

DPSE: Difference Power Space Extrapolation for π Computation in $O(\sqrt{n})$ Iterations

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Abstract

We present Difference Power Space Extrapolation (DPSE), a geometric method for computing the n decimal digits of π by combining multiple polygon observations in difference power space. The paper shows that DPSE should be understood as a higher-order development of the Difference-Power Cumulative Method (DPCM), rather than as an unrelated construction. For the polygon semiperimeter surrogate $S_k = n_k \sin(\pi/n_k)$, the geometric increments $\Delta S_k = S_{k+1} - S_k$ admit a structured boundary-layer expansion that can be cancelled by Richardson-type linear combinations. For fixed extrapolation order W , this yields the error law $E_{k,W} = O(4^{-(W+1)k})$, while the diagonal Neville sequence exhibits an approximately quadratic error exponent in the physical observation index k . Consequently, the required physical observation count for n correct decimal digits is reduced from linear growth to order $O(\sqrt{n})$, while preserving the low-memory character of geometric iteration. Numerical experiments include a direct million-digit fingerprint verification.

Keywords: π computation, Richardson extrapolation, difference power space, low-memory computation, Liu Hui, Zu Chongzhi

1 Introduction

1.1 Background

The computation of π has evolved along two broad routes: geometric methods and algebraic or analytic methods. The geometric route begins with Archimedes and the polygon method, and in the Chinese tradition was developed in remarkable detail by Liu Hui, annotated and transmitted by Li

Chunfeng, and pushed to higher accuracy by Zu Chongzhi [1, 2, 3, 4, 5, 6, 7]. Because the historical record preserves Zu Chongzhi’s final numerical result much more clearly than his detailed procedure, later scholars often assumed that he did not introduce a fundamentally new algorithm beyond Liu Hui, but instead carried the polygon computation farther with calculation tools such as counting rods or abacus-like devices. A modern reinterpretation of this situation suggests that the historically mentioned “difference-power method” (cha-mi fa) may correspond, in modern terms, to the Difference-Power Cumulative Method (DPCM) [8], namely a cumulative summation of polygon increments ΔS_k that improves numerical stability without changing the basic geometric orbit.

Pure polygon computation is structurally transparent, physically intuitive, and memory-light, but it suffers from linear convergence. As precision increases, the number of required polygon refinements grows essentially in direct proportion, which eventually made purely geometric computation too slow for very high precision. By contrast, the algebraic route, including Machin-type formulas, AGM methods [9, 10], and Chudnovsky-type series [11], is now the mainstream approach for large-scale π computation [12, 13, 14, 15, 16, 17], although pushing the precision frontier typically requires substantial memory and, at the largest scales, major computing resources.

This paper studies DPSE as a higher-order development of DPCM. The central idea is to treat the polygon sequence as repeated observations of the same limit and to combine the increments ΔS_k through structured extrapolation so as to cancel successive boundary-layer terms. In the Richardson-type realization analyzed here, this shifts the required physical observation count for n correct decimal digits from linear growth to order $\sqrt{\log_2(10)n}$, equivalently $O(\sqrt{n})$, while preserving the low-memory character of geometric iteration.

1.2 Definition of Symbols

To bridge the traditional geometric definitions (based on the number of sides n) with our algorithmic iteration, we introduce the iteration index $k \geq 0$. Let $n_k = 6 \cdot 2^k$ be the number of sides of the inscribed regular polygon at step k . As shown in Figure 1, let R be the radius of the circle ($R = 1$ in this study), l_k be the edge length of the n_k -gon, $L_k = n_k \cdot l_k$ be its perimeter, and denote by $S_k = L_k/2$ its semiperimeter surrogate for π . Let H_k be the distance from the circle’s center to the edge of the n_k -gon, and $\Delta H_k = R - H_k$ be the sagitta. The increment of the surrogate sequence

is denoted by $\Delta S_k = S_{k+1} - S_k$. For the algebraic iteration, we define the state variable $x_k = l_k^2$.

