mass M and the mass of the rest of the universe M_u, i.e., inertia becomes a function of all relevant mass distributions, in the spirit of Mach.

Brans' thought experiment illustrates how Mach's principle violates Newton's shell theorem[7]. According to Newton, a spherically symmetric massive shell does not contribute to the local dynamics of bodies within it when interior masses are present. In Brans' framework the acceleration depends on the shell's mass and radius.

A consequence of the shell theorem is that the gravitational field vanishes at the center of any spherically symmetric mass distribution. For any event, there exists a surrounding spherical region of the observable universe with approximately uniform matter density, for which the net gravitational field due to the mass of the universe vanishes. Therefore, the term M_u/R_u^2 in (2) (for which a_0 is effectively replaced) is not a field intensity or acceleration, but represents the directionless sum of the inverse-squared mass contributions from all volume elements in the universe at large.

$$a_0 = G\int_u \frac{dm}{r^2} = 4\pi G\int_0^{R_u} \rho_u\, dr = \frac{3GM_u}{R_u^2}. \tag{3}$$

This $a_0(M_u, R_u)$ scalar term matching $a_0 = (1.2 \pm 0.3) \times 10^{-10}$ m/s^2 (Zhao & Famaey 2012) will be used in the following sections as a constant at present epoch and simply denoted as a_0.

Newton's shell theorem follows from the inverse-square law. MOND breaks with the shell theorem in the MOND regime except

[4] Flat Minkowski spacetime is a solution to GR in absence of masses in which inertia behaves as usual. It contradicts Mach's principle since inertia should have exclusively an origin in the mass distribution at large.

[5] By dimensional analysis, the combinations $G \sim c^2/(M_u/R_u)$ and $a_0 \sim GM_u/R_u^2$ are essentially the only ways (up to additional dimensional constants) to construct constants that involve both a characteristic mass and distance, thereby encoding properties of a mass distribution.

[6] We also introduce the Machian gravitational constant defined in terms of the same parameters $G \sim c^2/(M_u/R_u)$, although not strictly necessary. Within the interpolating function, the gravitational constant cancels out, remaining only in the usual Newtonian force law.

[7] The gravitational potential inside an empty spherically symmetric massive shell is constant with vanishing field intensities. Consequently, a body inside such shell experiences no net gravitational force from the shell, regardless of the mass of the body or the shell, or the position within the shell.

for spherical symmetry[8], since the effective deep-MOND force tends to decay with the inverse of the distance at low accelerations, and superposition does not hold. Bekenstein's scale dependent critical acceleration in AQUAL and its generalization in EMOND based on a modification to a_0 in terms of potentials, both break with the shell theorem even in spherically symmetric cases, and masses outside a sphere can affect local dynamics inside it. This effect might even help with situations such as the Bullet cluster, in which the sources of the gravitational field seem to be outside the region expected to govern local dynamics in a Newtonian fashion. The external field effect (EFE) in Milgrom's MOND is also an example of how external far away mass distributions influence local dynamics in MOND.

A generalization of Newton's shell theorem occurs in GR for test particles inside a massive shell by Birkhoff's theorem, where they experience no accelerations from the shell. For off-centered massive particles, their presence inside the shell breaks the exact symmetry of spacetime, leading to weak accelerations toward the center of the shell. A stronger effect is found in AQUAL (Dai et al. 2010).

By identifying MOND's acceleration scale constant as coming from Brans' shell model of the universe $a_0 \sim GM_u/R_u^2$, we will argue in the following that a_0 is modified in clusters.

2.2 Machian MOND and a variable a_0

Machian MOND (2) is built under the assumption that $a_0 \sim GM_u/R_u^2$ while local gravitational field intensities are only accounted for in $a_N = GM/r^2$. This formulation raises the question of when a mass distribution should contribute to local field intensities and when it should instead contribute to a_0, and why Machian MOND appears to distinguish so clearly between the two.

Subsequently, it is conceivable that perhaps a_0 is also affected by large but local mass distributions analogous to that of the universe, such as in galaxy clusters. If the origin of a_0 is idealized as arising from an exterior spherically symmetric shell of mass M_u at distance R_u of the universe, such as Brans' model, then a galaxy within a galaxy cluster may likewise be regarded as being enclosed by an additional symmetric shell of mass M_s at a radius R_s, representing the cluster mass exterior to the enclosed mass within radius r of that galaxy (Figure 2).

