Dirac Equation for $q\bar{q}$ in a SU(2) Confining Potential

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Abstract

A Dirac equation for the $q\bar{q}$ system is formulated in the presence of the confining potential derived from a static solution of the SU(2) Yang-Mills equations. The resulting radial equations define an eigenvalue problem with singular behavior at $r = 0$ and $r = \rho$, leading to a discrete spectrum of confined states. The system reduces to four sets of coupled radial equations for each of quark and antiquark components, which do not appear to admit closed-form solutions in terms of standard special functions. An approximate treatment is therefore developed to determine the corresponding energy spectrum. The calculated bound-state energies are found to be largely insensitive to the current quark masses and are of the same order as the observed meson masses, suggesting that the dominant contribution to the meson masses originates from the confining dynamics described by the present Yang-Mills solution. The quark and antiquark spectra exhibit an asymmetry associated with the gauge orientation of the underlying monopole-like configuration, although the individual energy eigenvalues are not gauge invariant observables.

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I. INTRODUCTION

Solutions of SU(2) Yang-Mills equations suggest the following confining potential [1] for a quark at $\vec{r}$ with charge $Q = \pm i\frac{2\pi}{g}$ by an anti-quark or a quark at $\vec{r} = 0$ with charge $Q_0 = \mp i\frac{2\pi}{g}$.

$$QA_0^{\pm} = -\frac{2\pi}{g^2}\frac{\rho}{r(r-\rho)}\frac{\tau}{2}\cdot\hat{r} = V(r)\tau\cdot\hat{r}, \tag{1}$$

$$Q\vec{A} = \mp i\frac{2\pi}{g^2}\frac{\rho}{r(r-\rho)}\hat{r}\times\frac{\tau}{2} = \pm iV(r)\hat{r}\times\tau, \tag{2}$$

where $V(r)$ is

$$V(r) = -\frac{\pi}{g^2}\frac{\rho}{r(r-\rho)} \tag{3}$$

$V(r)$ is depicted in Fig. 1. This clearly indicates that $q\bar{q}$ can be confined between two infinite barriers at $r = 0$ and $r = \rho$. With these scalar and vector potentials, we write the following Dirac equation.

$$H\Psi = \{\vec{\alpha}\cdot(\vec{p} - Q\vec{A}) + \beta\mu + QA_0\}\Psi = E\Psi, \tag{4}$$

where μ is the reduced mass of q and $\bar{q}$, with $\mu = m/2$ when both q and $\bar{q}$ have the same rest mass m.

A constant of motion of the above equation is

$$\vec{J} = \vec{L} + \vec{S} + \vec{T} = \vec{L} + \frac{\vec{\sigma}}{2} + \frac{\vec{\tau}}{2}, \tag{5}$$

where $\vec{S}$ and $\vec{T}$ are spin and isospin, respectively. Since $\vec{J}$ is a constant of motion, $\vec{J}^2$ is also a constant of motion.

Let us see if $\vec{J}$ commutes with H.

$$[H,\vec{J}] = [\{\vec{\alpha}\cdot\vec{p} - \vec{\alpha}\cdot Q\vec{A} + \beta\mu + QA_0\}, (\vec{L} + \frac{\vec{\sigma}}{2} + \frac{\vec{\tau}}{2})] \tag{6}$$

$$[\beta\mu, (\vec{L} + \frac{\vec{\sigma}}{2} + \frac{\vec{\tau}}{2})] = 0 \tag{7}$$

$$\begin{aligned}[H,\vec{L}] &= [(\vec{\alpha}\cdot\vec{p} - Q\vec{\alpha}\cdot\vec{A} + QA_0), \vec{L}] = i\vec{p}\times\vec{\alpha} \\ &\pm V(r)(-\vec{\alpha}(\vec{\tau}\cdot\hat{r}) + \vec{\tau}(\vec{\alpha}\cdot\hat{r})) + iV(r)\hat{r}\times\vec{\tau}\end{aligned} \tag{8}$$

$$[H, \frac{\vec{\sigma}}{2}] = [(\vec{\alpha}\cdot\vec{p} - \vec{\alpha}\cdot Q\vec{A}), \frac{\vec{\sigma}}{2}] = i\vec{\alpha}\times\vec{p} \pm V(r)(\hat{r}(\vec{\alpha}\cdot\vec{\tau}) - \vec{\tau}(\vec{\alpha}\cdot\hat{r})) \tag{9}$$

$$\begin{aligned}[H, \frac{\vec{\tau}}{2}] &= [(-Q\vec{\alpha}\cdot\vec{A} + QA_0), \frac{\vec{\tau}}{2}] \\ &= \pm V(r)(\vec{\alpha}(\vec{\tau}\cdot\hat{r}) - \hat{r}(\vec{\alpha}\cdot\vec{\tau})) - iV(r)\hat{r}\times\vec{\tau}\end{aligned} \tag{10}$$

Adding Eqs. (7, 8, 9, 10), we have:

$$[H, \vec{J}] = 0, \tag{11}$$

and, hence,

$$[H, \vec{J}^2] = 0 \tag{12}$$

II. DIRAC EQUATION

Let us start with $Q_0 = -i\frac{2\pi}{g}$ at $\vec{r}_2$ and $Q = i\frac{2\pi}{g}$ and $\vec{r}_1$. Then, the Dirac equation turns out to be

$$\{\vec{\alpha}\cdot\vec{p} - i\vec{\alpha}\cdot(V(r)\hat{r}\times\vec{\tau}) + \beta\mu - E + V(r)\vec{\tau}\cdot\hat{r}\}\Psi_{JM} = 0, \tag{13}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$. $\Psi_{JM}(r,\theta,\phi)$ is the four-component Dirac spinor along with the isospin which is an eigenstate of energy E, $\vec{J}^2$ and J_z. Since $\vec{J}$ is the sum of $\vec{L}$, $\frac{\vec{\sigma}}{2}$ and $\frac{\vec{\tau}}{2}$, there are three coupling schemes between them, that is, $<\theta,\phi|j_{LS},T;JM>$, $<\theta,\phi|j_{LT},S;JM>$ and $<\theta,\phi|j_{ST},L;JM>$. $<\theta,\phi|j_{LS},T;JM>$ is the coupling the orbital and spin first, j_{LS}, and then the isospin. Likewise, the other coupling schemes are constructed. After constructing them, we will calculate Eq. (13) term by term.

Let us construct $<\theta,\phi|j_{LS},T;JM>$ in the same way as the $\phi^{\ell}_{j,m}$ of a hydrogen atom [2], as follows.

$$<\theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM> = \Phi^I_J + \Phi^{II}_J + \Phi^{III}_J + \Phi^{IV}_J, \tag{14}$$

where

$$\begin{aligned}\Phi^I_J &= \sqrt{\frac{J-M+1}{2J+2}}\phi^{J+1(-)}_{J+\frac{1}{2},M-\frac{1}{2}}\xi_+ - \sqrt{\frac{J+M+1}{2J+2}}\phi^{J+1(-)}_{J+\frac{1}{2},M+\frac{1}{2}}\xi_-, \\ \Phi^{II}_J &= \sqrt{\frac{J+M}{2J}}\phi^{J-1(+)}_{J-\frac{1}{2},M-\frac{1}{2}}\xi_+ + \sqrt{\frac{J-M}{2J}}\phi^{J-1(+)}_{J-\frac{1}{2},M+\frac{1}{2}}\xi_-, \\ \Phi^{III}_J &= -\sqrt{\frac{J-M+1}{2J+2}}\phi^{J(+)}_{J+\frac{1}{2},M-\frac{1}{2}}\xi_+ + \sqrt{\frac{J+M+1}{2J+2}}\phi^{J(+)}_{J+\frac{1}{2},M+\frac{1}{2}}\xi_-, \\ \Phi^{IV}_J &= -\sqrt{\frac{J+M}{2J}}\phi^{J(-)}_{J-\frac{1}{2},M-\frac{1}{2}}\xi_+ - \sqrt{\frac{J-M}{2J}}\phi^{J(-)}_{J-\frac{1}{2},M+\frac{1}{2}}\xi_-\end{aligned} \tag{15}$$

In Eq. (14) ξ_+ and ξ_- are isospin up and down, respectively. Moreover, Φ_J^I and Φ_J^{III} are valid for $J \geq 0$, while Φ_J^{II} and Φ_J^{IV} are defined for $J \geq 1$. $\phi^{J\pm1(\mp)}_{J\pm\frac{1}{2},M-\frac{1}{2}}$ and $\phi^{J(\pm)}_{J\pm\frac{1}{2},M-\frac{1}{2}}$ are

$$
\phi^{J+1(-)}_{J+\frac{1}{2},M-\frac{1}{2}} = \begin{pmatrix} \sqrt{\frac{J-M+2}{2J+3}}Y_{J+1,M-1} \\ -\sqrt{\frac{J+M+1}{2J+3}}Y_{J+1,M} \end{pmatrix}, \quad \phi^{J-1(+)}_{J-\frac{1}{2},M-\frac{1}{2}} = \begin{pmatrix} \sqrt{\frac{J+M-1}{2J-1}}Y_{J-1,M-1} \\ \sqrt{\frac{J-M}{2J-1}}Y_{J-1,M} \end{pmatrix}
$$

$$
\phi^{J(+)}_{J+\frac{1}{2},M-\frac{1}{2}} = \begin{pmatrix} \sqrt{\frac{J+M}{2J+1}}Y_{J,M-1} \\ \sqrt{\frac{J-M+1}{2J+1}}Y_{J,M} \end{pmatrix}, \qquad \phi^{J(-)}_{J-\frac{1}{2},M-\frac{1}{2}} = \begin{pmatrix} \sqrt{\frac{J-M+1}{2J+1}}Y_{J,M-1} \\ -\sqrt{\frac{J+M}{2J+1}}Y_{J,M} \end{pmatrix} \tag{16}
$$

and so on [2]. They also satisfy

$$
\vec{\sigma}\cdot\hat{r}\phi^{J(+)}_{J+\frac{1}{2},M\mp\frac{1}{2}} = \phi^{J+1(-)}_{J+\frac{1}{2},M\mp\frac{1}{2}}, \quad \vec{\sigma}\cdot\hat{r}\phi^{J(-)}_{J-\frac{1}{2},M\mp\frac{1}{2}} = \phi^{J-1(+)}_{J-\frac{1}{2},M\mp\frac{1}{2}} \tag{17}
$$