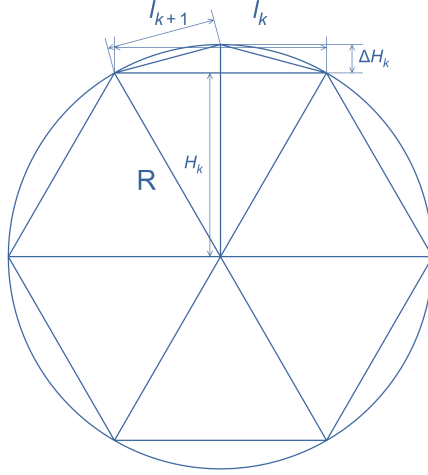


Figure 1: Definition of symbols for the inscribed polygon and circle.

1.3 Formal Definitions of DPCM and DPSE

DPCM (Difference-Power Cumulative Method) is the modern cumulative formulation described in [8], motivated by the historical interpretation that Zu Chongzhi may have used a difference-power idea. Instead of extracting π only from the last polygon state, DPCM reconstructs the semiperimeter surrogate cumulatively through the increments ΔS_k and writes

$$S_k = S_0 + \sum_{j=0}^{k-1} \Delta S_j. \quad (1.1)$$

This uses the whole observation history, avoids severe intermediate cancellation, and provides a cumulative implementation of the same geometric orbit.

DPSE (Difference Power Space Extrapolation) is the high-order extension of DPCM. Its key idea is to treat the polygon sequence as repeated observations of the same limit and to recombine them in difference power space. The Richardson filter [18, 19] studied here is one explicit realization. For fixed extrapolation order W it satisfies $E_{k,W} = O(4^{-(W+1)k})$;

the separate diagonal Neville sequence $\Pi_k = T_{k,k}$ then exhibits an approximately quadratic decimal error exponent in k , producing the observed super-exponential decay and an observation count of order $O(\sqrt{n})$ for n decimal digits.

1.4 Description of DPSE

The method starts with the semiperimeter surrogate of the inscribed regular hexagon ($S_0 = 3$) and iteratively doubles the number of sides. For the unit circle, the target value π is also the semiperimeter of the circle, so the polygon sequence can be compared directly with π . At the first nontrivial step, the classical polygon bracket is

$$S_1 < \pi < S_1 + \Delta S_0.$$

More generally, the doubled polygon sequence satisfies

$$S_{k+1} < \pi < S_{k+1} + \Delta S_k.$$

DPSE uses the sequence $\{S_k\}$ as a family of repeated observations of the same limit. Instead of keeping only the latest polygon, it combines several observations through the increment sequence ΔS_k , which isolates the new boundary layer created at each doubling step. In this sense the virtual estimate $\pi(k)$ in Fig. 2 is not a literal polygon but a synthesized limit estimate inside the classical geometric bracket.

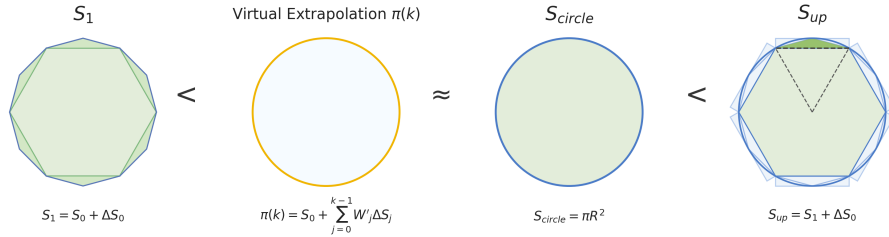


Figure 2: Geometric interpretation of DPSE in difference power space.

Note: In this figure, S_0 denotes the inscribed 6-gon state, and S_1 denotes the inscribed 12-gon state.

The basic geometric recurrence relations are:

$$l_0 = R = 1 \quad (1.2)$$

$$H_k = \sqrt{R^2 - (l_k/2)^2} \quad (1.3)$$

$$\Delta H_k = R - H_k \quad (1.4)$$

$$l_{k+1} = \sqrt{(l_k/2)^2 + (\Delta H_k)^2} \quad (1.5)$$

$$S_k = \frac{1}{2} n_k l_k \quad (1.6)$$

$$S_{k+1} = S_k + \Delta S_k \quad (1.7)$$

To avoid catastrophic cancellation as k grows large, we define a new state variable $x_k = l_k^2$, which represents the squared side length of the polygon. By substituting x_k into the geometric relations and eliminating the square roots involving differences, we obtain the un-squared algebraic recurrence:

$$x_{k+1} = \frac{x_k}{2 + \sqrt{4 - x_k}}, \quad x_0 = 1 \quad (1.8)$$

The semiperimeter at step k (which has $n_k = 6 \cdot 2^k$ sides) is given by $S_k = \frac{1}{2} n_k l_k = 3 \cdot 2^k \cdot \sqrt{x_k}$. Using the trigonometric identity $x_k = l_k^2 = (2R \sin(\pi/n_k))^2 = 4 \sin^2(\pi/n_k)$, we have $S_k = n_k \sin(\pi/n_k)$. The corresponding increment can be written without subtracting two nearby large values:

$$\Delta S_k = S_{k+1} - S_k = \frac{n_k x_{k+1}^2}{2(2\sqrt{x_{k+1}} + \sqrt{x_k})}. \quad (1.9)$$

The Taylor series expansion of $\sin(z)$ gives the explicit first terms

$$S_k = \pi - \frac{\pi^3}{6n_k^2} + \frac{\pi^5}{120n_k^4} - \frac{\pi^7}{5040n_k^6} + \cdots, \quad n_k = 6 \cdot 2^k,$$

where the error terms decay strictly as powers of $1/4$; the general form is the asymptotic expansion (2.4) below.

This exact algebraic structure of the error terms enables a Richardson-type extrapolation. For a sequence of $W + 1$ consecutive observations $S_k, S_{k+1}, \dots, S_{k+W}$, we consider an order- W linear estimate $\Pi_{k,W}$ whose coefficients cancel the first W modes in the expansion. The explicit weight formula and its transformation into difference power space are given later in Sections 3.2 and 3.3. At the structural level, this cancellation shifts the leading unresolved term to order $W + 1$, so for fixed W we obtain

$$\Pi_{k,W} = \pi + O\left(4^{-(W+1)k}\right), \quad (W \text{ fixed}). \quad (1.10)$$

This is the correct *fixed-order* Richardson estimate. Table 2 uses the diagonal values $\Pi_k = T_{k,k}$, so its entries are empirical diagonal errors rather than direct evaluations of a separate closed formula.

1.5 Residual Interpretation and Boundary Layers

Geometrically, the residual $R_k = \pi - S_k$ measures the remaining gap between the polygon semiperimeter surrogate and the circular limit. Traditional polygon methods rely on the final state alone; DPCM replaces this by cumulative recovery through ΔS_k ; DPSE further applies weighted combinations of these increments to cancel successive boundary-layer contributions. This is why the Richardson-type realization exhibits super-exponential error decay instead of the raw polygon method’s linear convergence.

2 Method Comparison

Liu Hui recorded a detailed polygon computation of π , and Li Chunfeng’s commentary preserves the historical line that later culminated in Zu Chongzhi’s sharper result. In the present paper, Liu’s two formulas are denoted Liu’s method 1 and Liu’s method 2. DPSE keeps the same geometric refinement orbit but changes how the observations are fused into a limit estimate. Table 1 summarizes the formula-level differences.

Table 1: Comparison of Liu’s methods, DPCM, and DPSE

Liu’s method 1	Liu’s method 2	DPCM (Zu Chongzhi?)	DPSE (This)
Common Geometric Recurrence (Steps 1-6)			
$l_0 = 1, \quad H_k = \sqrt{1 - (l_k/2)^2}, \quad \Delta H_k = 1 - H_k$ $l_{k+1} = \sqrt{(l_k/2)^2 + (\Delta H_k)^2}, \quad S_{k+1} = \frac{1}{2}n_{k+1} \times l_{k+1}$			
No correction	Empirical correction	Cumulative Method	Multi- observation Limit
$\pi \approx S_{k+1}$	$\pi \approx S_5 + \frac{36}{105} \Delta S_4$	$S_k =$ $S_0 + \sum_{j=0}^{k-1} \Delta S_j$	$\Pi_{k,W} =$ $S_k + \sum_{r=0}^{W-1} W'_r \cdot$ ΔS_{k+r}

Key differences. Liu’s method 1 keeps only the final semiperimeter and converges linearly without correction, while Liu’s method 2 applies a single empirical correction $\frac{36}{105} \Delta S_4$. DPCM replaces the last-state estimate with

a cumulative sum of the increments ΔS_k , avoiding intermediate precision loss without changing the linear convergence rate. DPSE goes further: it recombines the increments (or, equivalently, several observations) by a high-order filter, replacing linear convergence by the super-exponential diagonal error shown in Table 2.