We denote $a_s \sim GM_s/R_s^2$ the gravitational scalar sum of the inverse-squared masses of the cluster exterior to the radius r and expect a non-linear dependence of the effective acceleration scale constant $a_0'(M_u, R_u, M_s, R_s)$. We propose the following function as an Ansatz:

$$\mathbf{a} = \mathbf{a_N}\sqrt{1+\frac{a_0}{a_N}}\sqrt{1+\frac{a_s}{a_N}}\,, \quad a = \sqrt{(a_0+a_N)(a_s+a_N)}, \tag{4}$$

which is equivalent to Milgrom's MOND but replacing the acceleration scale constant $a_0 \to a_0' = a_0 + a_s + a_0 a_s/a_N$. Thus, a_0' is no longer a fundamental parameter, but an effective, position-dependent quantity arising from the mass distribution. Note that a_N is an input field determined by the mass distribution, independent of the test particle motion or its dynamics. It satisfies the following desired conditions of an extension for MOND:

[8] For spherically symmetric systems in the deep-MOND regime, the gravitational field is exactly zero at the very center; outside the spherically symmetric mass distribution, the gravitational field is the same as if all the mass were concentrated at the center.

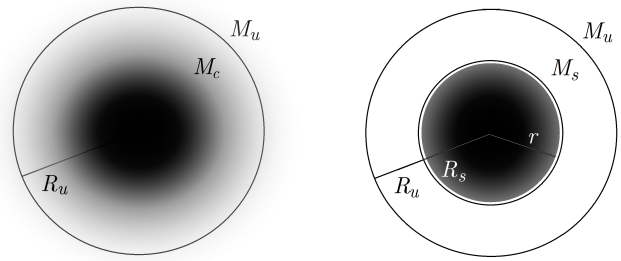

Figure 2. Left: a spherically symmetric schematic model for a galaxy cluster mass distribution (blurred) with total mass M_c inside Brans' shell of the universe M_u and R_u. Right: the mass distribution of the cluster exterior to a radius r is idealized as an additional shell with mass M_s and radius R_s. Here $R_s \neq r$ depends on the mass distribution outside r. Not to scale.

a) It does not introduce new constants or adjusted parameters, as required in other proposals such as Bekenstein's scale dependent critical acceleration or EMOND.

b) It reduces to Milgrom's MOND when the newly introduced $a_s \to 0$, i.e., when there is no external local shell. The EFE can be approximated as usual through a_N.

c) Removing the mass of the distant universe $M_u \to 0$, $a_0 \to 0$, $\mathbf{a} = \mathbf{a_N}\sqrt{1+\frac{a_s}{a_N}}$ and the shell of the cluster behaves as a new 'cosmic' shell, playing the same role in the dynamics as a_0 does in Machian MOND.

If $a_N \gg a_0, a_s$ then $a \approx a_N$ and the system is in the Newtonian regime. If $a_N \ll a_0$ and a_s is negligible, then $a \approx \sqrt{a_N a_0}$, corresponding to the usual deep-MOND regime. When $a_N \sim a_s \sim a_0$, equation (4) accounts for the combined contributions of both the cosmic and local external shells. In the following section, a simple toy model will show that such a modification to MOND produces a boost to a_0 in galaxy clusters of the order required by observations.

2.3 A toy model for galaxy clusters

We test the proposed modification to Milgrom's MOND (4) with a simple spherically symmetric model of standard galaxy clusters, in which $a_s(r)$ and $a_N(r)$ vary with distance from the cluster core r. Our aim is to show that such a modification generally matches the required boost in a_0 to fit clusters, while leaving MOND predictions for individual galaxies effectively unchanged, as discussed in the next section.

For any given distance r from the core of the cluster, the cluster's mass M_s exterior to r is treated as an additional shell, analogous to Brans' cosmic shell. Its field contribution $a_s(r)$ is computed[9] analogously to the definition of a_0: the scalar sum of the inverse-squared mass contributions

$$a_s(r) = G\int_{r<r'<R}\frac{\rho(r')}{|\mathbf{r}-\mathbf{r}'|^2}d^3r' = \frac{2\pi G}{r}\int_r^R \rho(r')r'\ln\left(\frac{r+r'}{r'-r}\right)dr', \tag{5}$$

[9] We used a simplified equation for (5) as $a_s(r) = G\int_r^R dm/r'^2$ in the published version, integrating the mass density divided by the distance to the center of the cluster r'^2 instead of to the point of evaluation $|r-r'|^2$. While not significantly altering the results in comparison to those presented here, we thank Hongsheng Zhao for suggesting this change.