Thus, Φ_J^I, Φ_J^{II}, Φ_J^{III}, and Φ_J^{IV} satisfy

$$
\vec{\sigma}\cdot\hat{r}\Phi_J^I = -\Phi_J^{III}, \quad \vec{\sigma}\cdot\hat{r}\Phi_J^{II} = -\Phi_J^{IV}, \tag{18}
$$

and, hence,

$$
\begin{aligned}
< \theta,\phi|\vec{\sigma}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM> &= -\Phi_J^{III} - \Phi_J^{IV} - \Phi_J^I - \Phi_J^{II} \\
&= - < \theta,\phi|j_{LS},T;JM>
\end{aligned} \tag{19}
$$

In analogy with the hydrogen atom [2], we write a four-component wave function Ψ_{JM} as a sum of four wave functions.

$$
\begin{aligned}
\Psi_{JM} &= \begin{pmatrix} i\frac{G_J^I}{r}(r)\Phi_J^I \\ \frac{F_J^I}{r}(r)\sigma\cdot\hat{r}\Phi_J^I \end{pmatrix} + \begin{pmatrix} i\frac{G_J^{II}}{r}(r)\Phi_J^{II} \\ \frac{F_J^I}{r}(r)\sigma\cdot\hat{r}\Phi_J^{II} \end{pmatrix} \\
&+ \begin{pmatrix} i\frac{G_J^{III}}{r}(r)\Phi_J^{III} \\ \frac{F_J^{III}}{r}(r)\sigma\cdot\hat{r}\Phi_J^{III} \end{pmatrix} + \begin{pmatrix} i\frac{G_J^{IV}}{r}(r)\Phi_J^{IV} \\ \frac{F_J^{IV}}{r}(r)\sigma\cdot\hat{r}\Phi_J^{IV} \end{pmatrix}
\end{aligned} \tag{20}
$$

Then, we calculate the upper and lower two components of the term of the Dirac equation Eq. (13), $\vec{\alpha}\cdot\vec{p}\Psi_{JM}$, as follows.

$$
\vec{\alpha}\cdot\vec{p}\Psi_{JM} = \begin{pmatrix} 0 & -i\sigma\cdot\hat{r}(\frac{\partial}{\partial r} - \frac{1}{r}\sigma\cdot\vec{L}) \\ -i\sigma\cdot\hat{r}(\frac{\partial}{\partial r} - \frac{1}{r}\sigma\cdot\vec{L}) & 0 \end{pmatrix}\Psi_{JM} \tag{21}
$$

Using the following relations:

$$
\begin{aligned}
&\sigma\cdot\vec{L}\Phi_J^I = -(J+2)\Phi_J^I, \quad \sigma\cdot\vec{L}\Phi_J^{II} = (J-1)\Phi_J^{II}, \\
&\sigma\cdot\vec{L}\Phi_J^{III} = J\Phi_J^{III}, \quad \sigma\cdot\vec{L}\Phi_J^{IV} = -(J+1)\Phi_J^{IV}, \\
&\sigma\cdot\vec{L}\sigma\cdot\hat{r}\Phi_J^I = J\sigma\cdot\hat{r}\Phi_J^I, \quad \sigma\cdot\vec{L}\sigma\cdot\hat{r}\Phi_J^{II} = -(J+1)\sigma\cdot\hat{r}\Phi_J^{II}, \\
&\sigma\cdot\vec{L}\sigma\cdot\hat{r}\Phi_J^{III} = -(J+2)\sigma\cdot\hat{r}\Phi_J^{III}, \quad \sigma\cdot\vec{L}\sigma\cdot\hat{r}\Phi_J^{IV} = (J-1)\sigma\cdot\hat{r}\Phi_J^{IV},
\end{aligned} \tag{22}
$$

we find that each of the upper two components of $\vec{\alpha}\cdot\vec{p}\Psi_{JM}$ is given by

$$\{\vec{\alpha}\cdot\vec{p}\Psi_{JM}\}^U = -i(\frac{d}{rdr}F_J^I - (J+1)\frac{1}{r^2}F_J^I)\Phi_J^I - i(\frac{d}{rdr}F_J^{II} + J\frac{1}{r^2}F_J^{II})\Phi_J^{II}$$
$$-i(\frac{d}{rdr}F_J^{III} + (J+1)\frac{1}{r^2}F_J^{III})\Phi_J^{III} - i(\frac{d}{rdr}F_J^{IV} - J\frac{1}{r^2}F_J^{IV})\Phi_J^{IV} \quad (23)$$

and each of the corresponding lower two components of $\vec{\alpha}\cdot\vec{p}\Psi_{JM}^{LS}$ is given by

$$\{\vec{\alpha}\cdot\vec{p}\Psi_{JM}\}^L = (\frac{d}{rdr}G_J^I + (J+1)\frac{1}{r^2}G_J^I)\vec{\sigma}\cdot\hat{r}\Phi_J^I + (\frac{d}{rdr}G_J^{II} - J\frac{1}{r^2}G_J^{II})\vec{\sigma}\cdot\hat{r}\Phi_J^{II}$$
$$+(\frac{d}{rdr}G_J^{III} - (J+1)\frac{1}{r^2}G_J^{III})\vec{\sigma}\cdot\hat{r}\Phi_J^{III} + (\frac{d}{rdr}G_J^{IV} + J\frac{1}{r^2}G_J^{IV})\vec{\sigma}\cdot\hat{r}\Phi_J^{IV} \quad (24)$$

For the term $QA_0 = V(r)\vec{\tau}\cdot\hat{r}$ in H of Eq. (4), we need to construct the coupling scheme $<\theta,\phi|j_{LT},S;JM>$. The structure of j_{LT} is exactly the same as that of j_{LS} if we replace two-component spinors ϕ's in Eq. (15) by two component isospinors ψ's. Let $\psi^{J\pm1(\mp)}_{J\pm\frac{1}{2},M\pm\frac{1}{2}}$ and $\psi^{J(\pm)}_{J\pm\frac{1}{2},M\pm\frac{1}{2}}$ be the two-component isospinors whose expressions are identical to those in Eq. (16) upon replacing the spinors with isospinors. The result is

$$<\theta,\phi|\sum_{j_{LT}}|j_{LT},S;JM> = \tilde{\Phi}_J^I + \tilde{\Phi}_J^{II} + \tilde{\Phi}_J^{III} + \tilde{\Phi}_J^{IV}, \quad (25)$$

where

$$\tilde{\Phi}_J^I = \sqrt{\frac{J-M+1}{2J+2}}\psi^{J+1(-)}_{J+\frac{1}{2},M-\frac{1}{2}}\chi_+ - \sqrt{\frac{J+M+1}{2J+2}}\psi^{J+1(-)}_{J+\frac{1}{2},M+\frac{1}{2}}\chi_-$$
$$\tilde{\Phi}_J^{II} = \sqrt{\frac{J+M}{2J}}\psi^{J-1(+)}_{J-\frac{1}{2},M-\frac{1}{2}}\chi_+ + \sqrt{\frac{J-M}{2J}}\psi^{J-1(+)}_{J-\frac{1}{2},M+\frac{1}{2}}\chi_-$$
$$\tilde{\Phi}_J^{III} = -\sqrt{\frac{J-M+1}{2J+2}}\psi^{J(+)}_{J+\frac{1}{2},M-\frac{1}{2}}\chi_+ + \sqrt{\frac{J+M+1}{2J+2}}\psi^{J(+)}_{J+\frac{1}{2},M+\frac{1}{2}}\chi_-$$
$$\tilde{\Phi}_J^{IV} = -\sqrt{\frac{J+M}{2J}}\psi^{J(-)}_{J-\frac{1}{2},M-\frac{1}{2}}\chi_+ - \sqrt{\frac{J-M}{2J}}\psi^{J(-)}_{J-\frac{1}{2},M+\frac{1}{2}}\chi_- \quad (26)$$

In Eq. (26) $\chi_\pm$ is spin up or down. Here again, $\tilde{\Phi}_J^I$ and $\tilde{\Phi}_J^{III}$ are valid for $J\geq 0$, while $\tilde{\Phi}_J^{II}$ and $\tilde{\Phi}_J^{IV}$ are defined for $J\geq 1$. $\psi^{J\pm1(\mp)}_{J\pm\frac{1}{2},M-\frac{1}{2}}$ and $\psi^{J(\pm)}_{J\pm\frac{1}{2},M-\frac{1}{2}}$ are

$$\psi^{J+1(-)}_{J+\frac{1}{2},M-\frac{1}{2}} = \begin{pmatrix} \sqrt{\frac{J-M+2}{2J+3}}Y_{J+1,M-1} \\ -\sqrt{\frac{J+M+1}{2J+3}}Y_{J+1,M} \end{pmatrix}_T$$
$$= \sqrt{\frac{J-M+2}{2J+3}}Y_{J+1,M-1}\xi_+ - \sqrt{\frac{J+M+1}{2J+3}}Y_{J+1,M}\xi_-$$
$$\psi^{J-1(+)}_{J-\frac{1}{2},M-\frac{1}{2}} = \begin{pmatrix} \sqrt{\frac{J+M-1}{2J-1}}Y_{J-1,M-1} \\ \sqrt{\frac{J-M}{2J-1}}Y_{J-1,M} \end{pmatrix}_T$$

