New Static Solutions of Yang-Mills Equations for n Point Sources and Their Implications for Quark Confinement

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Abstract

We present new static solutions of SU(2) Yang-Mills equations for n point sources aligned in a common isospin direction. The solutions possess a gauge field structure analogous to the BPST instanton while constituting a distinct class of static solutions, and satisfy the Coulomb gauge condition. They are either self dual or anti-self dual. Substitution of the solutions into the Yang-Mills equations yields localized source terms corresponding to a non-Abelian electric charge but also a magnetic monopole charge associated with each source. Furthermore, the resulting gauge fields generate a confining potential for quarks. These properties suggest that the present solutions provide a new class of nonperturbative configurations in Yang-Mills theory.

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I. INTRODUCTION

Solutions of Yang-Mills equations are always interesting. Most notable solutions are the instanton solutions [1]. They are solutions of Yang-Mills equations without sources. In this work I would like to present solutions of Yang-Mills equations with static point-sources.

Let's start with SU(2) Yang-Mills equations with sources.

$$D_\mu F^{\mu\nu} = j^\nu \tag{1}$$

In this work we normalize all fields as $F^{\mu\nu} = F_a^{\mu\nu}\frac{\tau_a}{2}$, $j^\nu = j_a^\nu \frac{\tau_a}{2}$ and so on.

We know the field strength tensor $F^{\mu\nu}$. However, I am writing it here because it is convenient to deal with Yang-Mills equations in terms of non-Abelian electric and magnetic fields, that is, $\vec{E}$ and $\vec{B}$.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{2}$$

The electric and magnetic fields in terms of A_0 and $\vec{A}$ are

$$\vec{E} = -\nabla A_0 - \frac{\partial \vec{A}}{\partial t} + ig[\vec{A}, A_0] \tag{3}$$

$$\vec{B} = \nabla \times \vec{A} - \frac{1}{2} ig[\vec{A} \stackrel{\times}{,} \vec{A}] \tag{4}$$

The Yang-Mills equations, Eq. (1), are written in terms of $\vec{E}$ and $\vec{B}$ fields along with A_0 and $\vec{A}$ as follows.

$$\nabla \cdot \vec{E} - ig[\vec{A} \stackrel{\cdot}{,} \vec{E}] = j^0 \tag{5}$$

$$-\frac{\partial}{\partial t}\vec{E} + \nabla \times \vec{B} - ig[A_0, \vec{E}] - ig[\vec{A} \stackrel{\times}{,} \vec{B}] = \vec{j} \tag{6}$$

In Eqs. (4, 5, 6), and subsequent equations, $\times$ and $\cdot$ in commutators mean vector cross and scalar products of space parts of $\vec{A}$, $\vec{B}$, and $\vec{E}$, respectively.

II. SOLUTIONS

After lengthy calculations and a careful analysis, I am able to obtain the following results for A_0 and $\vec{A}$, which satisfy the Yang-Mills equations Eqs. (5, 6).

$$A_0 = A_0^{\pm} = \mp \frac{i}{g} \frac{1}{\prod(\vec{r})} \sum_{i=1}^{n} \frac{\rho_i}{|\vec{r}-\vec{r}_i|^3} \frac{\vec{\tau}}{2} \cdot (\vec{r}-\vec{r}_i), \tag{7}$$

$$\vec{A}^{\pm} = \vec{A} = \frac{1}{g} \frac{1}{\prod(\vec{r})} \sum_{i=1}^{n} \frac{\rho_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i) \times \frac{\vec{\tau}}{2}, \tag{8}$$

where

$$\prod(\vec{r}) = C + \sum_{i=1}^{n} \frac{\rho_i}{|\vec{r}-\vec{r}_i|} \tag{9}$$

In Eqs. (7, 8, 9) ρ_i's are arbitrary and $C = 1$ or 0. We could start with C arbitrary. However, as long as C is not 0 it is absorbed by ρ_i's to make $C = 1$.

The structure of the above solution is very similar to the BPST instanton solution [1]. To see this, let us convert the static solution Eqs. (7, 8) into the Euclidean solution as follows.

$$iA_0 \to A_4^E, \quad \vec{A} \to \vec{A}^E \tag{10}$$

such that

$$A_4^{E\pm} = \pm \frac{1}{g} \frac{1}{\prod(\vec{r})} \sum_{i=1}^{n} \frac{\rho_i}{|\vec{r}-\vec{r}_i|^3} \frac{\vec{\tau}}{2} \cdot (\vec{r}-\vec{r}_i), \tag{11}$$

$$\vec{A}^{E\pm} = \vec{A}^E = \frac{1}{g} \frac{1}{\prod(\vec{r})} \sum_{i=1}^{n} \frac{\rho_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i) \times \frac{\vec{\tau}}{2}, \tag{12}$$