with R the virial radius, $\rho(r')$ the baryonic mass density of the cluster, so that $a_s \sim GM_s/R_s^2$. The virial radius marks the boundary of virial equilibrium. Although our models consider only the baryonic component of the cluster, we make use of the standard dark-matter-based virial radius, which serves as a convenient cutoff for integral and the definition of the cluster's total mass. Considering a larger bound than R in the integral leads to a very slight boost of $a_s(r)$ due to the $1/r^2$ dependence and the fact that those additional outer masses of the cluster lie very far from the center. For instance, for a greater bound of $4R$, the boost to $a_s(r)$ remains below 25% at the core of one of the toy models and only mildly increase $a_s(r)$ far away from the core (Figure 6). For even larger bounds, $a_s(r)$ converges, and such a boost can be considered contained in a_0 (3), since a boost to $a_s(r)$ is equivalent to a boost to a_0.

We choose to model a Coma-like, Virgo-like, and Fornax-like type clusters using a standard spherically symmetric β-model density profiles of baryonic matter $\rho(r) = \rho_0 \left[1 + (r/r_c)^2\right]^{-3\beta/2}$ with interior enclosed mass $M(< r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'$ and exterior mass $M(> r) = \int_r^R 4\pi r'^2 \rho(r')\, dr'$, Newtonian acceleration $a_N(r) = GM(< r)/r^2$, and $a_{\rm MOND} = \sqrt{a_N a_0}$ with $a_0 = 1.2 \times 10^{-10}$ m/s^2 as an approximation. While a multi-component mass model would be more accurate, a single standard β-model offers a good match for emission profiles of non-cool-core clusters with a flat surface brightness profile in their cores, such as the Coma cluster (Eckert, D. et al. 2012). For the cool-core clusters of Virgo and Fornax, a single β-model underestimates central density at $r < 300$ kpc and $r < 100$ kpc respectively, and their models serve as an order-of-magnitude consistency check only.

The adopted parameter values (Table 1) are chosen to be representative of low–redshift clusters and to lie within the observational ranges reported in large X–ray surveys of nearby systems. The parameter choices for a Coma-like cluster are guided by Briel et al. (1992) and typical intervals are $\beta \sim 0.60 - 0.80$, core radius $r_c \sim 0.25 - 0.40$ Mpc, central gas densities $\rho_0 \sim (5-7) \times 10^{-24}$ kg/m^{-3}, and virial radii $R \sim 2.6 - 3.0$ Mpc. For a Virgo-like cluster following Schindler et al. (1999), $\beta \sim 0.45 - 0.50$, $r_c \sim 0.05 - 0.1$ Mpc, $\rho_0 \sim (2-4)\times 10^{-24}$ kg/m^{-3}, and $R \sim 0.8-1.2$ Mpc. For a Fornax-like cluster (Machacek et al. 2005), $\beta \sim 0.40-0.55$, $r_c \sim 0.04-0.08$ Mpc, $\rho_0 \sim (2-4) \times 10^{-24}$ kg/m^{-3}, and $R \sim 0.5 - 1.0$ Mpc. The following selected values used for each cluster type fall within these envelopes and are intended to represent Coma–like, Virgo–like, and Fornax–like systems rather than their exact replicas:

Table 1: Parameters for the toy models of galaxy clusters.

Cluster type	β [−]	r_c [kpc]	ρ_0 [10^{-24} kg m^{-3}]	R [Mpc]	Baryonic mass [$10^{14} M_\odot$]
Coma–like	0.75	350	5.00	2.80	1.80
Virgo–like	0.50	80	3.75	1.00	0.10
Fornax–like	0.45	50	3.00	0.80	0.04

Across all three cluster type models, the proposed formulation (4) predicts accelerations that are notably higher than $a_{\rm MOND}$. Near their core, we generally find a boost in $a/a_{\rm MOND}$ of ~ 3 with an effective $a_0' \sim 10\, a_0$ at a distance of 0.1 Mpc from the center. At intermediate radii, we subsequently find a boost in $a/a_{\rm MOND}$ of ~ 1.5 with an effective $a_0' \sim 2\, a_0$. The estimated true acceleration reduces to $a_{\rm MOND}$ at their virial radius, as expected from the diminishing contribution of the cluster's outer shell mass with increasing distance from the core. Detailed data including $a/a_{\rm MOND}$ for different radii are provided as tables and plots in Appendix A.