$$
\begin{aligned}
&= \sqrt{\frac{J+M-1}{2J-1}} Y_{J-1,M-1}\xi_+ + \sqrt{\frac{J-M}{2J-1}} Y_{J-1,M}\xi_-, \\
\psi^{J(+)}_{J+\frac{1}{2},M-\frac{1}{2}} &= \begin{pmatrix} \sqrt{\frac{J+M}{2J+1}} Y_{J,M-1} \\ \sqrt{\frac{J-M+1}{2J+1}} Y_{J,M} \end{pmatrix}_T \\
&= \sqrt{\frac{J+M}{2J+1}} Y_{J,M-1}\xi_+ + \sqrt{\frac{J-M+1}{2J+1}} Y_{J,M}\xi_- \\
\psi^{J(-)}_{J-\frac{1}{2},M-\frac{1}{2}} &= \begin{pmatrix} \sqrt{\frac{J-M+1}{2J+1}} Y_{J,M-1} \\ -\sqrt{\frac{J+M}{2J+1}} Y_{J,M} \end{pmatrix}_T \\
&= \sqrt{\frac{J-M+1}{2J+1}} Y_{J,M-1}\xi_+ - \sqrt{\frac{J+M}{2J+1}} Y_{J,M}\xi_-, \qquad (27)
\end{aligned}
$$

and so on. Just like Eq. (17) they satisfy

$$
\vec{\tau}\cdot\hat{r}\psi^{J(+)}_{J+\frac{1}{2},M\mp\frac{1}{2}} = \psi^{J+1(-)}_{J+\frac{1}{2},M\mp\frac{1}{2}}, \quad \vec{\tau}\cdot\hat{r}\psi^{J(-)}_{J-\frac{1}{2},M\mp\frac{1}{2}} = \psi^{J-1(+)}_{J-\frac{1}{2},M\mp\frac{1}{2}} \qquad (28)
$$

With Eq. (27) we convert Eq. (26) into spinor times isospin $\xi_\pm$ as follows.

$$
\begin{aligned}
\tilde{\Phi}^I_J &= \sqrt{\frac{J-M+1}{2J+2}} \psi^{J+1(-)}_{J+\frac{1}{2},M-\frac{1}{2}}\chi_+ - \sqrt{\frac{J+M+1}{2J+2}} \psi^{J+1(-)}_{J+\frac{1}{2},M+\frac{1}{2}}\chi_- \\
&= \sqrt{\frac{J-M+1}{2J+2}} \phi^{J+1(-)}_{J+\frac{1}{2},M-\frac{1}{2}}\xi_+ - \sqrt{\frac{J+M+1}{2J+2}} \phi^{J+1(-)}_{J+\frac{1}{2},M+\frac{1}{2}}\xi_- = \Phi^I_J, \\
\tilde{\Phi}^{II}_J &= \sqrt{\frac{J+M}{2J}} \psi^{J-1(+)}_{J-\frac{1}{2},M-\frac{1}{2}}\chi_+ + \sqrt{\frac{J-M}{2J}} \psi^{J-1(+)}_{J-\frac{1}{2},M+\frac{1}{2}}\chi_- \\
&= \sqrt{\frac{J+M}{2J}} \phi^{J-1(+)}_{J-\frac{1}{2},M-\frac{1}{2}}\xi_+ + \sqrt{\frac{J-M}{2J}} \phi^{J-1(+)}_{J-\frac{1}{2},M+\frac{1}{2}}\xi_- = \Phi^{II}_J, \\
\tilde{\Phi}^{III}_J &= -\begin{pmatrix} \sqrt{\frac{(J-M+1)(J+M)}{(2J+2)(2J+1)}} Y_{J,M-1} \\ -\sqrt{\frac{(J+M+1)(J+M+1)}{(2J+2)(2J+1)}} Y_{J,M} \end{pmatrix} \xi_+ - \begin{pmatrix} \sqrt{\frac{(J-M+1)(J-M+1)}{(2J+2)(2J+1)}} Y_{J,M} \\ -\sqrt{\frac{(J+M+1)(J-M)}{(2J+2)(2J+1)}} Y_{J,M+1} \end{pmatrix} \xi_-, \\
\tilde{\Phi}^{IV}_J &= -\begin{pmatrix} \sqrt{\frac{(J+M)(J-M+1)}{(2J)(2J+1)}} Y_{J,M-1} \\ \sqrt{\frac{(J-M)(J-M)}{2J(2J+1)}} Y_{J,M} \end{pmatrix} \xi_+ + \begin{pmatrix} \sqrt{\frac{(J+M)(J+M)}{2J(2J+1)}} Y_{J,M} \\ \sqrt{\frac{(J-M)(J+M+1)}{2J(2J+1)}} Y_{J,M+1} \end{pmatrix} \xi_- \qquad (29)
\end{aligned}
$$

In analogy with Eq. (18), $\tilde{\Phi}^I_J = \Phi^I_J$, $\tilde{\Phi}^{II}_J = \Phi^{II}_J$, $\tilde{\Phi}^{III}_J$, and $\tilde{\Phi}^{IV}_J$ satisfy

$$
\vec{\tau}\cdot\hat{r}\Phi^I_J = -\tilde{\Phi}^{III}_J, \quad \vec{\tau}\cdot\hat{r}\Phi^{II}_J = -\tilde{\Phi}^{IV}_J \qquad (30)
$$

Thus, from Eqs. (26, 28, 29) we obtain:

$$
\begin{aligned}
<\theta,\phi|\vec{\tau}\cdot\hat{r}|\sum_{j_{LT}}|j_{LT},S;JM> &= -\tilde{\Phi}^{III}_J - \tilde{\Phi}^{IV}_J - \Phi^I_J - \Phi^{II}_J \\
&= -<\theta,\phi|\sum_{j_{LT}}|j_{LT},S;JM>, \qquad (31)
\end{aligned}
$$

In order to calculate the term $QA_0(\vec{r})\Psi_{J.M} = V(r)\vec{\tau}\cdot\hat{r}\Psi_{JM}$ in Eq. (13), we use the well known the vector-coupling coefficients, that is, Clebsh-Gordan coefficients [3].

$$
\begin{aligned}
&< \theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM> = < r,\theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM>< j_{LS},T;JM| \\
&\times\int d\Omega'|\theta',\phi'>< \theta',\phi'|\vec{\tau}\cdot\hat{r}\sum_{j_{LT}}|j_{LT},S;JM>< j_{LT},S;JM| \\
&\times\int d\Omega''|\theta'',\phi''>< \theta'',\phi''|\sum_{j_{LS}}|j_{LS},T;JM>
\end{aligned}
\tag{32}
$$

Here, it is interesting to note the following completeness relations:

$$
\begin{aligned}
&< \theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM>< j_{LS},T;JM|\theta',\phi'> \\
&= \Phi_J^I(\theta,\phi)\Phi_J^I(\theta',\phi')^+ + \Phi_J^{II}(\theta,\phi)\Phi_J^{II}(\theta',\phi')^+ \\
&+\Phi_J^{III}(\theta,\phi)\Phi_J^{III}(\theta',\phi')^+\Phi_J^{IV}(\theta,\phi)\Phi_J^{IV}(\theta',\phi')^+ = \delta(\theta-\theta')\delta(\phi-\phi') \\
&< \theta,\phi|\sum_{j_{LT}}|j_{LT},S;JM>< j_{LT},S;JM|\theta',\phi'> \\
&= \Phi_J^I(\theta,\phi)\Phi_J^I(\theta',\phi')^+ + \Phi_J^{II}(\theta,\phi)\Phi_J^{II}(\theta',\phi')^+ \\
&+\tilde{\Phi}_J^{III}(\theta,\phi)\tilde{\Phi}_J^{III}(\theta',\phi')^+\tilde{\Phi}_J^{IV}(\theta,\phi)\tilde{\Phi}_J^{IV}(\theta',\phi')^+ = \delta(\theta-\theta')\delta(\phi-\phi')
\end{aligned}
\tag{33}
$$

The detailed calculation for the term $QA_0(\vec{r})\Psi_{JM} = V(r)\vec{\tau}\cdot\hat{r}\Psi_{JM}$ is shown in Appendix A, which yields the following result.

$$
\begin{aligned}
QA_0\Psi_{JM} &= V(r)\vec{\tau}\cdot\hat{r}\Psi_{JM} = V(r)\begin{pmatrix}\vec{\tau}\cdot\hat{r} & 0 \\ 0 & \vec{\tau}\cdot\hat{r}\end{pmatrix}\Psi_{JM} \\
&= \frac{1}{2J+1}(1-\sqrt{2J(2J+2)})V(r)\begin{pmatrix} i\frac{G_J^I}{r}\Phi_J^I \\ \frac{F_J^I}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^I\end{pmatrix} \\
&-\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})V(r)\begin{pmatrix} i\frac{G_J^{II}}{r}\Phi_J^{II} \\ \frac{F_J^{II}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{II}\end{pmatrix} \\
&+\frac{1}{2J+1}(1-\sqrt{2J(2J+2)})V(r)\begin{pmatrix} i\frac{G_J^{III}}{r}\Phi_J^{III} \\ \frac{F_J^{III}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{III},\end{pmatrix} \\
&-\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})V(r)\begin{pmatrix} i\frac{G_J^{IV}}{r}\Phi_J^{IV} \\ \frac{F_J^{IV}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{IV}\end{pmatrix}
\end{aligned}
\tag{34}
$$

Calculating the term $\vec{\alpha}\cdot(-Q\vec{A})$ is somewhat subtle. In order to calculate the term $\vec{\alpha}\cdot(-Q\vec{A})\Psi_{JM}$ we have to construct $< \theta,\phi|j_{ST},L;JM>$.