For compactness, Table 2 retains only DPCM and DPSE. Its DPSE Error column reports the empirical diagonal error $|\Pi_k - \pi|$ with $\Pi_k = T_{k,k}$. The historical comparison with Liu’s method 1 and Liu’s method 2 is already summarized above and can be traced back to the reference discussion in [8].

Table 2: Compact comparison between DPCM and DPSE

Iter k	N-polygon	π (DPCM)	DPSE Error $ \Pi_k - \pi $
0	$n = 6$	-	-
1	$n = 12$	3.105828541	4.9×10^{-4}
2	$n = 24$	3.132628613	2.0×10^{-7}
3	$n = 48$	3.139350203	1.2×10^{-11}
4	$n = 96$	3.141031951	1.2×10^{-16}
5	$n = 192$	3.141452472	2.0×10^{-22}
9	$n = 3072$	3.141592106	1.2×10^{-52}
10	$n = 6144$	3.141592517	6.2×10^{-62}
11	$n = 12288$	3.141592619	6.7×10^{-72}

For practical computation, the extrapolation can be organized as a right-pointing Neville-type triangle. The basic polygon surrogates are placed in the first column, and each later entry is obtained from the two adjacent entries immediately to its left.

Let the first column be $T_{k,0} = S_k$. The next columns ($m \geq 1$) are calculated by:

$$T_{k,m} = T_{k,m-1} + \frac{T_{k,m-1} - T_{k-1,m-1}}{4^m - 1}. \quad (2.1)$$

As shown in Table 3, this recursion builds a right-pointing triangle. The final diagonal value at depth W is its tip, namely $\Pi_W = T_{W,W}$.

The first two columns are

$$T_{k,1} = S_k + \frac{S_k - S_{k-1}}{3}. \quad (2.2)$$

Table 3: The Extrapolation Triangle

Observation	$\mathbf{m} = 0$	$\mathbf{m} = 1$	$\mathbf{m} = 2$	$\mathbf{m} = 3$	$\cdots$	$\mathbf{m} = \mathbf{W}$
$k = 0$	$T_{0,0}$					
		$T_{1,1}$				
$k = 1$	$T_{1,0}$		$T_{2,2}$			
		$T_{2,1}$		$T_{3,3}$		
$k = 2$	$T_{2,0}$		$T_{3,2}$		$\ddots$	
		$T_{3,1}$		$\ddots$		$\mathbf{T}_{\mathbf{W},\mathbf{W}} = \mathbf{\Pi}_{\mathbf{W}}$
$k = 3$	$T_{3,0}$		$\ddots$			
$\vdots$	$\vdots$	$\ddots$				

and

$$T_{k,2} = T_{k,1} + \frac{T_{k,1} - T_{k-1,1}}{15}. \quad (2.3)$$

More generally, the divisor follows the pattern $4^m - 1$, giving the sequence 3, 15, 63, 255, $\dots$.

In arithmetic-cost terms, under the high-precision model used in our implementations, Liu Hui’s classical polygon iteration and DPCM both remain in an $O(n^2)$ regime for n decimal digits, and the corresponding physical observation depth grows linearly with n . By contrast, the Richardson-type DPSE realization considered here reduces the practical observation count to about $O(\sqrt{n})$; at the million-digit scale, the direct verified run below succeeds already at $k_{\max} = 1823$, exactly matching the rounded diagonal scaling-law prediction. Its synthesis stage is better described as $O(n \log n)$ arithmetic in the present implementation, so arithmetic cost and physical observation count should be understood as related but distinct notions.

2.1 Error Structure of Geometric Iterations

The foundation of DPSE lies in the strict algebraic structure of the error terms in the geometric polygon sequence.