3 DISCUSSION

The model described above essentially relies on the same characteristic that gives galaxy clusters their large gravitational potential depths (and inspired modifications to MOND in terms of these), which is their very large total mass distributed over an extended, relatively uniform volume. However, in contrast to potential-based modifications such as Bekenstein's scale-dependent critical acceleration and EMOND, our model addresses Milgrom's MOND cluster problem in terms of Newtonian accelerations and a directionless sum of inverse-square field intensities. While the EFE in MOND from an imposed external acceleration tends to 'Newtonianize' motion, our outer field dependence of a_0 can only enhance the MONDian effect (given that no shielding is possible).

We reproduce the required distance-from-the-core dependence of the MOND enhancement: strongest in the core and gradually diminishing with distance. In the central regions, the boost is somewhat smaller than needed, with 'true' accelerations on the order of a_0, rather than several times a_0. This may reflect the simplicity of the adopted density model for two reasons. Firstly, the core enhancement is particularly sensitive to the density–profile parameter β in our model, as well as the total baryonic mass estimation, and we use a very simple model for the entire cluster that is not accurate enough to describe the core. Secondly, we have assumed the simplest form of the interpolating function for the contribution of a_s, which could actually be steeper at the cores (the same happens for the chosen MOND interpolating function (1), which is required to be steeper to satisfy constraints in the Solar System). One may argue that an arbitrary complex function of measurable quantities is as unsatisfactory as a simpler function containing free parameters. However, since the required function appears to be already present in Milgrom's MOND in a similar form, we do not consider this to be a significant issue.

Given that the outer shell of clusters modifies the motion of galaxies inside clusters, we check whether such an effect is significant in the internal dynamics of galaxies themselves. This would entail that our model could also enhance rotation curves at low-acceleration regimes if $a_s \sim a_N$. In spiral galaxies, we consider a possible gentle outer shell effect from the galaxy itself because of their highly concentrated baryonic mass distribution. For a razor-thin exponential disk of the spiral Milky Way-like galaxy, $a_N \geq a_s$ holds for all regimes, and the cross-term $a_0 a_s/a_N$ is negligible. For inner radius $r \sim 5$ kpc corresponding to the Newtonian transition regime in Milgrom's MOND, $a_0' < 1.5 a_0$ and an effective boost to a_0' of $< 50\%$ is acceptable (Zhao & Famaey 2012). Such a boost can be accommodated by the uncertainty between the different proposed interpolating functions, the stellar population mass-to-light uncertainty estimations of about 10%, and by the uncertainty of $a_0 = \left(1.2^{+0.5}_{-0.3}\right) \times 10^{-10}$ m/s^2 (Zhao & Famaey 2012). For smaller radii, a stronger transition of the interpolating function can accommodate any deviations from Newtonian predictions except for the singularity at the very center. For mid and larger radii $r \geq 10$ kpc, $a_0' < 1.1 a_0$ and the boost can easily be accommodated by the uncertainties mentioned above. For $r \to \infty$, MOND is effectively recovered.

But the shell effect from the cluster still boosts the effective a_0' inside galaxies. Considering a rough estimate for the EFE as a sum of a_N of the cluster and a_N of the galaxy, we find that the effective a_0' for a rotation curve of a galaxy is only sensitive to the cluster's a_s at the core of the cluster, and only at the outer galactic radii of the rotation curve. Such an effect is also found in Bekenstein's scale dependent acceleration and EMOND, and recent studies suggest deviations from standard MOND predictions in galaxies at the cores of

clusters (Bílek et al. 2026). However, in contrast to these approaches, the effective boost of a_0' in our framework is partly suppressed by both the Newtonian acceleration a_N of the galaxy and the cluster when considering the internal dynamics of galaxies, whereas only the cluster's a_N suppresses a_0' in the dynamics of galaxies within the cluster. This effect could nevertheless help MOND in galaxies with more evenly distributed baryonic mass profiles, such as UDGs (Milgrom 2015).