$$
< \theta,\phi|\sum_{j_{ST}}|j_{ST},L;JM> = \delta_{j_{ST}=0}\bar{\Phi}_J^{IV} + \delta_{j_{ST}=1}\{\Phi_J^I + \Phi_J^{II} + \bar{\Phi}_J^{III}\},
\tag{35}
$$

where

$$\bar{\Phi}_J^{IV} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ Y_{J,M} \end{pmatrix}\xi_+ + \frac{1}{\sqrt{2}}\begin{pmatrix} Y_{J,M} \\ 0 \end{pmatrix}\xi_-,$$

$$\bar{\Phi}_J^{III} = \begin{pmatrix} \sqrt{\frac{(J-M+1)(J+M)}{2J(J+1)}}Y_{J,M-1} \\ -\frac{M}{\sqrt{2J(J+1)}}Y_{J,M} \end{pmatrix}\xi_+ - \begin{pmatrix} \frac{M}{\sqrt{2J(J+1)}}Y_{J,M} \\ \sqrt{\frac{(J+M+1)(J-M)}{2J(J+1)}}Y_{J,M+1} \end{pmatrix}\xi_- \tag{36}$$

Here again, we have another completeness relation:

$$\begin{aligned} &< \theta,\phi|\sum_{j_{ST}}|j_{ST},L;JM >< j_{ST},L;JM|\theta',\phi' >= \delta(\theta-\theta')\delta(\phi-\phi') \\ &= \delta_{j_{ST}=0}\bar{\Phi}_J^{IV}(\theta,\phi)\bar{\Phi}_J^{IV}(\theta',\phi')^+ + \delta_{j_{ST}=1}\{\Phi_J^{I}(\theta,\phi)\Phi_J^{I}(\theta',\phi')^+ \\ &+ \Phi_J^{II}(\theta,\phi)\Phi_J^{II}(\theta',\phi')^+ + \bar{\Phi}_J^{III}(\theta,\phi)\bar{\Phi}_J^{III}(\theta',\phi')^+\} \end{aligned} \tag{37}$$

We also note:

$$\vec{\sigma}\cdot\vec{\tau} = 2j_{ST}(j_{ST}+1) - 3 = -3\delta_{j_{ST}=0} + \delta_{j_{ST}=1} \tag{38}$$

$$\begin{aligned} \vec{\alpha}\cdot(-Q\vec{A}) &= -i\vec{\alpha}\cdot(V(r)\hat{r}\times\vec{\tau}) = \begin{pmatrix} 0 & -iV(r)\vec{\sigma}\cdot(\hat{r}\times\vec{\tau}) \\ -iV(r)\vec{\sigma}\cdot(\hat{r}\times\vec{\tau}) & 0 \end{pmatrix} \\ &= -iV(r)\begin{pmatrix} 0 & -i\vec{\tau}\cdot\hat{r} + i(\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ i\vec{\tau}\cdot\hat{r} - i\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix} \\ &= -V(r)\begin{pmatrix} 0 & \vec{\tau}\cdot\hat{r} \\ -\vec{\tau}\cdot\hat{r} & 0 \end{pmatrix} + V(r)\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix}, \end{aligned} \tag{39}$$

where we used the following relation:

$$\vec{\sigma}\cdot(\hat{r}\times\vec{\tau}) = -i\vec{\tau}\cdot\hat{r} + i(\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} = i\vec{\tau}\cdot\hat{r} - i\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) \tag{40}$$

Here, it is important to observe that Eq. (39) contains no factor of i. The presence of such a factor would make the energy eigenvalue non-real. With Eqs. (36-40) we calculate the term $\vec{\alpha}\cdot(-Q\vec{A})\Psi_{JM}$. the detailed calculation is shown in Appendix B. The result is:

$$\begin{aligned} \vec{\alpha}\cdot(-Q\vec{A})\Psi_{JM} &= -V(r)\begin{pmatrix} 0 & \vec{\tau}\cdot\hat{r} \\ -\vec{\tau}\cdot\hat{r} & 0 \end{pmatrix}\Psi_{JM} \\ &+ V(r)\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix}\Psi_{JM} \end{aligned}$$

$$
\begin{aligned}
&= [1 + \frac{1}{2J+1}(1 - \sqrt{2J(2J+2)})]V(r) \begin{pmatrix} i\frac{G_J^I}{r}\Phi_J^I \\ -\frac{F_J^I}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^I \end{pmatrix} \\
&+ [1 - \frac{1}{2J+1}(1 + \sqrt{2J(2J+2)})]V(r) \begin{pmatrix} i\frac{G_J^{II}}{r}\Phi_J^{II} \\ -\frac{F_J^{II}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{II} \end{pmatrix} \\
&+ [-1 - \frac{1}{2J+1}]V(r) \begin{pmatrix} i\frac{G_J^{III}}{r}\Phi_J^{III} \\ -\frac{F_J^{III}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{III} \end{pmatrix} \\
&+ [-1 + \frac{1}{2J+1}]V(r) \begin{pmatrix} i\frac{G_J^{IV}}{r}\Phi_J^{IV} \\ -\frac{F_J^{IV}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{IV} \end{pmatrix}
\end{aligned}
\tag{41}
$$

Adding all terms, the Dirac equation Eq. (13) is reduced to the following four sets of two radial equations.

$$
\begin{aligned}
&\frac{d}{dr}F_J^I - (J+1)\frac{1}{r}F_J^I - (\mu - E)G_J^I + A_J^I V(r) G_J^I = 0, \\
&\frac{d}{dr}G_J^I + (J+1)\frac{1}{r}G_J^I - (\mu + E)F_J^I + B_J^I V(r) F_J^I = 0,
\end{aligned}
\tag{42}
$$

where

$$
\begin{aligned}
&A_J^I = -1 - \frac{2}{2J+1}(1 - \sqrt{2J(2J+2)}), \\
&B_J^I = -1
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&\frac{d}{dr}F_J^{II} + J\frac{1}{r}F_J^{II} - (\mu - E)G_J^{II} + A_J^{II} V(r) G_J^{II} = 0, \\
&\frac{d}{dr}G_J^{II} - J\frac{1}{r}G_J^{II} - (\mu + E)F_J^{II} + B_J^{II} V(r) F_J^{II} = 0,
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
&A_J^{II} = -1 + \frac{2}{2J+1}(1 + \sqrt{2J(2J+2)}), \\
&B_J^{II} = -1
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&\frac{d}{dr}F_J^{III} + (J+1)\frac{1}{r}F_J^{III} - (\mu - E)G_J^{III} + A_J^{III} V(r) G_J^{III} = 0, \\
&\frac{d}{dr}G_J^{III} - (J+1)\frac{1}{r}G_J^{III} - (\mu + E)F_J^{III} + B_J^{III} V(r) F_J^{III} = 0,
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
&A_J^{III} = 1 + \frac{1}{2J+1}\sqrt{2J(2J+2)}, \\
&B_J^{III} = 1 + \frac{1}{2J+1}(2 - \sqrt{2J(2J+2)})
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&\frac{d}{dr}F_J^{IV} - J\frac{1}{r}F_J^{IV} - (\mu - E)G_J^{IV} + A_J^{IV}V(r)G_J^{IV} = 0,\\
&\frac{d}{dr}G_J^{IV} + J\frac{1}{r}G_J^{IV} - (\mu + E)F_J^{IV} + B_J^{IV}V(r)F_J^{IV} = 0,
\end{aligned} \tag{48}
$$

where

$$
\begin{aligned}
A_J^{IV} &= 1 + \frac{1}{2J+1}\sqrt{2J(2J+2)},\\
B_J^{IV} &= 1 - \frac{1}{2J+1}(2 + \sqrt{2J(2J+2)})
\end{aligned} \tag{49}
$$

Now, let us write the Dirac equation for $\bar{q}$ with charge $Q_0 = -i\frac{2\pi}{g}$ at $\vec{r}_2$ in the potential due to q with charge $Q = i\frac{2\pi}{g}$ at $\vec{r}_1$. The reason why we have to do this is because there is an asymmetry arising from the interaction between a charge and the monopole configuration.

$$
\begin{aligned}
&\{\vec{\alpha}\cdot(\vec{p'} - Q_0\vec{A}(\vec{r'}) + \beta\mu - E + Q_0 A_0(\vec{r'})\}\Psi'_{JM}(\vec{r'}) = 0\\
&\{\vec{\alpha}\cdot\vec{p'} + i\vec{\alpha}\cdot(V(r')\hat{r'}\times\vec{\tau}) + \beta\mu - E + V(r')\vec{\tau}\cdot\hat{r'}\}\Psi'_{JM}(\vec{r'}) = 0,
\end{aligned} \tag{50}
$$

where $\vec{r'} = \vec{r}_2 - \vec{r}_1$.

