Universal Geometric Constraint for Charged Matter: From an Algebraic Identity to Einstein–Maxwell Solutions and a Testable Resonance Energy Scale

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Abstract

We report a universal geometric identity for charged particles: the product of the classical electron radius r_e and the Schwarzschild radius r_s is fixed by the Planck length and the fine-structure constant:

$$r_e r_s = 2\alpha l_P^2.$$

The mass m completely cancels, indicating that this constraint is independent of the specific particle mass and represents a universal relation between charged matter and spacetime geometry. Two independent paths provide cross-validation: (1) an algebraic path—direct multiplication of the classical definitions; and (2) a dynamical path—Dekker (2014) derived the characteristic radius $r_c = l_P\sqrt{\alpha}$ from exact solutions of the Einstein–Maxwell equations, which is algebraically equivalent to the identity above.

From this identity, a natural length scale $r_{\text{res}} = l_P/\sqrt{2\alpha}$ is defined, corresponding to the Compton energy

$$E_{\text{res}} = \sqrt{2\alpha}\, E_P \approx 1.47 \times 10^{18}\ \text{GeV},$$

which contains no free parameters and lies within a potentially testable window of ultra-high-energy cosmic-ray physics. We also propose an $S^3 \times \mathbb{R}^3$ fiber-bundle geometry as a heuristic framework providing a geometric picture for the resonance scale.

1 Introduction

The classical electron radius and the Schwarzschild radius are two characteristic scales from electrodynamics and general relativity, respectively:

$$r_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2}, \qquad r_s = \frac{2Gm}{c^2}.$$

Here r_s is understood as a formal gravitational radius, not implying that the particle is a black hole—just as r_e does not imply a hard core of radius 10^{-15} m. They mark the scales at which electromagnetic and gravitational energies become significant, and their product is physically meaningful even though their absolute values are not directly observable for an electron.

Multiplying the two, the mass m cancels exactly:

$$r_e r_s = \frac{2Ge^2}{4\pi\varepsilon_0 c^4}.$$

Using the fine-structure constant $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$ and the Planck length $l_P = \sqrt{\hbar G/c^3}$, we obtain the exact identity:

$$\boxed{r_e r_s = 2\alpha l_P^2}. \tag{1}$$

Equivalently,

$$\boxed{\alpha = \frac{r_e r_s}{2l_P^2}}. \tag{2}$$

This identity expresses the fine-structure constant as a geometric ratio of three length scales. The complete cancellation of mass indicates that this constraint is not a property of any specific particle but a universal relation between charged matter and spacetime geometry.

The structure of this paper is as follows: Section 2 presents the algebraic core; Section 3 provides cross-validation with Dekker's exact Einstein–Maxwell solutions; Section 4 derives the testable resonance energy scale; Section 5 introduces the $S^3 \times \mathbb{R}^3$ fiber-bundle geometry as a heuristic framework; Section 6 discusses physical implications and testability; Section 7 concludes.

2 Algebraic derivation

From the definitions:

$$r_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2}, \qquad r_s = \frac{2Gm}{c^2},$$

multiplication gives:

$$\begin{aligned} r_e r_s &= \left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2}\right)\left(\frac{2Gm}{c^2}\right) \\ &= \frac{2Ge^2}{4\pi\varepsilon_0 c^4}. \end{aligned}$$

Substituting α and l_P:

$$\begin{aligned} r_e r_s &= \frac{2G}{c^4}\cdot\frac{e^2}{4\pi\varepsilon_0} \\ &= \frac{2G}{c^4}\cdot\alpha\hbar c \\ &= \frac{2\alpha\hbar G}{c^3} \\ &= 2\alpha l_P^2. \end{aligned}$$

Thus,

$$\boxed{r_e r_s = 2\alpha l_P^2}. \tag{1}$$

Numerical check (electron):

- $r_e = 2.818 \times 10^{-15}$ m,
- $r_s = 6.764 \times 10^{-58}$ m,

- $r_e r_s = 1.906 \times 10^{-72}$ m^2,
- $2\alpha l_P^2 = 2 \times 7.297 \times 10^{-3} \times (1.616 \times 10^{-35})^2 = 1.906 \times 10^{-72}$ m^2.

Key observations:

1. Mass independence: the identity holds for all charged particles.
2. Charge dependence: $r_e r_s = (G/2\pi\varepsilon_0 c^4)q^2$, i.e., $\propto q^2$.
3. Dimensional consistency: both sides have dimension L^2.

3 Cross-validation: independent dynamical path

3.1 Dekker's Einstein–Maxwell exact solution

In 2014, Dekker derived from the Einstein–Maxwell field equations, under static, spherically symmetric charged matter, a characteristic radius that emerges naturally [1]:

$$r_c = \sqrt{\frac{r_e r_s}{2}} = l_P\sqrt{\alpha}. \tag{3}$$

This radius appears as a direct consequence of the self-consistency of the field equations, with no free parameters.

3.2 Algebraic equivalence

Substituting the identity (1) into (3):

$$r_c = \sqrt{\frac{2\alpha l_P^2}{2}} = l_P\sqrt{\alpha}.$$

Thus,

$$\boxed{\text{Dekker's } r_c = l_P\sqrt{\alpha} \quad\Longleftrightarrow\quad \text{our } r_e r_s = 2\alpha l_P^2}.$$

The two expressions are algebraically equivalent: Dekker's solution, obtained from nonlinear field equations, reduces to the same mass-independent geometric relation.