The key asymptotic expansion is

$$S_k = \pi + \sum_{m=1}^{\infty} C_m 4^{-mk}, \quad C_m = (-1)^m \frac{\pi^{2m+1}}{(2m+1)! 6^{2m}}. \quad (2.4)$$

Proof: Using the trigonometric identity $x_k = 4 \sin^2(\theta_k)$ where $\theta_k = \pi/n_k$, we have $S_k = n_k \sin(\pi/n_k)$. Expanding the sine function in its Taylor series

and substituting $n_k = 6 \cdot 2^k$ gives

$$S_k = \pi + \sum_{m=1}^{\infty} (-1)^m \frac{\pi^{2m+1}}{(2m+1)! 6^{2m}} 4^{-mk}, \quad (2.5)$$

so the error is strictly composed of modes proportional to 4^{-mk} .

2.2 Construction of Analytical Weights

Using the expansion established above, the Richardson weights are determined by cancelling the first W error modes. For order W , let

$$\Pi_{k,W} = \sum_{j=0}^W w_j S_{k+j}, \quad \sum_{j=0}^W w_j = 1, \quad \sum_{j=0}^W w_j (4^{-j})^m = 0 \quad (m = 1, \dots, W). \quad (2.6)$$

This is a Vandermonde system with nodes 4^{-j} , so the weights admit the closed form

$$w_j = \frac{(-1)^{W-j} 4^{-(W-j)(W-j+1)/2}}{p_j p_{W-j}}, \quad \text{where } p_m = \prod_{i=1}^m \left(1 - \frac{1}{4^i}\right) \text{ and } p_0 = 1. \quad (2.7)$$

Thus the weight generation cost drops from generic $O(W^3)$ elimination to $O(W)$ arithmetic.

2.3 Transformation to Difference Power Space

The same extrapolant can be rewritten in increment form.

Let $\Pi_{k,W} = \sum_{j=0}^W w_j S_{k+j}$ be any linear limit extrapolant formed from $W+1$ consecutive polygon observations, with $\sum_{j=0}^W w_j = 1$. Then

$$\Pi_{k,W} = S_k + \sum_{r=0}^{W-1} W'_r \cdot \Delta S_{k+r}, \quad W'_r = \sum_{j=r+1}^W w_j. \quad (2.8)$$

Proof: Write each later semiperimeter as

$$S_{k+j} = S_k + \sum_{r=0}^{j-1} \Delta S_{k+r}. \quad (2.9)$$

Substituting this into $\Pi_{k,W} = \sum_{j=0}^W w_j S_{k+j}$, using $\sum_j w_j = 1$, and interchanging the order of summation yields the stated increment representation.

This form is the practical one used in DPSE. The direct formula for ΔS_k given earlier avoids large-number subtraction and preserves numerical stability.

2.4 Error Analysis and Complexity

For the coefficient choice that annihilates the first W terms of the expansion (2.4), the resulting fixed-order extrapolated estimate satisfies

$$E_{k,W} = \Pi_{k,W} - \pi = O\left(4^{-(W+1)k}\right), \quad k \rightarrow \infty, W \text{ fixed.} \quad (2.10)$$

Proof: After the first W error modes have been annihilated, the dominant unresolved contribution comes from the $(W+1)$ -st term of the expansion, namely $C_{W+1}4^{-(W+1)k}$. The Richardson coefficients depend on W but not on k , so for fixed W they are absorbed into a constant A_W . Therefore

$$\Pi_{k,W} - \pi = A_W 4^{-(W+1)k} + O\left(4^{-(W+2)k}\right),$$

which proves the estimate (2.10). The diagonal Neville sequence $\Pi_k = T_{k,k}$ used in Table 2 is a separate object and is analyzed below through (2.11)–(2.13).

2.5 Incremental Bit Gain Along the Diagonal

The diagonal orbit $\Pi_k = T_{k,k}$ is especially important in practice because it measures how much precision is gained when one more *physical* polygon observation is made and the extrapolation order is raised by the same amount.

With C_m as in (2.4), the leading diagonal error satisfies

$$\Pi_k - \pi \approx (-1)^k C_{k+1} 4^{-k(k+1)/2}. \quad (2.11)$$

Hence the effective bit count

$$B_k = -\log_2 |\Pi_k - \pi| \quad (2.12)$$

obeys

$$B_k \approx k(k+1) + \log_2 \left(\frac{(2k+3)! 6^{2k+2}}{\pi^{2k+3}} \right). \quad (2.13)$$

Sketch of proof. For the diagonal Richardson filter, the response polynomial on the error mode $x = 4^{-m}$ is

$$R_k(x) = \prod_{r=1}^k \frac{x - 4^{-r}}{1 - 4^{-r}}, \quad R_k(1) = 1, \quad R_k(4^{-r}) = 0 \quad (1 \leq r \leq k).$$

Therefore the first unresolved mode is $m = k + 1$, and

$$R_k(4^{-(k+1)}) = (-1)^k 4^{-k(k+1)/2}.$$

Substituting the explicit coefficient C_{k+1} gives the stated error law and the formula for B_k .