Using 't Hooft's $\eta^{\pm}_{a\mu\nu}$ [1], we rewrite the Euclidean solution:

$$\begin{aligned} A^{E\pm}_{\mu=1,2,3,4} &= -\frac{1}{g} \frac{\tau_a}{2} \eta^{\mp}_{a\mu\alpha} \partial_\alpha ln \prod \\ &= -\frac{1}{2g} \tau^{\mp}_{\mu\alpha} \partial_\alpha ln \prod = -\frac{1}{2g} \tau^{\mp}_{\mu\alpha} \frac{1}{\prod} \prod{}_{,\alpha}, \end{aligned} \tag{13}$$

where

$$\tau^{\mp}_{\mu\alpha} = \tau_a \eta^{\mp}_{a\mu\alpha} = \tau_a (\epsilon_{a\mu\alpha} \pm \delta_{\mu 4} \delta_{a\alpha} \mp \delta_{\alpha 4} \delta_{a\mu}) \tag{14}$$

and

$$\prod(\vec{r}) = 1 + \sum_{i=1}^{n} \frac{\rho}{|\vec{r}-\vec{r}_i|} \tag{15}$$

If we substitude the $\prod$ in Eq. (13) with the $\prod$ given by

$$\prod(x) = 1 + \sum_{i=1}^{n} \frac{\lambda^2}{|x - x_i|^2}, \tag{16}$$

we obtain the BPST instanton slution [1]. In Eq. (16) $|x - x_i|^2$ means

$$|x - x_i|^2 = \sum_{\mu=1}^{4} (x_\mu - (x_i)_\mu)^2 \tag{17}$$

As we can see in Eqs. (13, 16, 15), the static solution indeed resembles the BPST instanton solution. The only difference is that $\prod$ for instanton has the fourth component $x_{\mu=4}$, that is, time component in Minkowsky space, while $\prod$ for our static solution does not have the Euclidean fourth component.

In order to see our solutions given by Eqs. (7, 8) satisfy Yang-Mills equations with $\vec{j} = 0$, we substitute our solutions into Eqs. (5, 6). Now, we calculate $\vec{E}$ and $\vec{B}$. Since calculations are not affected by the values of C's, we consider either $C = 1$ or $C = 0$. We should note that in Eqs. (7, 8) $\pm$ represents self dual or anti-self dual field. In Eq. (8) $\vec{A}^{\pm} = \vec{A}$ means that $\vec{A}$ is the same for both self dual and anti-self dual fields. Inserting these A_0 and $\vec{A}$ into Eqs. (3, 4) we have the $\vec{E}$ and $\vec{B}$ as follows.

$$\begin{aligned} \vec{E}^{\pm} = \pm \frac{i}{g} \{ & \frac{2}{\prod^2} \sum_{i,j} \frac{\rho_i \rho_j}{|\vec{r} - \vec{r}_i|^3 |\vec{r} - \vec{r}_j|^3} (\vec{r} - \vec{r}_j) \frac{\tau}{2} \cdot (\vec{r} - \vec{r}_i) \\ & - \frac{3}{\prod} \sum_{i=1} \frac{\rho_i}{|\vec{r} - \vec{r}_i|^5} (\vec{r} - \vec{r}_i) \frac{\tau}{2} \cdot (\vec{r} - \vec{r}_i) \\ & + [- \frac{1}{\prod^2} \sum_{i,j} \frac{\rho_i \rho_j}{|\vec{r} - \vec{r}_i|^3 |\vec{r} - \vec{r}_j|^3} (\vec{r} - \vec{r}_i) \cdot (\vec{r} - \vec{r}_j) + \frac{1}{\prod} \sum_{i} \frac{\rho_i}{|\vec{r} - \vec{r}_i|^3}] \frac{\tau}{2} \} \end{aligned} \tag{18}$$

$$\begin{aligned} \vec{B}^{\pm} = \vec{B} = \frac{1}{g} \{ & \frac{2}{\prod^2} \sum_{i,j} \frac{\rho_i \rho_j}{|\vec{r} - \vec{r}_i|^3 |\vec{r} - \vec{r}_j|^3} (\vec{r} - \vec{r}_j) \frac{\tau}{2} \cdot (\vec{r} - \vec{r}_i) \\ & - \frac{3}{\prod} \sum_{i=1} \frac{\rho_i}{|\vec{r} - \vec{r}_i|^5} (\vec{r} - \vec{r}_i) \frac{\tau}{2} \cdot (\vec{r} - \vec{r}_i) \\ & + [- \frac{1}{\prod^2} \sum_{i,j} \frac{\rho_i \rho_j}{|\vec{r} - \vec{r}_i|^3 |\vec{r} - \vec{r}_j|^3} (\vec{r} - \vec{r}_i) \cdot (\vec{r} - \vec{r}_j) + \frac{1}{\prod} \sum_{i} \frac{\rho_i}{|\vec{r} - \vec{r}_i|^3}] \frac{\tau}{2} \} \end{aligned} \tag{19}$$