As we can see in the above equation, the interaction of the vector potential $\vec{A}(\vec{r'})$ with charge $Q_0 = -i\frac{2\pi}{g}$ at $\vec{r'} = \vec{r}_2 - \vec{r}_1 = -\vec{r}$ is different from that in Eq. (13). The Dirac equation Eq. (50) yields another four sets of two radial equations, which are identical to Eq. (13) except for A_J's and B_J's.

$$
\begin{aligned}
&\frac{d}{dr'}{F'}_J^I - (J+1)\frac{1}{r'}{F'}_J^I - (\mu - E){G'}_J^I + {A'}_J^I V(r'){G'}_J^I = 0,\\
&\frac{d}{dr'}{G'}_J^I + (J+1)\frac{1}{r'}{G'}_J^I - (\mu + E){F'}_J^I + {B'}_J^I V(r'){F'}_J^I = 0,
\end{aligned} \tag{51}
$$

where

$$
\begin{aligned}
{A'}_J^I &= 1,\\
{B'}_J^I &= 1 + \frac{2}{2J+1}(1 - \sqrt{2J(2J+2)})
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&\frac{d}{dr'}{F'}_J^{II} + J\frac{1}{r'}{F'}_J^{II} - (\mu - E){G'}_J^{II} + {A'}_J^{II}V(r'){G'}_J^{II} = 0,\\
&\frac{d}{dr'}{G'}_J^{II} - J\frac{1}{r'}{G'}_J^{II} - (\mu + E){F'}_J^{II} + {B'}_J^{II}V(r'){F'}_J^{II} = 0,
\end{aligned} \tag{53}
$$

where

$$
\begin{aligned}
A'^{II}_J &= 1, \\
B'^{II}_J &= 1 - \frac{2}{2J+1}(1 + \sqrt{2J(2J+2)})
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
&\frac{d}{dr'}F'^{III}_J + (J+1)\frac{1}{r'}F'^{III}_J - (\mu - E)G'^{III}_J + A'^{III}_J V(r')G'^{III}_J = 0, \\
&\frac{d}{dr'}G'^{III}_J - (J+1)\frac{1}{r'}G'^{III}_J - (\mu + E)F'^{III}_J + B^{III}_J V(r')F'^{III}_J = 0,
\end{aligned}
\tag{55}
$$

where

$$
\begin{aligned}
A'^{III}_J &= -1 - \frac{1}{2J+1}(2 - \sqrt{2J(2J+2)}), \\
B'^{III}_J &= -1 - \frac{1}{2J+1}\sqrt{2J(2J+2)}
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
&\frac{d}{dr'}F'^{IV}_J - J\frac{1}{r'}F'^{IV}_J - (\mu - E)G'^{IV}_J + A'^{IV}_J V(r')G'^{IV}_J = 0, \\
&\frac{d}{dr'}G'J^{IV} + J\frac{1}{r'}G'^{IV}_J - (\mu + E)F'^{IV}_J + B'^{IV}_J V(r')F'^{IV}_J = 0,
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
A'^{IV}_J &= -1 + \frac{1}{2J+1}(2 + \sqrt{2J(2J+2)}), \\
B'^{IV}_J &= -1 - \frac{1}{2J+1}\sqrt{2J(2J+2)}
\end{aligned}
\tag{58}
$$

III. SOLVING RADIAL EQUATIONS

Let us try to solve Eq. (13). The potential $V(r) = -\frac{\pi}{g^2}\frac{\rho}{r(r-\rho)}$ given by Eq. (3) diverges at $r = 0$ and $r = \rho$, leading to a discrete eigenvalue problem describing a particle confined between two infinite walls. We rewrite the first equation of Eq. (13) by changing variable r to $x = r/\rho$ as

$$
\begin{aligned}
&\frac{d}{dx}F - (J+1)\frac{1}{x}F + [(E-\mu)\rho + \frac{\pi}{g^2}A_J\frac{1}{x(x-1)}]G = 0 \\
&\frac{d}{dx}G + (J+1)\frac{1}{x}G - [(E+\mu)\rho - \frac{\pi}{g^2}B_J\frac{1}{x(x-1)}]F = 0
\end{aligned}
\tag{59}
$$

Naturally, in analogy with the hydrogen atom, we may try:

$$
F(x) = e^{\gamma x}x^s\sum_{n=0} a_n x^n, \qquad G(x) = e^{\gamma x}x^s\sum_{n=0} b_n x^n
\tag{60}
$$

These expansions lead us three-term recurrence relation for a_n and b_n. To our best knowledge, no exact solution in closed form is known for this kind set of equations. If we try to make Eq. (59) a second order differential equation for $F(x)$ by eliminating $G(x)$, we have:

$$P_5(x)\frac{d^2}{dx^2}y + P_2(x)\frac{d}{dx}y + P_6(x)y = 0, \tag{61}$$

where $F(x) = x^{J+1}y(x)$ and each $P_i(x)$ is a polynomial of degree i. This equation does not belong to any known differential equation such as the Heun differential equation [4] which is a natural extension of Riemann hypergeometric differential equation [5]. This confirms that no exact solution in closed form is available.

Since no closed-form solution in terms of standard special functions is known yet, the energy eigenvalues will be obtained approximately. Inspection of Fig. 1 shows that the potential $V(r)$ does not vary rapidly in the interior region and is symmetric about $r = \rho/2$. Moreover, it diverges very rapidly to infinity at end points $r = 0$ and $r = \rho$. Therefore, it is reasonable to approximate it by a constant equal to its value $\kappa = 4\pi/(g^2\rho)$ at midpoint. This approximation captures the leading contribution to the discrete spectrum, which is governed mainly by the infinite barriers for the eigenvalue spectrum. The approximate square well potential is also depicted in Fig. 1 as the horizontal line segment κ.

Now, we rewrite the approximate equation for Eq. (42) by replacing $V(r)$ with $4\pi/(g^2\rho)$ as follows.

$$\begin{aligned}
&\frac{d}{dr}F - (J+1)\frac{1}{r} - (\mu - E)G + \frac{4\pi}{g^2\rho}A_J G = 0, \\
&\frac{d}{dr}G + (J+1)\frac{1}{r}G - (\mu + E)F + \frac{4\pi}{g^2\rho}B_J F = 0,
\end{aligned} \tag{62}$$

The boundary condition we impose is either $F(0) = 0, F(\rho) = 0$ or $G(0) = 0, G(\rho) = 0$ but not both. This is because $F(r) = 0$ and $G(r) = 0$ depend on each other. If they were entirely independent of one another, we could impose the boundary condition on both. Obviously, the eigenvalues obtained from the boundary condition $F(0) = 0, F(\rho) = 0$ are different from those obtained from the boundary condition $G(0) = 0, G(\rho) = 0$. As we can see later, each eigenvalue is pretty close to the other one obtained from the other boundary condition. This property inherits from the Dirac equation for a confining potential. In the case of hydrogen atom, this kind neighboring eigenvalues do not arise because the electron is not confined in a finite region. This property has nothing to do with our approximate treatment of the confining potential. Even if we know exact analytic solution of Eq. (59), The boundary condition should be imposed either on $G(x)$ or $F(x)$.

By eliminating G in Eq. (62), we have a second order differential equation for $F(r)$.

$$\frac{d^2}{dr^2}F - \frac{J(J+1)}{r^2}F + \mathcal{E}^2 F = 0, \tag{63}$$

where $\mathcal{E}^2 = [E - \mu + (\frac{4\pi}{g^2\rho})A_J][E + \mu - (\frac{4\pi}{g^2\rho})B_J]$. Let $\mathcal{E} = k$, $x = kr$ and $F(r) = r^{1/2}U(x) = r^{1/2}U(kr)$. Then, Eq. (63) turns out to be:

$$\frac{d^2}{dx^2}U(x) + \frac{1}{x}\frac{d}{dx}U(x) + (1 - \frac{\nu^2}{x^2})U(x) = 0, \tag{64}$$

where $\nu = J + \frac{1}{2}$. The solution of above equation is very well known.

$$U(x) = U(kr) \sim \sqrt{r}j_J(kr), \tag{65}$$

and, hence,

$$F(r) = N_I r j_J(kr) \tag{66}$$

Here, $j_J(kr)$ is a spherical Bessel function. From Eq. (62), $G(r)$ is obtained as

$$G(r) = N_I(\frac{k}{[E - \mu + (\frac{4\pi}{g^2\rho})A_J]})r j_{J+1}(kr) \tag{67}$$

where N_I is the normalization constant. By imposing the boundary condition on either $F(r)$, we have energy eigenvalues. The boundary condition for $F^I_J(r)$ at $r = \rho$, that is, $j_J(k\rho) = 0$ yields discrete k-values. $k\rho = k_{J,n}$'s for $j_J(k_{J,n}) = 0$ are well known. For instance, $k_{0,n} = n\pi$, $k_{1,1} = 4.4934$, $k_{1,2} = 7.7252$, $k_{2,1} = 5.7634$, $k_{2,2} = 9.0951$, and so on. If we eliminate F instead of G in Eq. (62), we obtain the same solution as Eqs. (66, 67) but with different k values. Thus, we have neighboring solutions and eigenvalues.

The boundary condition on either $G(r)$ or $F(r)$ yields $k\rho = k_{J,n}$ which are the values for $j_J(k_{J+1,n}) = 0$ or $j_J(k_{J,n}) = 0$. With those $k = \frac{k_{J,n}}{\rho}$ values the energy eigenvalues are determined by $\mathcal{E}^2 = k^2$

$$\mathcal{E}^2 - \frac{k_{J,n}^2}{\rho^2} = [E_{J,n} - \mu + (\frac{4\pi}{g^2\rho})A_J][E_{J,n} + \mu - (\frac{4\pi}{g^2\rho})] - \frac{k_{J,n}^2}{\rho^2} = 0, \tag{68}$$

which leads

$$E_{J,n} = \frac{1}{2}\frac{4\pi}{g^2\rho}(B_J - A_J) + \sqrt{[\mu - \frac{1}{2}\frac{4\pi}{g^2\rho}(A_J + B_J)]^2 + (\frac{k_{J,n}}{\rho})^2} \tag{69}$$

This eigenvalue is obtained by solving the quadratic equation Eq. (68). Since we want the energy eigenvalue to be positive, we choose only positive sign in front of the square root. Here, we note that there are infinitely many eigenvalues for a given J. Moreover, as mentioned before, for each eigenvalue there is neighboring eigenvalue. This is a general property for eigenvalues of a particle confined in a region. As stated before, if we require that the boundary conditions be imposed on both $F(r)$ and $G(r)$, we must have $j_J(k_{J,n}) = 0$ and $j_{J+1}(k_{J,n}) = 0$ simultaneously, which is impossible to meet.