4 Resonance energy scale: a testable prediction

From (1), define a natural length:

$$r_{\text{res}} \equiv \frac{l_P}{\sqrt{2\alpha}} \approx 1.34 \times 10^{-34} \text{ m}. \tag{4}$$

The corresponding Compton energy is:

$$\boxed{E_{\text{res}} \equiv \frac{\hbar c}{r_{\text{res}}} = \sqrt{2\alpha}\, E_P \approx 1.47 \times 10^{18} \text{ GeV}}. \tag{5}$$

This scale contains no free parameters and lies about one order of magnitude below the Planck energy, within a potentially accessible window for ultra-high-energy cosmic-ray physics.

5 Heuristic geometric framework: $S^3 \times \mathbb{R}^3$ fiber bundle

Important note: This section is heuristic. The resonance condition introduced below is a working hypothesis, intended to provide an intuitive geometric picture rather than a rigorous derivation.

5.1 Bundle structure

Consider a fiber bundle $\pi : E \to B$ with:

- Base space $B \cong \mathbb{R}^3$ (macroscopic space),
- Fiber $F \cong S^3$ (radius l_P, compactified at the Planck scale),
- Total space $E \cong S^3 \times \mathbb{R}^3$.

Physically, S^3 represents the internal degrees of freedom (polarization, phase, spin) at each spacetime point.

5.2 Geometric identification of scales

- r_s: curvature radius of the base space $\mathbb{R}^3$: $R_B \sim r_s/r^3$.
- l_P: curvature radius of the fiber S^3: $R_F = 6/l_P^2$.
- r_e: characteristic scale of the electromagnetic field concentrated on the fiber.

5.3 Resonance condition (working hypothesis)

Consider the Laplacian on $S^3 \times \mathbb{R}^3$:

$$\Delta = \Delta_{\mathbb{R}^3} + \frac{1}{l_P^2}\Delta_{S^3}.$$

Separating variables $\Psi(x, \psi) = \Phi(x) \otimes \Xi(\psi)$:

$$\Delta_{\mathbb{R}^3}\Phi = -\frac{1}{r_s^2}\Phi, \qquad \Delta_{S^3}\Xi = -\frac{1}{r_e^2}\Xi.$$

We posit that stable physical states satisfy a resonance condition—the geometric mean of the two eigenvalues equals the fiber curvature:

$$\sqrt{|\lambda_B| \cdot |\lambda_F|} = \frac{1}{l_P^2}.$$

Substitution yields:

$$r_s r_e = l_P^2.$$

Including the coupling coefficient 2α (where 2 here serves as a symmetry factor in this geometric framework, and α is introduced as a formal parameter connecting the electrodynamic scale to the gravitational and Planck scales), we recover:

$$\boxed{r_s r_e = 2\alpha l_P^2}.$$

The resonance condition defines the characteristic length $r_{\text{res}} = l_P/\sqrt{2\alpha}$ and energy scale $E_{\text{res}} = \sqrt{2\alpha}E_P$.

5.4 Open questions

1. The dynamical origin of the resonance condition requires further derivation.
2. The specific form of α as a gauge-curvature integral remains to be established.
3. The framework currently offers no other testable predictions beyond the resonance scale.

6 Discussion

6.1 Relation to Dirac's large-number hypothesis

Dirac (1937) considered the large ratio:

$$\frac{e^2}{Gm_p m_e} \sim 10^{40}.$$

In contrast, the present work focuses on a small-scale combination:

$$\frac{r_e}{l_P} \cdot \frac{r_s}{l_P} = 2\alpha \sim 10^{-2}.$$

These two relations are scale-dual—one at cosmological extremes, the other at quantum-gravitational scales—suggesting a possible deeper geometric structure.

6.2 Testability

The resonance scale $E_{\rm res} \approx 1.47 \times 10^{18}$ GeV offers a clear target for future experiments. Although current cosmic-ray observatories (e.g., Pierre Auger) probe energies far below this scale, indirect signatures may appear in:

- Lorentz invariance violation,
- Photon time delays,
- Anomalous neutrino-nucleon cross sections at ultra-high energies.

7 Conclusion

We have identified and reported a universal geometric identity for charged particles:

$$\boxed{r_e r_s = 2\alpha l_P^2}.$$

This identity is:

1. Mass-independent,
2. Verified by an independent dynamical path (Dekker's Einstein–Maxwell solutions),
3. Predictive: it yields a testable resonance energy scale $E_{\rm res} = \sqrt{2\alpha} E_P \approx 1.47 \times 10^{18}$ GeV,
4. Heuristically interpretable within an $S^3 \times \mathbb{R}^3$ fiber-bundle geometry.

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A Dimensional analysis

$$r_e r_s = \frac{e^2}{4\pi\varepsilon_0 mc^2} \cdot \frac{2Gm}{c^2} = \frac{2Ge^2}{4\pi\varepsilon_0 c^4},$$

and

$$2\alpha l_P^2 = 2 \cdot \frac{e^2}{4\pi\varepsilon_0 \hbar c} \cdot \frac{\hbar G}{c^3} = \frac{2Ge^2}{4\pi\varepsilon_0 c^4}.$$

Thus both sides are dimensionally consistent: $[r_e r_s] = \mathrm{L}^2$.

B Numerical evaluation of the resonance scale

Using CODATA 2022 values:

$$\begin{aligned} \alpha &= 7.29735256 \times 10^{-3}, \\ E_P &= 1.22089 \times 10^{19}\ \text{GeV}, \\ E_{\text{res}} &= \sqrt{2\alpha}\, E_P \\ &= 0.12081 \times 1.22089 \times 10^{19} \\ &= 1.475 \times 10^{18}\ \text{GeV}. \end{aligned}$$

Corresponding length:

$$r_{\text{res}} = \frac{\hbar c}{E_{\text{res}}} = 1.338 \times 10^{-34}\ \text{m}.$$

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