When we calculate above equations, we used the following formula.

$$[(\vec{r} - \vec{r}_i) \times \frac{\vec{\tau}}{2}, \frac{\vec{\tau}}{2} \cdot (\vec{r} - \vec{r}_j)] = i \{ (\vec{r} - \vec{r}_j) \frac{\vec{\tau}}{2} \cdot (\vec{r} - \vec{r}_i) - (\vec{r} - \vec{r}_i) \cdot (\vec{r} - \vec{r}_j) \frac{\vec{\tau}}{2} \}, \tag{20}$$

$$[(\vec{r}-\vec{r}_i)\times\frac{\vec{\tau}}{2} \stackrel{\times}{;} (\vec{r}-\vec{r}_j)\times\frac{\vec{\tau}}{2}] = \hat{e}_k\epsilon_{kij}[\epsilon_{i\ell a}(\vec{r}-\vec{r}_i)_\ell\frac{\tau_a}{2}, \epsilon_{jmb}(\vec{r}-\vec{r}_j)_m\frac{\tau_b}{2}]$$
$$= \hat{e}_k(\vec{r}-\vec{r}_i)_\ell(\vec{r}-\vec{r}_j)_m\epsilon_{kij}\epsilon_{i\ell a}\epsilon_{jmb}i\epsilon_{abc}\frac{\tau_c}{2}$$
$$= i\{(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_j) + (\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)\}, \tag{21}$$

where $\hat{e}_k$ is the unit vector in the k-th direction. In the calculations of Eqs. (18, 19) and subsequent equations, we frequently interchange dummy indices i, j.

From above equations we observe

$$\vec{E}^{\pm} = \pm i\vec{B} \tag{22}$$

Self duality or anti-self duality in Minkowsky space are realized by Eq. (22). $\vec{B}$ is the same for both self dual and anti-self dual fields because $\vec{A}$ is the same for both cases.

With $\vec{E}^{\pm}$ and $\vec{A}$ we calculate $\nabla\cdot\vec{E}$ and $-ig[\vec{A}\,;\,\vec{E}]$ in Eq. (5).

$$\nabla\cdot\vec{E}^{\pm} = \pm i\frac{4\pi}{g}\{\frac{2}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^2}\delta^3(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|})$$
$$-\frac{3}{\prod}\sum_i\frac{\rho_i}{|\vec{r}-\vec{r}_i|}\delta^3(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|})\}$$
$$\pm\frac{i}{g}\{\frac{2}{\prod^3}\sum_{i,j,k}\frac{\rho_i\rho_j\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k)$$
$$-\frac{3}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^5|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)$$
$$+\frac{1}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)\}, \tag{23}$$

The detailed calculations of the above equation is shown in Appendix A.

$$-ig[\vec{A}\,;\,\vec{E}^{\pm}] = -ig\{\frac{1}{g}\frac{1}{\prod}\sum_k\frac{\rho_k}{|\vec{r}-\vec{r}_i|^3}[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2}\,;\,\vec{E}^{\pm}]\} \tag{24}$$

Using the following equalities:

$$[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2}\,;\,(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)] = \epsilon_{abc}(\vec{r}-\vec{r}_k)_b(\vec{r}-\vec{r}_j)_a[\frac{\vec{\tau}_c}{2},\frac{\vec{\tau}_d}{2}(\vec{r}-\vec{r}_i)_d]$$
$$= i\epsilon_{abc}\epsilon_{cde}(\vec{r}-\vec{r}_k)_b(\vec{r}-\vec{r}_j)_a(\vec{r}-\vec{r}_i)_d\frac{\vec{\tau}_e}{2}$$
$$= i\{(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}_e}{2}\cdot(\vec{r}-\vec{r}_k) - (\vec{r}-\vec{r}_k)\cdot(\vec{r}-\vec{r}_i)\frac{\vec{\tau}_e}{2}\cdot(\vec{r}-\vec{r}_j)\}, \tag{25}$$

$$[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2}\,;\,\frac{\vec{\tau}}{2}] = \epsilon_{iab}(\vec{r}-\vec{r}_k)_a[\frac{\tau_b}{2},\frac{\tau_i}{2}]$$
$$= \epsilon_{iab}(\vec{r}-\vec{r}_k)_a i\epsilon_{bic}\frac{\tau_c}{2} = 2i(\vec{r}-\vec{r}_k)\cdot\frac{\vec{\tau}}{2} \tag{26}$$

we have

$$-ig[\vec{A}\,;\,\vec{E}^{\pm}] = \pm i\frac{1}{g}\{-\frac{2}{\prod^3}\sum_{i,j,k}\frac{\rho_i\rho_j\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k)$$
$$+\frac{3}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^5|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)$$
$$-\frac{1}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)\} \tag{27}$$