The approximate four-component wave function is

$$\Psi_{JM} = \Psi^{I}_{JM} + \Psi^{II}_{JM} + \Psi^{III}_{JM} + \Psi^{IV}_{JM}, \tag{70}$$

where

$$\begin{aligned}
\Psi^{I}_{JM} &= N_I \begin{pmatrix} (\frac{k}{[E-\mu+(\frac{4\pi}{g^2\rho})A^I_J]}) i r j_{J+1}(kr)\Phi^I_J \\ r j_J(kr)\sigma\cdot\hat{r}\Phi^I_J \end{pmatrix}, \\
\Psi^{II}_{JM} &= N_{II} \begin{pmatrix} (\frac{k}{[E-\mu+(\frac{4\pi}{g^2\rho})A^{II}_J]}) i r j_{J-1}(kr)\Phi^{II}_J \\ -r j_J(kr)\sigma\cdot\hat{r}\Phi^{II}_J \end{pmatrix}, \\
\Psi^{III}_{JM} &= N_{III} \begin{pmatrix} (\frac{k}{[E-\mu+(\frac{4\pi}{g^2\rho})A^{III}_J]}) i r j_{J}(kr)\Phi^{III}_J \\ -r j_{J+1}(kr)\sigma\cdot\hat{r}\Phi^{III}_J \end{pmatrix}, \\
\Psi^{IV}_{JM} &= N_{IV} \begin{pmatrix} (\frac{k}{[E-\mu+(\frac{4\pi}{g^2\rho})A^{IV}_J]}) i r j_{J}(kr)\Phi^{IV}_J \\ -r j_{J-1}(kr)\sigma\cdot\hat{r}\Phi^{IV}_J \end{pmatrix}
\end{aligned} \tag{71}$$

In the above solutions, there are corresponding neighboring solutions which have different k values.

The procedure for solving other radial equations are the same as that for Eq. (42). Thus, for the solution of Eq. (50), the radial equations given by Eqs. (51-58) have exactly same structures as Eqs. (42-49), and hence, their solutions have the same expressions identical to those in Eq. (71) upon replacing A_J's, B_J's, and so on, with the corresponding primed A'_J's, B'_J's, and so on. The energy eigenvalues have also the same expression as Eq. (69) only with different A'_J's and B'_J's, that is,

$$E'_{J,n} = \frac{1}{2}\frac{4\pi}{g^2\rho}(B'_J - A'_J) + \sqrt{[\mu - \frac{1}{2}\frac{4\pi}{g^2\rho}(A'_J + B'_J)]^2 + (\frac{k'_{J,n}}{\rho})^2} \tag{72}$$

IV. DISCUSSIONS

As we have already mentioned, the interaction between a charge $i\frac{2\pi}{g}$ with the magnetic charge $-\frac{2\pi}{g}$ is different from the interaction between a charge $-i\frac{2\pi}{g}$ and the same magnetic charge. This results in different energy eigenvalues given by Eq. (69) and Eq. (72). Even though this is an expected asymmetry, it is still a little hard to imagine two interacting particles in their ground states with the same rest mass have different energies. Let us put it another way. A quark with isospin up interacts with another quark with isospin down. The charge of isospin up particle is $-i\frac{2\pi}{g}$ and the other particle with isospin down has a charge $i\frac{2\pi}{g}$. Still, they have different energies. The isospin up or down is our choice, which should not affect physics. Nevertheless, this asymmetry arises from the interaction between non-Abelian charges and the magnetic charge, and is therefore unavoidable.

The energy eigenvalues given by Eqs. (69) and (72) include the rest mass energy of each q and $\bar{q}$. Therefore, to obtain the energy arising purely from the confining dynamics, we must subtract the corresponding rest mass contribution. By analogy with the case of positronium, we then obtain the following expression for the total mass of each confined particle. Here again, it should be noted that the energies of q and $\bar{q}$ are not identical. this asymmetry originates from the monoploe contribution to the confining gauge field. Thus, the masses of q and $\bar{q}$ must be evaluated separately before obtaining the total mass of $q\bar{q}$ system.

$$\begin{aligned}\mathcal{M}_{J,n} &= m + (E_{J,n} - \mu) \\ &= \frac{m}{2} + \frac{1}{2}\frac{4\pi}{g^2\rho}(B_J - A_J) + \sqrt{[\frac{m}{2} - \frac{1}{2}\frac{4\pi}{g^2\rho}(A_J + B_J)]^2 + (\frac{k_{J,n}}{\rho})^2}\end{aligned} \tag{73}$$

and

$$\begin{aligned}\mathcal{M}'_{J,n} &= m + (E'_{J,n} - \mu) \\ &= \frac{m}{2} + \frac{1}{2}\frac{4\pi}{g^2\rho}(B'_J - A'_J) + \sqrt{[\frac{m}{2} - \frac{1}{2}\frac{4\pi}{g^2\rho}(A'_J + B'_J)]^2 + (\frac{k'_{J,n}}{\rho})^2}\end{aligned} \tag{74}$$

The total mass of $q\bar{q}$ is then obtained by adding these two masses.

To compute energy eigenvalues of Ψ_{JM}'s and Ψ'_{JM}'s, that is, $E_{J,n}$ and $E'_{J,n}$, we have to know the coupling constant $\alpha_S = \frac{g^2}{4\pi}$ and the size of a $q\bar{q}$ confined state, that is, a meson. We consider the radius of a nucleon as an approximate radius of a confining sphere, ~ 1 fm. In nuclear physics, the radius of a nucleon is about 1.2 fm $\sim$ 1.5 fm. Thus, for the maximum distance between q and $\bar{q}$, that is, the diameter of a confining sphere, ρ in Eqs.

(69) and (72), we take it to be 2 fm. We also have to know the mass of a quark, which is believed to be 1 MeV $\sim$ 9 MeV for light quarks, u- and d- quarks, [6]. We take the quark mass to be 5 MeV. For the strong coupling constant $\alpha_S = \frac{g^2}{4\pi}$, we take 0.1 $\sim$ 1.1 [7]. Energy eigenvalues are computed for the lowest values of J, namely $J = 0$ and $J = 1$, with $n = 1$ and $n = 2$. Then, we compute each mass $\mathcal{M}_{J,n}$ and $\mathcal{M}'_{J,n}$ and are listed in Tables I and II only for $n = 1$. They are computed by imposing the boundary condition on $G(r)$ or $F(r)$, that is, $G(\rho) = 0$ or $F(\rho) = 0$. We note that the values in Tables I and II are pretty close to each other.

In Tables I and II, some of energy eigenvalues are negative. Since we want only positive eigenvalues, we discard them, which means $\alpha_S = \frac{g^2}{4\pi} \gtrsim 0.5$. Let us take $\alpha_S = \frac{g^2}{4\pi}$ to be 0.5 for a phenomenological reason. Then, a mass of $q\bar{q}$, presumably, the sum of masses of each quark, that is, $\mathcal{M}_{J,n}$ and $\mathcal{M}'_{J,n}$, yields $\mathcal{M}^{II}_{J=1,n=1} + \mathcal{M}'^{II}_{J=1,n=1} = 64.9 + 65.7 = 130.6$ MeV in Table I, and $\mathcal{M}^{IV}_{J=1,n=1} + \mathcal{M}'^{IV}_{J=1,n=1} = 86.3 + 88.2 = 174.5$ MeV in Table II. These values are close to pion masses 135 $\sim$ 140 MeV. It also yields $\mathcal{M}^{IV}_{J=1,n=1} + \mathcal{M}'^{IV}_{J=1,n=1} = 212.1 + 213.5 = 425.6$ MeV in Table I, and $\mathcal{M}^{II}_{J=1,n=1} + \mathcal{M}'^{II}_{J=1,n=1} = 390.4$ MeV in Table II, which are $\sim$ 100 MeV smaller than K-on masses, 493.7 $\sim$ 497.7 MeV, and $\sim$ 150 MeV smaller than the mass of η, 547.3 MeV. Thus, let us include $n = 2$ states. Then, for $G(\rho) = 0$, $\mathcal{M}^{II}_{J=1,n=2} = 372.2$ MeV, $\mathcal{M}'^{II}_{J=1,n=2} = 372.6$ MeV, and for $F(\rho) = 0$, $\mathcal{M}^{IV}_{J=1,n=2} = 383.3$ MeV, $\mathcal{M}'^{II}_{J=1,n=2} = 384.3$ MeV. Combining one of these states with one of $n = 1$ states, for instance, $\mathcal{M}^{IV}_{J=1,n=1} + \mathcal{M}'^{II}_{J=1,n=2} = 86.3 + 384.3 = 470.6$ MeV and so on, we have 430 $\sim$ 470 MeV, which are fairly close to K-on masses. On the other hand, other combinations such as $\mathcal{M}^{IV}_{J=1,n=1} + \mathcal{M}'^{II}_{J=1,n=2} = 212.1 + 372.6 = 584.7$ MeV yield 560 $\sim$ 580 MeV. These values are near to the mass of η. This way, we may generate the masses of all other mesons [6] in a comparable range. At this point, we might construct a model in an *ad hoc* manner for two free quarks confined within a sphere of radius R. The resulting mass of $q\bar{q}$ is greater than 1237.8 MeV. Thus, this kind of *ad hoc* model does not yield masses of light mesons.

The results obtained in this work for the masses of $q\bar{q}$ systems, *i.e.*, mesons, indicate the existence of infinitely many states, including closely neighboring excited states. Since such excited mesonic states have not yet been experimentally observed, one may regard these results as a limitation of the present approach in describing meson masses. On the other hand, there is currently neither experimental nor theoretical evidence that definitively excludes the existence of excited meson states. The purpose of this work is therefore not to claim their existences, but rather to provide a possible clue toward understanding the dynamics of confined quarks inside hadrons, particularly mesons.

It is worth noting that calculations show that the contribution of the current quark masses to the masses of $q\bar{q}$ is negligible. Even if we take the rest mass of a confined quark to be zero, the results do not change much. Thus, our results suggest that the dominant contribution to the masses of mesons originates from the confining Yang-Mills dynamics rather than from the rest masses of individual quarks. In other words, our results should be understood as a statement about the confining energy scale rather than about the rest masses of individual quarks.

Another point worth mentioning is that, although $\alpha_S = 0.5$ and $\rho = 2$ fm have been chosen phenomenologically in the present work, these parameters may be correlated. In the limiting case where $\alpha_S \to \infty$ and $\rho \to \infty$ the energy eigenvalues obtained from Eqs. (69, 72) approach the rest mass of a quark. Physically, this corresponds to two noninteracting quarks, for which confinement is absent. In contrast, decreasing α_S and ρ increases the energy eigenvalues, suggesting a stronger confining interaction between the quarks.