We also acknowledge that the concept of mass exterior to a certain radius, differentiating between a_N and the cluster's shell a_s, is not well-defined except for the spherically symmetric case, even within an integral formulation. It must be replaced by another more general concept that plays the same role in producing an effectively varying a_0'. To address this issue, we may generalize (4) by interpreting that a_s only comes from the external shell because it is the total directionless sum of inverse-squared distance local mass contributions minus the modulus of the Newtonian acceleration. Introducing a new scalar matter field

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} d^3r', \tag{6}$$

$a_\phi = G\phi(\mathbf{r})$ such that $a_\phi \geq a_N$ and assuming linearity $a_0' = a_0 + a_s$, then $a_0' = a_\phi - a_N$[10] and MOND is generalized for all regimes as

$$\mathbf{a} \sim \mathbf{a_N}\sqrt{1+\frac{a_\phi - a_N}{a_N}}\,, \quad a \sim \sqrt{a_N a_\phi}\,, \tag{7}$$

where a_ϕ is the inverse-squared distance sum of total directionless field intensities $a_\phi \sim a_0 + a_s + a_N$, similar to a_N but without vector cancellations. This formulation also satisfies the desired conditions for an extension of MOND listed in § 2.2. For the Solar System, $a_\phi \sim a_N + a_0$ and we recover MOND's Newtonian regime, except for a possible but negligible boost to a_0' due to the cancellation of gravitational accelerations between planets. For galaxies, the more the difference between a_N and a_ϕ (i.e., the more cancellation of Newtonian fields), the bigger the departure expected from the Newtonian regime.

4 CONCLUSION

Milgrom's MOND, while resolving the need for physical dark matter in galaxies, falls short in explaining the mass discrepancies found in galaxy clusters, showing more disagreement at their cores than in their outer regions.

Machian MOND is not merely a reinterpretation of Milgrom's MOND with $a_0 \sim GM_u/R_u^2$, but it is also expected to feature a variable a_0 in response to local mass distributions, rather than being exclusively determined by the scalar sum of inverse-squared distance mass contributions of the universe at large. The term M_u/R_u^2 does not represent a field intensity or an acceleration, both of which vanish at the center of any spherically symmetric mass distribution, as the observable universe is to good approximation.

In analogy with Carl Brans' shell model of the universe with mass M_u and radius R_u, we idealize the spherically symmetric mass distribution in galaxy clusters exterior to a given radius from the core as a shell of M_s and R_s additional to the cosmic shell M_u and R_u. We then consider a variable a_0 in terms of the cluster's exterior shell $a_s \sim GM_s/R_s^2$, using the same interpolating function of Milgrom's MOND in (4), with an effective $a_0' = a_0 + a_s + a_0 a_s/a_N$. This parameter-free modification to Milgrom's MOND produces boosts of the same order as those typically required to resolve the mass discrepancies that MOND leaves across the cores and mid regions of galaxy clusters, depicted by simple spherically symmetric models of some clusters implementing a_0'. The result is a boost of $a_0' \sim 10\,a_0$ at the galaxy cluster cores, approaching standard MOND in the outer regions, and barely modifying Milgrom's MOND a_0 in galaxies except for the ones in the cluster's core and only at their outer radii. This matches the expected ten times larger $a_0' \sim 10a_0$ required in the cores of clusters, and particularly in the Coma cluster, the several tens times $a_0' \sim 30a_0$ (Milgrom 2015).

Even though we do not specify a precise mechanism for how shells affect local dynamics, we start to generalize the shell formulation by introducing a scalar field $\phi = \sum_i m_i/|\mathbf{r}-\mathbf{r_i}|^2$ as the directionless sum of inverse-squared distance mass contributions, $a_\phi = G\phi$, and set $a_0' = a_\phi - a_N$ in MOND's interpolating function converting from Newtonian accelerations. Then MOND is simply $a \sim \sqrt{a_N a_\phi}$ for all regimes. The field a_ϕ gives $3GM_u/R_u^2 \sim a_0$ when evaluated on the universe at large, and $\sim 3GM_u/R_u^2 + a_s + a_N$ when evaluated inside a local spherically symmetric mass distribution within the universe, such as inside a galaxy cluster. Under this generalization, the enhancement of a_0', which increases with greater directional cancellation of Newtonian field intensities, is smaller than that presented in the Appendix due to dropping nonlinearity of the cross-term $a_0 a_s/a_N$ of a_0' in (4). It is interesting to note that while the scalar field $\Phi = \sum_i m_i/|\mathbf{r}-\mathbf{r_i}|$ defines the gravitational constant $G = c^2/\Phi$ in Machian theories such as that of Reissner, Schrödinger, Sciama and Treder (1972), the field $\phi = \sum_i m_i/|\mathbf{r}-\mathbf{r_i}|^2$ defines the acceleration scale constant $a_0' = G\phi - a_N$ in Machian MOND. Especially when the deep-MOND limit is controlled only by $\mathcal{A}_0 = Ga_0$. At cluster scales, a direct test for the proposed model would be to search for an environmental dependence of the effective MOND acceleration scale in galaxies located in different cluster environments. An Earth-based experiment boosting a_0' appears impractical given the limited precision of current measurements of the gravitational constant, as it would require a Cavendish torsion balance inside a hollow sphere weighing over tens of tonnes.