Now, it is clear that Eq. (27) cancels the terms without δ-fuctions in Eq. (23). Thus, we have the following.

$$\nabla\cdot\vec{E}^{\pm} - ig[\vec{A}\,;\,\vec{E}^{\pm}] = \pm i\frac{4\pi}{g}\{\frac{2}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^2}\delta^3(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|})$$
$$-\frac{3}{\prod}\sum_i\frac{\rho_i}{|\vec{r}-\vec{r}_i|}\delta^3(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|})\} \tag{28}$$

In the above equation, only the term $\frac{\rho_j}{|\vec{r}-\vec{r}_j|}$ in $\prod^2$ and the term $\frac{\rho_i}{|\vec{r}-\vec{r}_i|}$ in $\prod$ do not vanish because of the δ-functions, which leads to the following result.

$$\nabla\cdot\vec{E}^{\pm} - ig[\vec{A}\,;\,\vec{E}^{\pm}] = j^0$$
$$= \pm i\frac{4\pi}{g}\{2\sum_j\delta^3(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|}) - 3\sum_i\delta^3(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|})\}$$
$$= \mp i\frac{4\pi}{g}\{\sum_i\delta^3(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|})\} = \mp i\frac{4\pi}{g}\{\sum_i\delta^3(\vec{r}-\vec{r}_i)\frac{\tau_3}{2}\} \tag{29}$$

The last equality is established by the following useful equality:

$$\delta^3(\vec{r})\vec{\tau}\cdot\hat{r} = \frac{\delta(r)\delta(\theta)\delta(\phi)}{r^2\sin\theta}(\tau_1\sin\theta\cos\phi + \tau_2\sin\theta\sin\phi + \tau_3\cos\theta)$$
$$= \frac{\delta(r)\delta(\theta)\delta(\phi)}{r^2\sin\theta}\tau_3\cos\theta = \delta^3(\vec{r})\tau_3 \tag{30}$$

We have just shown that Eqs. (7, 8) satisfy Eq. (5). The n point sources are all aligned in the same direction τ_3, each carrying charge $\mp i\frac{2\pi}{g}$ or aligned in the opposite direction, each carrying charge $\pm i\frac{2\pi}{g}$. Since the eigenvalue of $\frac{\tau_3}{2}$ is either $\frac{1}{2}$ or $-\frac{1}{2}$, the electric charge is $\mp i\frac{2\pi}{g}$ rather than $\mp i\frac{4\pi}{g}$. The finding that the the non-Abelian charge is imaginary rather than real is particularly remarkable. It implies that the non-Abelian charge by an individual source is not directly measurable. If each source is interpreted as a quark, then the physical properties of individual quarks have not been directly observed experimentally. This may be viewed as a consequence of quark confinement. It is particularly important to emphasize that this result arises directly from the new static solutions of the Yang-Mills equations presented here.

According to Eq. (3, 5), removing the factor i from $A^{\pm}$ in Eq. (7), thereby making $A^{\pm}$ real, leads to real non-Abelian charge. However, the resulting solution fails to satisfy Eq. (6) with $\vec{j}=0$. Therefore, the factor i in $A_0^{\pm}$ is required, as in Eq. (7). Later we will show that there is an exceptional single-source solution which has a real non-Abelian electric charge.

Now, let us see if Eqs. (7, 8) satisfy Eq. (6) with $\vec{j}=0$. Since A_0 and $\vec{A}$ are static, we need to show that they satisfy

$$\nabla \times \vec{B} - ig[A_0, \vec{E}] - ig[\vec{A} \overset{\times}{,} \vec{B}] = \vec{j} = 0 \tag{31}$$

We calculate each term in Eq. (31) as follows.