Finally, we note that from Fig. 1 the confining potential outside the confining region behaves $\sim -1/(r(r-\rho))$. This suggests that mesons attract each other with the potential $\sim -1/(r(r-\rho))$, which is an attractive short-range potential, and there is a hard core that other meson cannot penetrates. In nuclear physics, we know that for the nucleon-nucleon interaction there is a hard core that another nucleon cannot penetrates and the attractive nuclear potential is short ranged. Of course, nucleons are made of three quarks. We restrict ourselves to quark-antiquark systems; extending the analysis to baryons would require a genuine three-body treatment and is beyond the scope of this work.

Appendix A: Derivation of Eq. (34)

From Eqs. (14, 20) the upper two component of the wave function Ψ_{JM} is

$$< r,\theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM >^U = i\frac{G_J^I}{r}\Phi_J^I + i\frac{G_J^{II}}{r}\Phi_{II}^J + i\frac{G_J^{III}}{r}\Phi_{III}^J + i\frac{G_J^{IV}}{r}\Phi_{IV}^J, \qquad \text{(A1)}$$

and the lower two component of the wave function Ψ_{JM} is

$$< r,\theta,\phi|\sigma\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^L = \frac{F_J^I}{r}\sigma\cdot\hat{r}\Phi_J^I + \frac{F_J^{II}}{r}\sigma\cdot\hat{r}\Phi_J^{II}$$

TABLE I. Energy eigenvalues (in units of MeV) for lowest J values and $n=1$ satisfying $G(\rho)=0$.

$\alpha_S=\frac{g^2}{4\pi}$	0.1	0.3	0.5	0.7	0.9	1.1
$\mathcal{M}^{I}_{J=0,n=1}$	3009.1	1124.8	793.7	669.1	606.8	570.4
$\mathcal{M}^{II}_{J=1,n=1}$	-844.1	-94.7	64.9	134.5	173.5	198.5
$\mathcal{M}^{III}_{J=0,n=1}$	2979.2	1054.5	698.5	559.8	489.5	448.3
$\mathcal{M}^{IV}_{J=1,n=1}$	-464.7	76.2	212.1	274.8	311.0	334.5
$\mathcal{M}'^{I}_{J=0,n=1}$	3004.2	1120.7	790.4	666.4	604.6	568.6
$\mathcal{M}'^{II}_{J=1,n=1}$	-840.8	-93.3	65.7	135.1	174.0	198.9
$\mathcal{M}'^{III}_{J=0,n=1}$	2984.1	1059.0	702.5	563.2	492.4	450.8
$\mathcal{M}'^{IV}_{J=1,n=1}$	-460.5	78.4	213.5	275.9	311.8	335.2

TABLE II. Energy eigenvalues (in units of MeV) for lowest J values and $n=1$ satisfying $F(\rho)=0$.

$\alpha_S=\frac{g^2}{4\pi}$	0.1	0.3	0.5	0.7	0.9	1.1
$\mathcal{M}^{I}_{J=0,n=1}$	2984.1	1059.0	702.4	563.2	492.4	450.8
$\mathcal{M}^{II}_{J=1,n=1}$	-736.3	34.8	196.7	267.0	306.3	331.4
$\mathcal{M}^{III}_{J=0,n=1}$	3004.2	1120.7	790.4	666.4	604.6	568.6
$\mathcal{M}^{IV}_{J=1,n=1}$	-530.8	-38.9	86.3	145.6	180.2	202.9
$\mathcal{M}'^{I}_{J=0,n=1}$	2979.2	1054.5	698.5	559.8	489.5	448.3
$\mathcal{M}'^{II}_{J=1,n=1}$	-733.7	35.8	197.3	267.5	306.6	331.7
$\mathcal{M}'^{III}_{J=0,n=1}$	3009.0	1124.8	793.7	669.1	606.8	570.4
$\mathcal{M}'^{IV}_{J=1,n=1}$	-526.3	-36.0	88.2	147.0	181.3	203.9

$$+\frac{F_J^{III}}{r}\sigma\cdot\hat{r}\Phi_J^{III}+\frac{F_J^{IV}}{r}\sigma\cdot\hat{r}\Phi_J^{IV} \tag{A2}$$

$$V(r)\begin{pmatrix}\vec{\tau}\cdot\hat{r} & 0\\ 0 & \vec{\tau}\cdot\hat{r}\end{pmatrix}\Psi_{JM}=V(r)\begin{pmatrix}<r,\theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LT}}|j_{LT},S;JM>^U\\ <r,\theta,\phi|\vec{\tau}\cdot\hat{r}\vec{\sigma}\cdot\hat{r}\sum_{j_{LT}}|j_{LT},S;JM>^L\end{pmatrix} \tag{A3}$$

$$\begin{aligned}&V(r)<r,\theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM>^U=V(r)<r,\theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM>^U\\&\times<j_{LS},T;JM|d\Omega'|\theta',\phi'><\theta',\phi'|\vec{\tau}\cdot\hat{r}\sum_{j_{LT}}|j_{LT},S;JM>\\&\times<j_{LT},S;JM|d\Omega''|\theta'',\phi''><\theta'',\phi''|\sum_{j_{LS}}|j_{LS},T;JM>\}\end{aligned} \tag{A4}$$

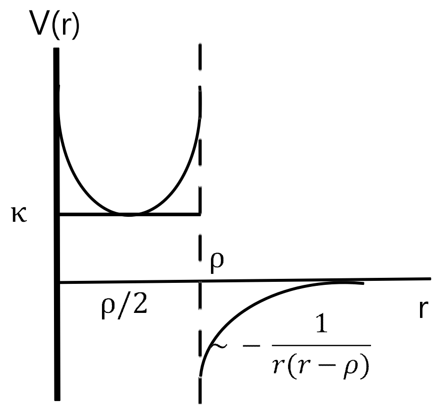

FIG. 1. $V(r) = -\frac{\pi}{g^2}\frac{\rho}{r(r-\rho)}$, $\kappa = \frac{4\pi}{g^2\rho}$

From Eqs. (30, 33) we are using

$$
\begin{aligned}
&< \theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LT}}|j_{LT},S;JM >< j_{LT},S;JM|\theta',\phi' > \\
&= -\tilde{\Phi}_J^{III}(\theta,\phi)\Phi_J^{I}(\theta',\phi')^{+} - \tilde{\Phi}_J^{IV}(\theta,\phi)\Phi_J^{II}(\theta',\phi')^{+} \\
&\quad -\Phi_J^{I}(\theta,\phi)\tilde{\Phi}_J^{III}(\theta',\phi')^{+} - \Phi_J^{II}(\theta,\phi)\tilde{\Phi}_J^{IV}(\theta',\phi')^{+}
\end{aligned}
\tag{A5}
$$

Then,

$$
\begin{aligned}
&V(r) < r,\theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^{U} = \\
&= V(r)\{i\frac{G_J^{I}}{r}\Phi_J^{I}\int d\Omega(-1)\tilde{\Phi}_J^{III}(\theta,\phi)^{+}(\Phi_J^{III}(\theta,\phi)+\Phi_J^{IV}(\theta,\phi)) \\
&+ i\frac{G_J^{II}}{r}\Phi_J^{II}\int d\Omega(-1)\tilde{\Phi}_J^{IV}(\theta,\phi)^{+}(\Phi_J^{III}(\theta,\phi)+\Phi_J^{IV}(\theta,\phi)) \\
&+ i\frac{G_J^{III}}{r}\Phi_J^{III}\int d\Omega(-1)\Phi_J^{III}(\theta,\phi)^{+}(\tilde{\Phi}_J^{III}(\theta,\phi)+\tilde{\Phi}_J^{IV}(\theta,\phi)) \\
&+ i\frac{G_J^{IV}}{r}\Phi_J^{IV}\int d\Omega(-1)\Phi_J^{IV}(\theta,\phi)^{+}(\tilde{\Phi}_J^{III}(\theta,\phi)+\tilde{\Phi}_J^{IV}(\theta,\phi))\} \\
&= \frac{1}{2J+1}(1-\sqrt{2J(2J+2)})V(r)i\frac{G_J^{I}}{r}\Phi_J^{I} \\
&\quad -\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})V(r)i\frac{G_J^{II}}{r}\Phi_J^{II}
\end{aligned}
$$

$$+\frac{1}{2J+1}(1-\sqrt{2J(2J+2)})V(r)i\frac{G_J^{III}}{r}\Phi_J^{III}$$
$$-\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})V(r)i\frac{G_J^{IV}}{r}\Phi_J^{IV} \tag{A6}$$

Similarly, $V(r) < r,\theta,\phi|\vec{\tau}\cdot\hat{r}\vec{\sigma}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^L$ is calculated and the result is included in Eq. (34).