In contrast to the extensions of MOND based on gravitational potential depths, our boost on scalar field intensities does not require tweaking to avoid high boosts at deep potential regimes such as neutron stars. This framework can set the basis for a new Lagrangian-based formulation of extended MOND with varying a_0 not based on gravitational potential depths.

It would be interesting to investigate if a_0 evolves over cosmological time, given that its definition depends on the mass enclosed M_u within the radius R_u of the observable universe, in a cosmological model without physical dark matter.

ACKNOWLEDGMENTS

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[10] Far outside an spherical mass distribution in the high acceleration regime, $a_\phi \approx a_N + a_0 \sim a_N$. At the center of a uniform solid sphere in the high acceleration regime, $a_\phi = 3GM/R^2 + a_0 \sim 3GM/R^2$ while $a_N = 0$. At any point inside a spherical mass distribution at the high acceleration regime, $a_\phi \geq a_N$ while they approach each other further away from the source.

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A DATA TABLES AND PLOTS

Table 2: Coma type cluster: radius in [Mpc], baryonic mass in [$10^{12} M_{\odot}$], accelerations in [10^{-10} m/s^2]

Radius	$M(<r)$	$M(>r)$	a_N	a_s	a'_0	a (4)	a/a_{MOND}
0.05	0.04	179.96	0.02	0.54	32.63	0.83	5.22
0.10	0.29	179.71	0.04	0.49	16.48	0.81	3.71
0.25	3.56	176.44	0.08	0.36	7.02	0.75	2.43
0.50	16.80	163.20	0.09	0.21	4.10	0.63	1.87
0.75	34.43	145.57	0.09	0.13	3.18	0.53	1.65
1.00	53.16	126.84	0.07	0.09	2.72	0.46	1.53
1.25	71.93	108.07	0.06	0.06	2.42	0.40	1.44
1.50	90.39	89.61	0.06	0.04	2.19	0.35	1.37
1.75	108.45	71.55	0.05	0.03	2.00	0.32	1.31
2.00	126.09	53.91	0.04	0.02	1.83	0.29	1.25
2.25	143.33	36.67	0.04	0.01	1.66	0.26	1.19
2.50	160.21	19.79	0.04	0.00	1.48	0.23	1.12
2.75	176.73	3.27	0.03	0.00	1.26	0.21	1.04
2.80	179.99	0.01	0.03	0.00	1.20	0.20	1.00

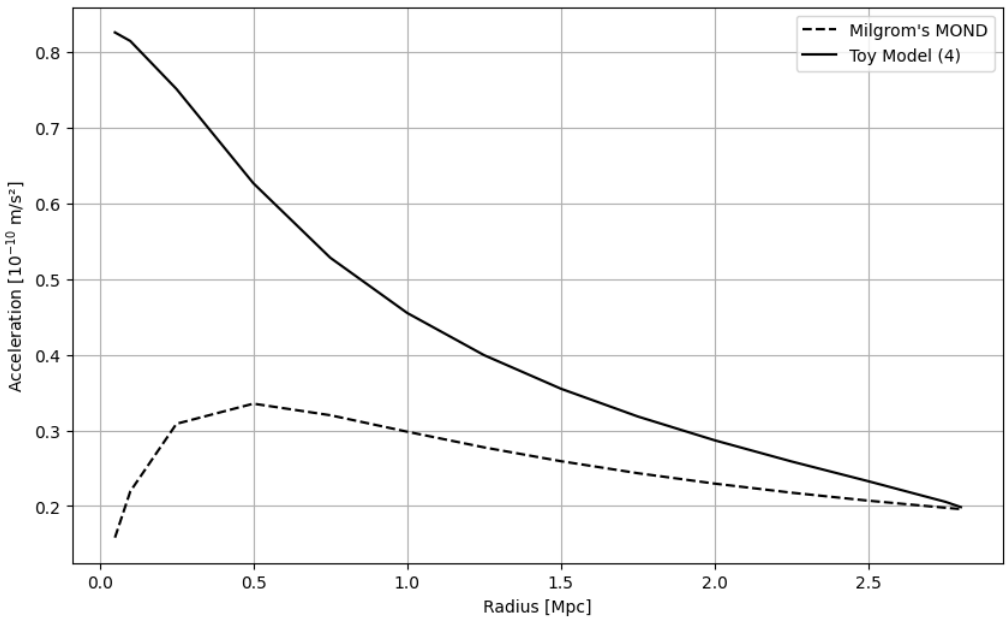

Figure 3: Plot for the Coma cluster toy model showing the acceleration predicted by MOND a_{MOND} (1) and by the proposed a (4).