$$\begin{aligned}
\nabla \times \vec{B} = \frac{1}{g}\{ & -\frac{3}{\prod^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k) \\
& + \frac{3}{\prod^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^3}\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k) \\
& + \frac{2}{\prod^3}\sum_{i,j,k}\frac{\rho_i\rho_j\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k) \\
& - \frac{6}{\prod^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_k)\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k)\}
\end{aligned} \tag{32}$$

$$\begin{aligned}
-ig[A_0, \vec{E}^{\pm}] &= -ig(\mp i\frac{1}{g})\frac{1}{\prod}\{\sum_k \frac{\rho_k}{|\vec{r}-\vec{r}_k|^3}[\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k), \vec{E}^{\pm}]\} \\
&= \frac{1}{g}\{\frac{3}{\prod^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}(\vec{r}-\vec{r}_k)[(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)]\cdot\frac{\vec{\tau}}{2} \\
&- \frac{1}{\prod^3}\sum_{i,j,k}\frac{\rho_i\rho_j\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k) \\
&+ \frac{1}{\prod^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^3}\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k)\}
\end{aligned} \tag{33}$$

$$-ig[\vec{A} \overset{\times}{,} \vec{B}] = -ig(\frac{1}{g}\frac{1}{\prod})\{\sum_k \frac{\rho_k}{|\vec{r}-\vec{r}_k|^3}[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2} \overset{\times}{,} \vec{B}]\}$$

$$
\begin{aligned}
&= \frac{1}{g}\{\frac{3}{\Pi^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_k)\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k)] \\
&- \frac{1}{\Pi^3}\sum_{i,j,k}\frac{\rho_i\rho_j\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k) \\
&- \frac{1}{\Pi^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^3}\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k)\},
\end{aligned} \tag{34}
$$

where we are using the following equalities:

$$
\begin{aligned}
&[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2}\overset{\times}{,}(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_j)] \\
&= i(\vec{r}-\vec{r}_j)\times(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k) - i(\vec{r}-\vec{r}_k)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_i),
\end{aligned} \tag{35}
$$

$$
[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2}\overset{\times}{,}\frac{\vec{\tau}}{2}] = -i\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k) \tag{36}
$$

Adding Eqs. (32, 33, 34) we have

$$
\begin{aligned}
&\nabla\times\vec{B} - ig[A_0,\vec{E}] - ig[\vec{A}\overset{\times}{,}\vec{B}] = \vec{j} \\
&= \frac{1}{g}\{\frac{3}{\Pi^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}\{(\vec{r}-\vec{r}_k)[(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)]\cdot\frac{\vec{\tau}}{2} \\
&- [(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)]\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k)\} \\
&- \frac{3}{\Pi^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}[(\vec{r}-\vec{r}_k)\cdot(\vec{r}-\vec{r}_i) - |\vec{r}-\vec{r}_k|^2]\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k)\}
\end{aligned} \tag{37}
$$

To show that $\vec{j}=0$ in the above equation, we recall the formula for the cross product of three vectors, that is, $\vec{A}\times(\vec{B}\times\vec{C})$ to obtain the following equality.

$$
\begin{aligned}
&\frac{\vec{\tau}}{2}\times\{(\vec{r}-\vec{r}_k)\times[(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)]\} \\
&= (\vec{r}-\vec{r}_k)[(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)]\cdot\frac{\vec{\tau}}{2} - [(\vec{r}-\vec{r}_k)\times(\vec{r}-\vec{r}_i)]\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_k) \\
&= \frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k)[(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_k)] - |\vec{r}-\vec{r}_k|^2\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_i)
\end{aligned} \tag{38}
$$

Then, Eq. (37) becomes

$$
\begin{aligned}
&\nabla\times\vec{B} - ig[A_0,\vec{E}] - ig[\vec{A}\overset{\times}{,}\vec{B}] = \vec{j} \\
&= \frac{1}{g}\frac{3}{\Pi^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^5}\{-|\vec{r}-\vec{r}_k|^2\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_k) \\
&+|\vec{r}-\vec{r}_k|^2\frac{\vec{\tau}}{2}\times(\vec{r}-\vec{r}_i)\} = 0
\end{aligned} \tag{39}
$$

This equality holds if we exchange the indices i and k in the second term. At last, we have shown that A_0 and $\vec{A}$ given by Eqs. (7, 8) satisfy Eq. (6) with $\vec{j} = 0$. If $A_0^{\pm}$ were real, then the term $-ig[A_0, \vec{E}]$ would change sign so that Eq. (39) cannot be satisfied, and hence, imaginary $A_0^{\pm}$ is required.

Now, let us see the vector potential $\vec{A}$ satisfies the Coulomb gauge condition as follows.