The incremental bit gain per new physical observation is then

$$\Delta B_k = B_{k+1} - B_k \approx 2k + 2 + \log_2 \left(\frac{36(2k+5)(2k+4)}{\pi^2} \right) = 2k + 2 \log_2 k + O(1). \quad (2.14)$$

This shows that the *marginal* information gained by each new polygon refinement increases slowly with k . In other words, later observations become more valuable after they are filtered through the full diagonal DPSE construction. This is one of the main conceptual differences between diagonal DPSE and many algebraic series algorithms, where the bit gain per additional term is approximately constant.

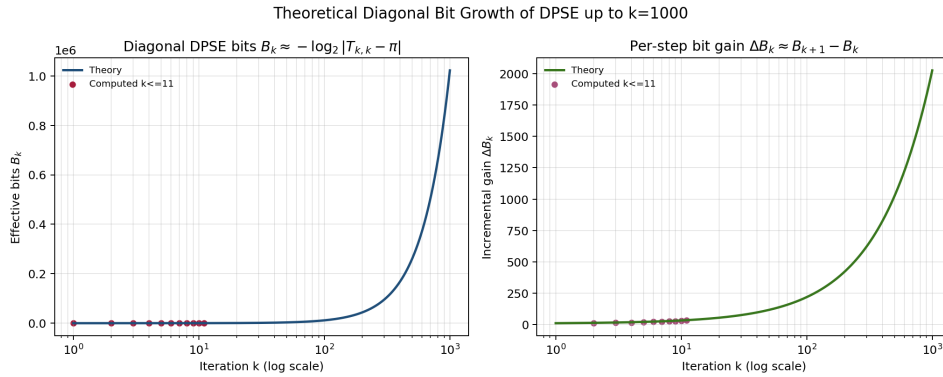


Figure 3: Theoretical diagonal bit-growth law of DPSE up to $k = 1000$ using a logarithmic horizontal scale. The left panel shows the total effective bits B_k , while the right panel shows the incremental gain $\Delta B_k = B_{k+1} - B_k$. The dots for $k \leq 11$ are computed from the exact Neville diagonal, and the curves show the theoretical continuation.

Convergence comparison. The fastest practical algebraic methods for high-precision π computation, especially Chudnovsky-type series combined with binary splitting, have a nearly fixed gain per added term: each new term contributes approximately a constant number of correct digits. AGM-type iterations likewise have a fixed convergence order per iteration.

DPSE is not asymptotically faster than these record-setting algebraic algorithms, but its behavior is qualitatively different from most geometric schemes: along the diagonal realization, the marginal gain $\Delta B_k = B_{k+1} - B_k$ is not constant and instead grows slowly with k . This gradually increasing return per observation is one of the distinctive advantages of DPSE within the geometric-computation route.

3 Numerical Experiments

3.1 Reproducible Reference Code

For direct reproducibility, we state the reference implementation in algorithmic form. The actual rerun in the project uses `gmpy2` with the GMP/MPFR backend, and the pseudocode below computes the same four-digit fingerprint at decimal positions 999,997–1,000,000.

Algorithm 1. Reference DPSE fingerprint computation

1. Set $\text{PREC} = 3,400,000$ and $W = 1823$.
2. Generate weights: $p_0 = 1$ and $p_m = p_{m-1}(1 - 4^{-m})$ for $m = 1, \dots, W$;
then set $w_j = \frac{(-1)^{W-j} 4^{-(W-j)(W-j+1)/2}}{p_j p_{W-j}}$ for $j = 0, \dots, W$.
3. Initialize $x = 1$, $\text{acc} = 0$, and $\text{pow2} = 1$.
4. For $j = 0, 1, \dots, W$ do:
 $s = 3 \text{ pow2} \sqrt{x}$;
 $\text{acc} = \text{acc} + w_j(s - 3)$;
 $x = \frac{x}{2 + \sqrt{4 - x}}$ and $\text{pow2} = 2 \text{ pow2}$.
5. Set $\pi_{\text{est}} = 3 + \text{acc}$, expand to at least 1,000,005 decimal places,
and output the four digits in positions 999,997–1,000,000.