Table 3: Virgo type cluster: radius in [Mpc], baryonic mass in [$10^{12} M_\odot$], accelerations in [10^{-10} m/s^2]

Radius	$M(<r)$	$M(>r)$	a_N	a_s	a_0'	a (4)	$a/a_{\rm MOND}$
0.05	0.02	9.98	0.01	0.13	12.40	0.41	3.22
0.10	0.15	9.85	0.02	0.10	7.12	0.38	2.44
0.25	0.98	9.02	0.02	0.05	4.17	0.30	1.87
0.50	3.29	6.71	0.02	0.03	2.87	0.23	1.55
0.75	6.35	3.65	0.02	0.01	2.07	0.18	1.32
1.00	10.00	0.01	0.01	0.00	1.20	0.13	1.00

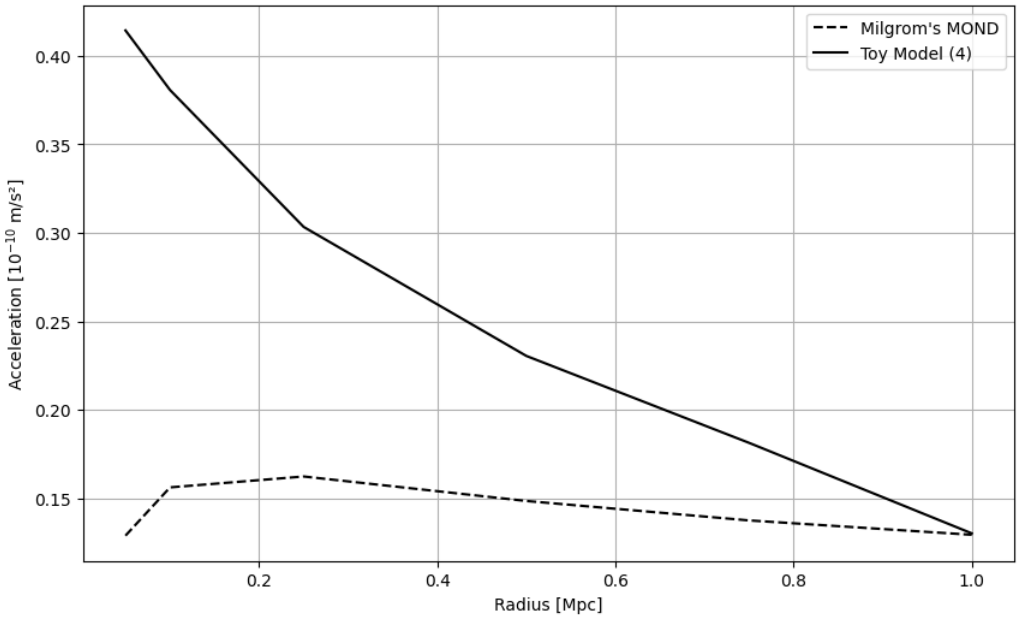

Figure 4: Plot for the Virgo cluster toy model showing the acceleration predicted by MOND $a_{\rm MOND}$ (1) and by the proposed a (4).

Table 4: Fornax type cluster: radius in [Mpc], baryonic mass in [$10^{12} M_\odot$], accelerations in [10^{-10} m/s^2]

Radius	$M(<r)$	$M(>r)$	a_N	a_s	a'_0	a (4)	a/a_{MOND}
0.05	0.02	3.98	0.01	0.07	9.87	0.31	2.87
0.10	0.09	3.91	0.01	0.05	6.24	0.28	2.28
0.25	0.53	3.47	0.01	0.03	3.94	0.22	1.81
0.50	1.80	2.20	0.01	0.01	2.56	0.16	1.46
0.75	3.59	0.41	0.01	0.00	1.50	0.12	1.12
0.80	4.00	0.01	0.01	0.00	1.20	0.10	1.00