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{1}{g}\{\frac{1}{\prod^2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_k)\cdot[(\vec{r}-\vec{r}_i)\times\frac{\vec{\tau}}{2}] \\
&-\frac{3}{\prod}\sum_i\frac{\rho_i}{|\vec{r}-\vec{r}_i|^5}(\vec{r}-\vec{r}_i)\cdot[(\vec{r}-\vec{r}_i)\times\frac{\vec{\tau}}{2}] \\
&+\frac{3}{\prod}\sum_i\frac{\rho_i}{|\vec{r}-\vec{r}_i|^3}\nabla\cdot[(\vec{r}-\vec{r}_i)\times\frac{\vec{\tau}}{2}]\} \\
&= \frac{1}{g}\frac{1}{\prod^2}\frac{1}{2}\sum_{i,k}\frac{\rho_i\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_k|^3}\{(\vec{r}-\vec{r}_k)\cdot[(\vec{r}-\vec{r}_i)\times\frac{\vec{\tau}}{2}] \\
&+(\vec{r}-\vec{r}_i)\cdot[(\vec{r}-\vec{r}_k)\times\frac{\vec{\tau}}{2}]\} = 0
\end{aligned} \tag{40}$$

III. DUAL TENSOR AND MAGNETIC CHARGE

The dual field strength tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{41}$$

where $\epsilon^{0123} = 1$ and $\epsilon^{\mu\nu\lambda\rho} = -\epsilon_{\mu\nu\lambda\rho}$. If we replace $\vec{B}$ by $\mp i\vec{E}$ and $\vec{E}$ by $\pm i\vec{B}$ in Eq. (41), we have

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \mp iF^{\mu\nu} \tag{42}$$

The equation for the dual tensor is:

$$\begin{aligned}
D_\mu\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}D_\mu F_{\alpha\beta} = j_m{}^\nu \\
&= \delta^{\nu,0}(\nabla\cdot\vec{B} - ig[\vec{A}\,;\,\vec{B}]) \\
&+ \delta^{\nu\neq 0}\{-\frac{\partial}{\partial t}\vec{B} - \nabla\times\vec{E} - ig[A_0, \vec{B}] + ig[\vec{A} \overset{\times}{;} \vec{E}]\},
\end{aligned} \tag{43}$$

where $j_m{}^\nu$ is a 4-vector magnetic charge and current. From Eq. (22), $\vec{E}^{\pm} = \pm i\vec{B}$, we convert Eq. (43) to the following equation.

$$\begin{aligned}
D_\mu\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}D_\mu F_{\alpha\beta} = j_m{}^\nu \\
&= \mp i\{\delta^{\nu,0}(\nabla\cdot\vec{E} - ig[\vec{A}\,;\,\vec{E}])
\end{aligned}$$

$$+\,\delta^{\nu\neq 0}(-\frac{\partial}{\partial t}\vec{E}+\nabla\times\vec{B}-ig[A_0,\vec{E}]-ig[\vec{A}\overset{\times}{,}\vec{B}])\}$$
$$=\delta^{\nu,0}\{-\frac{2\pi}{g}[\sum_i\delta^3(\vec{r}-\vec{r}_i)\tau_3]\}+\delta^{\nu\neq 0}\cdot 0 \tag{44}$$

The last equality in the above equation is due to Eqs. (29, 39). Eq. (44) further reveals that each source carries a magnetic monopole charge of $-\frac{2\pi}{g}$. This is yet another remarkable consequence of the new static solutions of the Yang-Mills equations. If each source is identified with a quark again, these equations imply that every quark carries not only an imaginary non-Abelian electric charge charge, $\mp i\frac{2\pi}{g}$, but also a magnetic monopole charge of $-\frac{2\pi}{g}$.

IV. EXCEPTIONAL SINGLE SOURCE SOLUTION

For a single point source ($n = 1$), the Yang-Mills equations admit additional solutions for which the non-Abelian electric charge may be real, pure imaginary, or more generally, complex. These solutions are distinct from the family of imaginary-charge solutions that exist for arbitrary $n \geq 2$. In this section, we examine these exceptional solutions. Let us consider the solution with $n = 1$ and $C = 0$ in Eq. (9)

$$A_0^{\pm}=\mp\frac{i}{gr}\frac{\tau}{2}\cdot\hat{r},\quad \vec{A}=\frac{1}{gr}\hat{r}\times\frac{\tau}{2}, \tag{45}$$

which yield $\vec{E}$ and $\vec{B}$:

$$\vec{E}^{\pm}=\mp\frac{i}{gr^2}\hat{r}(\frac{\tau}{2}\cdot\hat{r}),\quad \vec{B}=-\frac{1}{gr^2}\hat{r}(\frac{\tau}{2}\cdot\hat{r}) \tag{46}$$

The solution obtained by deleting the factor i in Eq. (45) is also a solution.

$$A_0^{\pm}=\mp\frac{1}{gr}\frac{\tau}{2}\cdot\hat{r},\quad \vec{A}=\frac{1}{gr}\hat{r}\times\frac{\tau}{2}, \tag{47}$$

The $\vec{E}$ and $\vec{B}$ are given by

$$\vec{E}^{\pm}=\mp\frac{1}{gr^2}\hat{r}(\frac{\tau}{2}\cdot\hat{r}),\quad \vec{B}=-\frac{1}{gr^2}\hat{r}(\frac{\tau}{2}\cdot\hat{r}) \tag{48}$$

The electric charge and magnetic charge in Eq. (47) are $\mp\frac{2\pi}{g}$ and $-\frac{2\pi}{g}$, respectively. The electric charge is not imaginary but real.