Appendix B: Derivation of Eq. (41)

$$\vec{\alpha}\cdot(-Q\vec{A})\Psi_{JM} = -V(r)\begin{pmatrix} 0 & \vec{\tau}\cdot\hat{r} \\ -\vec{\tau}\cdot\hat{r} & 0 \end{pmatrix}\Psi_{JM}$$
$$+V(r)\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix}\Psi_{JM} \tag{B1}$$

We calculate the term with $\vec{\tau}\cdot\hat{r}$ first.

$$V(r)\begin{pmatrix} 0 & -\vec{\tau}\cdot\hat{r} \\ \vec{\tau}\cdot\hat{r} & 0 \end{pmatrix}\Psi_{JM}$$
$$= V(r) < r,\theta,\phi|\begin{pmatrix} 0 & -\vec{\tau}\cdot\hat{r} \\ \vec{\tau}\cdot\hat{r} & 0 \end{pmatrix}\begin{pmatrix} \sum_{j_{LS}}|j_{LS},T;JM > \\ \vec{\sigma}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM > \end{pmatrix}$$
$$= V(r)\begin{pmatrix} -< r,\theta,\phi|\vec{\tau}\cdot\hat{r}\vec{\sigma}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^U \\ < r,\theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^L \end{pmatrix} \tag{B2}$$

From Eqs. (30, A5) we have

$$-V(r) < r,\theta,\phi|\vec{\tau}\cdot\hat{r}\vec{\sigma}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^U$$
$$= -V(r) < r,\theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM >^U < j_{LS},T;JM|\int d\Omega'\vec{\sigma}\cdot\hat{r}|\theta',\phi' >$$
$$\times < \theta',\phi'|\vec{\tau}\cdot\hat{r}\sum_{j_{LT}}|j_{LT},S;JM >< j_{LT},S;JM|\int d\Omega''|\theta'',\phi'' >$$
$$\times < \theta'',\phi''|\sum_{j_{LS}}|j_{LS},T;JM >$$
$$= -V(r)i\frac{G_J^I}{r}\Phi_J^I\int d\Omega(\Phi_J^{III})^+(\tilde{\Phi}_J^{III}+\tilde{\Phi}_J^{IV})$$
$$-V(r)i\frac{G_J^{II}}{r}\Phi_J^{II}\int d\Omega(\Phi_J^{IV})^+(\tilde{\Phi}_J^{III}+\tilde{\Phi}_J^{IV})$$

$$
\begin{aligned}
&-V(r)i\frac{G_J^{III}}{r}\Phi_J^{III}\int d\Omega(\tilde{\Phi}_J^{III})^+(\Phi_J^{III}+\Phi_J^{IV}) \\
&-V(r)i\frac{G_J^{IV}}{r}\Phi_J^{IV}\int d\Omega(\tilde{\Phi}_J^{IV})^+(\Phi_J^{III}+\Phi_J^{IV}) \\
=\ & V(r)\frac{1}{2J+1}(1-\sqrt{2J(2J+2)})i\frac{G_J^{I}}{r}\Phi_J^{I} \\
&-V(r)\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})i\frac{G_J^{II}}{r}\Phi_J^{II} \\
&+V(r)\frac{1}{2J+1}(1-\sqrt{2J(2J+2)})i\frac{G_J^{III}}{r}\Phi_J^{III} \\
&-V(r)\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})i\frac{G_J^{IV}}{r}\Phi_J^{IV}
\end{aligned} \tag{B3}
$$

Similarly, $V(r) < r,\theta,\phi|\vec{\tau}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM >^L$ is calculated and the result is included in the following equation.

$$
\begin{aligned}
& V(r)\begin{pmatrix} 0 & -\vec{\tau}\cdot\hat{r} \\ \vec{\tau}\cdot\hat{r} & 0 \end{pmatrix}\Psi_{JM} \\
&= \frac{1}{2J+1}(1-\sqrt{2J(2J+2)})V(r)\begin{pmatrix} \frac{iG_J^{I}}{r}\Phi_J^{I} \\ -\frac{F_J^{I}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{I} \end{pmatrix} \\
&-\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})V(r)\begin{pmatrix} \frac{iG_J^{II}}{r}\Phi_J^{II} \\ -\frac{F_J^{II}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{II} \end{pmatrix} \\
&+\frac{1}{2J+1}(1-\sqrt{2J(2J+2)})V(r)\begin{pmatrix} \frac{iG_J^{III}}{r}\Phi_J^{III} \\ -\frac{F_J^{III}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{III} \end{pmatrix} \\
&-\frac{1}{2J+1}(1+\sqrt{2J(2J+2)})V(r)\begin{pmatrix} \frac{iG_J^{IV}}{r}\Phi_J^{IV} \\ -\frac{F_J^{IV}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{IV} \end{pmatrix}
\end{aligned} \tag{B4}
$$

Next, we calculate the term with $(\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r}$.

$$
\begin{aligned}
& V(r)\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix}\Psi_{JM} \\
&= V(r) < r,\theta,\phi|\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix}\begin{pmatrix} \sum_{j_{LS}}|j_{LS},T;JM > \\ \vec{\sigma}\cdot\hat{r}\sum_{j_{LS}}|j_{LS},T;JM > \end{pmatrix} \\
&= V(r)\begin{pmatrix} < r,\theta,\phi|\vec{\sigma}\cdot\vec{\tau}\sum_{j_{LS}}|j_{LS},T;JM >^U \\ - < r,\theta,\phi|\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau})\sum_{j_{LS}}|j_{LS},T;JM >^L \end{pmatrix}
\end{aligned} \tag{B5}
$$

$$
V(r) < r,\theta,\phi|\vec{\sigma}\cdot\vec{\tau}\sum_{j_{LS}}|j_{LS},T;JM >^U
$$

$$= V(r) < r,\theta,\phi|\sum_{j_{LS}}|j_{LS},T;JM >^U < J_{LS},T;JM|\int d\Omega'|\theta',\phi' >$$
$$\times < \theta',\phi'|\vec{\sigma}\cdot\vec{\tau}\sum_{j_{ST}}|j_{ST},L;JM >< j_{ST},L;JM|\int d\Omega''|\theta'',\phi'' >$$
$$\times < \theta'',\phi''|\sum_{j_{LS}}|j_{LS},T;JM > \tag{B6}$$

From Eqs. (37, 38) we are going to use:

$$< \theta,\phi|\vec{\sigma}\cdot\vec{\tau}\sum_{j_{ST}}|j_{ST},L;JM >< j_{ST},L;JM|\theta',\phi' >$$
$$= \Phi_J^I(\theta,\phi)\Phi_J^I(\theta',\phi')^+\Phi_J^{II}(\theta,\phi)\Phi_J^{II}(\theta',\phi')^+$$
$$+ \bar{\Phi}_J^{III}(\theta,\phi)\bar{\Phi}_J^{III}(\theta',\phi')^+ - 3\bar{\Phi}_J^{IV}(\theta,\phi)\bar{\Phi}_J^{IV}(\theta',\phi')^+ \tag{B7}$$

Then,

$$V(r) < r,\theta,\phi|\vec{\sigma}\cdot\vec{\tau}\sum_{j_{LS}}|j_{LS},T;JM >^U$$
$$= V(r)i\frac{G_J^I}{r}\Phi_J^I\int d\Omega(\Phi_J^I)^+\Phi_J^I + V(r)i\frac{G_J^{II}}{r}\Phi_J^{II}\int d\Omega(\Phi_J^I)^+\Phi_J^I$$
$$+V(r)i\frac{G_J^{III}}{r}\Phi_J^{III}[\int d\Omega(\Phi_J^{III})^+\bar{\Phi}_J^{III}(\int d\Omega'\bar{\Phi}_J^{III}(\theta',\phi')^+(\Phi_J^{III}(\theta',\phi') + \Phi_J^{IV}(\theta',\phi')))$$
$$-3\int d\Omega(\Phi_J^{III})^+\bar{\Phi}_J^{IV}(\int d\Omega'\bar{\Phi}_J^{IV}(\theta',\phi')^+(\Phi_J^{III}(\theta',\phi') + \Phi_J^{IV}(\theta',\phi')))]$$
$$+V(r)i\frac{G_J^{IV}}{r}\Phi_J^{IV}[\int d\Omega(\Phi_J^{IV})^+\bar{\Phi}_J^{III}(\int d\Omega'\bar{\Phi}_J^{III}(\theta',\phi')^+(\Phi_J^{III}(\theta',\phi') + \Phi_J^{IV}(\theta',\phi')))$$
$$-3\int d\Omega(\Phi_J^{IV})^+\bar{\Phi}_J^{IV}(\int d\Omega'\bar{\Phi}_J^{IV}(\theta',\phi')^+(\Phi_J^{III}(\theta',\phi') + \Phi_J^{IV}(\theta',\phi')))]$$
$$= V(r)i\frac{G_J^I}{r}\Phi_J^I + V(r)i\frac{G_J^{II}}{r}\Phi_J^{II}$$
$$-[1 + \frac{1}{2J+1}(2 - \sqrt{2J(2J+2)})]V(r)i\frac{G_J^{III}}{r}\Phi_J^{III}$$
$$[-1 + \frac{1}{2J+1}(2 + \sqrt{2J(2J+2)})]V(r)i\frac{G_J^{IV}}{r}\Phi_J^{IV} \tag{B8}$$

Similarly, $V(r) < r,\theta,\phi|\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau})\sum_{j_{LS}}|j_L S,T;JM >^L$ is calculate and its result is in the following equation.

$$V(r)\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{\tau})\vec{\sigma}\cdot\hat{r} \\ -\vec{\sigma}\cdot\hat{r}(\vec{\sigma}\cdot\vec{\tau}) & 0 \end{pmatrix}\Psi_{JM}$$

$$
\begin{aligned}
&= V(r)\begin{pmatrix} i\frac{G_J^I}{r}\Phi_J^I \\ -\frac{F_J^I}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^I \end{pmatrix} + V(r)\begin{pmatrix} i\frac{G_J^{II}}{r}\Phi_J^{II} \\ -\frac{F_J^{II}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{II} \end{pmatrix} \\
&-[1+\frac{1}{2J+1}(2-\sqrt{2J(2J+2)})]V(r)\begin{pmatrix} i\frac{G_J^{III}}{r}\Phi_J^{III} \\ -\frac{F_J^{III}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{III} \end{pmatrix} \\
&+[-1+\frac{1}{2J+1}(2+\sqrt{2J(2J+2)})]V(r)\begin{pmatrix} i\frac{G_J^{IV}}{r}\Phi_J^{IV} \\ -\frac{F_J^{IV}}{r}\vec{\sigma}\cdot\hat{r}\Phi_J^{IV} \end{pmatrix}\}
\end{aligned} \tag{B9}
$$

The sum of Eq. (B4) and Eq. (B9) is given by Eq. (41).

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