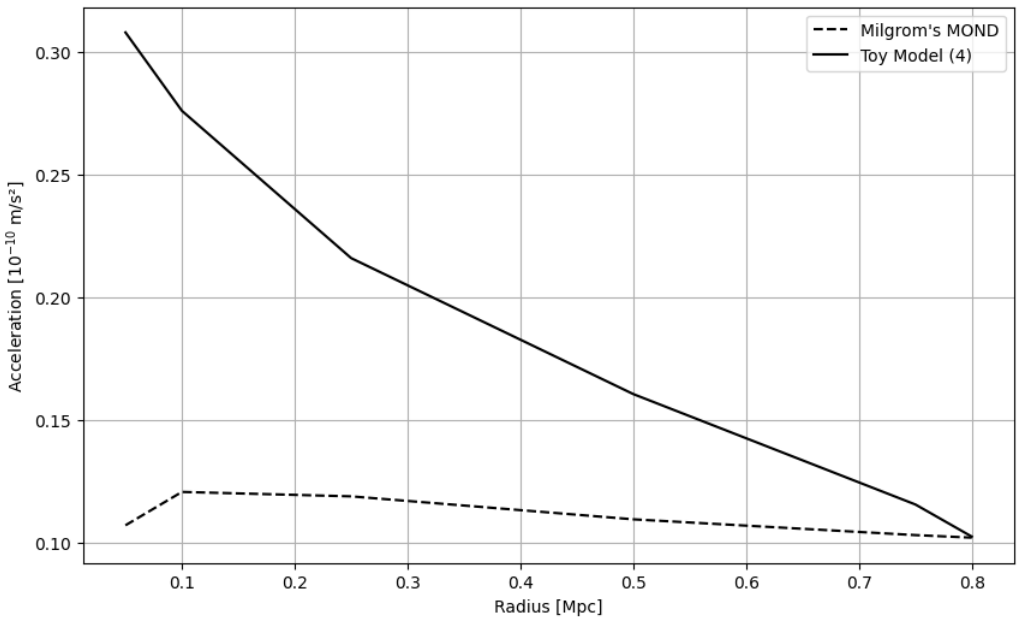

Figure 5: Plot for the Fornax cluster toy model showing the acceleration predicted by MOND a_{MOND} (1) and by the proposed a (4).

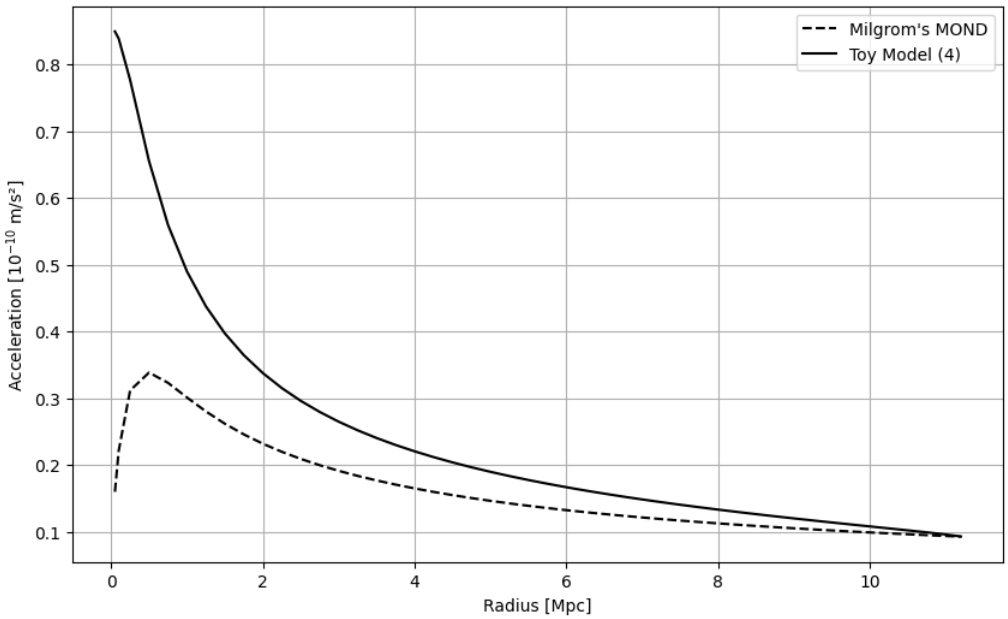

Figure 6: Plot for the Coma cluster toy model showing the acceleration predicted by MOND a_{MOND} (1) and by the proposed a (4) with a virial radius of $4R$ as the upper bound of the integral (5).