Moreover, there exists a general complex solution.

$$A_0^{\pm}=\frac{e^{i\varphi}}{gr}\frac{\tau}{2}\cdot\hat{r},\quad \vec{A}=\frac{1}{gr}\hat{r}\times\frac{\tau}{2}, \tag{49}$$

The $\vec{E}$ and $\vec{B}$ fields are

$$\vec{E}^{\pm} = \frac{e^{i\varphi}}{gr^2}\hat{r}(\frac{\tau}{2}\cdot\hat{r}), \quad \vec{B} = -\frac{1}{gr^2}\hat{r}(\frac{\tau}{2}\cdot\hat{r}) \tag{50}$$

The electric charge and magnetic charge in Eq. (50) are $\frac{e^{i\varphi}}{g}$ and $-\frac{2\pi}{g}$, respectively. The electric charge is complex. Naturally, the purely imaginary and real solutions Eqs. (46, 47) are special cases of the general complex solution.

V. DISCUSSIONS

From our solutions of the Yang-Mills equations for n point sources, we find that each source carries not only a non-Abelian electric charge, $\mp i\frac{2\pi}{g}$, but also a magnetic monopole charge of $-\frac{2\pi}{g}$. This result is independent of whether $C = 1$ or $C = 0$. As we saw just before, there are exceptional solutions for which the non-Abelian electric charge may be real, pure imaginary, or more generally, complex. These solutions are distinct from the family of imaginary-charge solutions that exist for arbitrary $n \geq 2$. The solutions with $C = 0$, including the exceptional cases without a purely imaginary solution do not give rise to a particularly interesting potential. In contrast, the solutions with $C = 1$ appear to lead to an interesting potential.

For a single source at the origin with an electric charge $Q_0 = \mp i\frac{2\pi}{g}$, A_0 and $\vec{A}$ are given by

$$A_0^{\pm} = \mp i\frac{1}{g}\frac{\rho}{(1+\frac{\rho}{r})r^2}\frac{\tau}{2}\cdot\hat{r}, \quad \vec{A} = \frac{1}{g}\frac{\rho}{(1+\frac{\rho}{r})r^2}\hat{r}\times\frac{\tau}{2} \tag{51}$$

Since ρ_i's in the solution given by Eqs. (7, 8)) are arbitrary and, hence, ρ here is also arbitrary. Now, let us replace ρ by $-\rho$ such that the above A_0 and $\vec{A}$ become

$$\begin{aligned} A_0^{\pm} &= \pm i\frac{1}{g}\frac{\rho}{(1-\frac{\rho}{r})r^2}\frac{\tau}{2}\cdot\hat{r} = \pm i\frac{1}{g}\frac{\rho}{r(r-\rho)}\frac{\tau}{2}\cdot\hat{r}, \\ \vec{A} &= -\frac{1}{g}\frac{\rho}{(1-\frac{\rho}{r})r^2}\hat{r}\times\frac{\tau}{2} = -\frac{1}{g}\frac{\rho}{r(r-\rho)}\hat{r}\times\frac{\tau}{2} \end{aligned} \tag{52}$$

Consider an interaction of Q_0 with an antiparticle or a particle aligned in the opposite direction, that is, $Q = \pm i\frac{2\pi}{g}$. Then,

$$QA_0^{\pm} = -\frac{2\pi}{g^2}\frac{\rho}{r(r-\rho)}\frac{\tau}{2}\cdot\hat{r} = V(r)\tau\cdot\hat{r}, \tag{53}$$

$$Q\vec{A} = \mp i\frac{2\pi}{g^2}\frac{\rho}{r(r-\rho)}\hat{r}\times\frac{\tau}{2} = \pm iV(r)\hat{r}\times\tau, \tag{54}$$

where $V(r)$ is

$$V(r) = -\frac{\pi}{g^2}\frac{\rho}{r(r-\rho)} \tag{55}$$

$V(r)$ is depicted in Fig. 1. This indicates that two interacting particles with charges $Q = \pm i\frac{2\pi}{g}$ and $Q_0 = \mp i\frac{2\pi}{g}$, respectively, may be confined between two infinite barriers at $r = 0$ and $r = \rho$. Then, ρ is no longer arbitrary and it is the maximum distance between two particles, that is, the diameter of a confining sphere. Therefore, if we call the sources quarks, the static solution presented here may provide a confining potential for quarks. In the accompanying manuscript, we employ the confining potential here in the Dirac equations for $q\bar{q}$ system [2].

It has already been shown that the solution given by Eqs. (7, 8, 9) closely resembles the BPST instanton solution [1]. So, it is natural to ask if the solution here has any topological invariant [3]. To answer this question we should convert our solution into the Euclidean solution given by Eq. (13) and, then, insert it into the formula in the textbook of Weinberg [3]. Calculations show that any topological invariant is undefinable. We know the BPST instanton solution results in $\frac{8\pi^2}{g^2}n$. On the other hand, the solution here yields ∞. Therefore, there is no topological invariant associated with the solution here.

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Appendix A: Calculation of $\nabla\cdot\vec{E}^{\pm}$ in Eq. (23)

$$\nabla\cdot\vec{E}^{\pm} = \pm\frac{i}{g}\{\nabla\cdot[\frac{2}{\Pi^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)]$$
$$-\nabla\cdot[\frac{3}{\Pi}\sum_{i}\frac{\rho_i}{|\vec{r}-\vec{r}_i|^5}(\vec{r}-\vec{r}_i)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)]$$

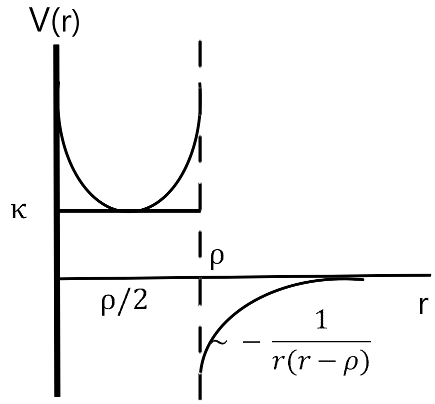

FIG. 1. $V(r) = -\frac{\pi}{g^2}\frac{\rho}{r(r-\rho)}, \quad \kappa = \frac{4\pi}{g^2\rho}$

$$
\begin{aligned}
&-(\nabla[\frac{1}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)]) \cdot \frac{\vec{\tau}}{2} \\
&+(\nabla[\frac{1}{\prod}\sum_{i}\frac{\rho_i}{|\vec{r}-\vec{r}_i|^3}]) \cdot \frac{\vec{\tau}}{2}\}
\end{aligned}
\tag{A1}
$$

To obtain the δ-function terms in Eq. (23), we calculate the first divergence term as follows.

$$
\begin{aligned}
&\nabla \cdot [\frac{2}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)] \\
&= \sum_{i,j}\{[\nabla(\frac{2}{\prod^2}\frac{\rho_i}{|\vec{r}-\vec{r}_i|^3}] \cdot [\rho_j\frac{(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_i|^3}]\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i) \\
&+(\frac{2}{\prod^2}\sum_{i,j}\frac{\rho_i}{|\vec{r}-\vec{r}_i|^3})[\nabla\cdot(\rho_j\frac{(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_i|^3})]\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i) \\
&+[\frac{2}{\prod^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)]\cdot[\nabla\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)]
\end{aligned}
\tag{A2}
$$

Using the formula:

$$
\nabla \cdot (\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}) = 4\pi\delta^3(\vec{r}-\vec{r}_i) \tag{A3}
$$

and straightforward calculations lead to

$$\nabla \cdot [\frac{2}{\Pi^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)]$$
$$= \{\frac{4}{\Pi^3}\sum_{i,j,k}\frac{\rho_i\rho_j\rho_k}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3|\vec{r}-\vec{r}_k|^3}(\vec{r}-\vec{r}_k)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)$$
$$-\frac{6}{\Pi^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^5|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_i)\cdot(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)$$
$$+\frac{2}{\Pi^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3}4\pi\delta^3(\vec{r}-\vec{r}_j)\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)$$
$$+\frac{2}{\Pi^2}\sum_{i,j}\frac{\rho_i\rho_j}{|\vec{r}-\vec{r}_i|^3|\vec{r}-\vec{r}_j|^3}\frac{\vec{\tau}}{2}\cdot(\vec{r}-\vec{r}_i)\} \tag{A4}$$

In Eq. (23), the other term gives rise to the δ-function is calculated in a similar manner. All remaining terms are evaluated straightforwardly, leading to Eq. (23).