3.2 Result

Running the implementation corresponding to Algorithm 1 prints 8151. In our rerun, this agrees with the gold-value script based on `gmpy2.const_pi()`, which gives the same decimal suffix 8151 at positions 999,997–1,000,000. An independent audit script in the project reproduces the same fingerprint again, so the million-digit result is verified by two separate computational paths. In this simplified reference implementation, the synthesis starts from S_0 and uses order $W = 1823$, that is, the observations $S_0, \dots, S_{1823}$.

3.3 Error Analyses and Comparison

For the Richardson-type realization analyzed in this paper, the fixed-order family satisfies $E_{k,W} = O(4^{-(W+1)k})$, while the diagonal Neville sequence $\Pi_k = T_{k,k}$ exhibits an empirical super-exponential decay in k . To keep the counting convention uniform, we use $k_{\max}$ to denote the deepest physical observation. Direct diagonal reruns give $-\log_{10} |T_{51,51} - \pi| = 995.09$ and $-\log_{10} |T_{52,52} - \pi| = 1031.02$, so the threshold of 1000 correct decimal digits is first crossed at $k_{\max} = 52$. At the million-digit scale, a direct diagonal run at $k_{\max} = 1823$ reproduces the decimal fingerprint **8151** at positions 999,997–1,000,000, confirming the rounded heuristic count in practice.

For n decimal digits we have $B \approx \log_2(10)n$, so the required physical observation count is on the order of $\sqrt{\log_2(10)n}$, equivalently $O(\sqrt{n})$. Table 4 lists rounded heuristic values from this diagonal scaling law. For the million-digit case, the rounded heuristic value $k_{\max} = 1823$ is also the directly verified observation depth reported in Algorithm 1. The simplified reference code above follows the same from- S_0 counting convention.

Table 4: Rounded heuristic diagonal observation counts from the scaling law

Precision (digits)	$\log_2(10)n$ (bits)	$\sqrt{\log_2(10)n}$	Rounded heuristic $k_{\max}$
10	33	5.8	6
1,000	3,322	58	58
100,000	332,193	576	576
1,000,000	3,321,928	1,822	1,823

4 Conclusions and Outlook

4.1 Conclusions

The numerical and theoretical results in this paper show that DPSE is an effective high-order extension of the classical polygon method. First, it accelerates convergence substantially: in the Richardson-type realization, the required physical observation count for n correct decimal digits is reduced from linear growth to a scale proportional to $\sqrt{\log_2(10)n}$, equivalently $O(\sqrt{n})$. Second, the method retains the low-memory character of geometric iteration, since the working state remains small and no large coefficient tables or long integer products are required during the basic orbit construction.

In a period dominated by algebraic algorithms, DPSE provides a different direction for the computation of π . It does not abandon the geo-

metric route, but further develops it from local correction into high-order structured synthesis. This viewpoint shows that geometric algorithms still contain unexplored room for acceleration when multiple observations are combined systematically. At the same time, DPSE also has a clear practical limitation: it relies on high-precision square-root evaluation and repeated computation of ΔS_k , so its actual running speed is still slower than record-oriented algebraic methods such as Chudnovsky-type implementations.

4.2 Outlook

More broadly, DPSE suggests a new computational principle for π : instead of relying on a single final approximation, one may collect multiple observations of variables related to π , whether independent or not, and then combine them through a structured synthesis to obtain higher-order, non-linearly increasing accuracy. In the present paper these observations come from polygon increments, but the idea itself should not be limited to circle partition. Similar observation-combination frameworks may exist for series, recurrences, integral transforms, or other approximation or measurement processes.

This opens several directions for future work. If a better synthesis rule can be found, then a geometric–algebraic hybrid scheme may eventually challenge, or even surpass, the current Chudnovsky standard in some precision ranges. If square-root evaluation or the direct computation of ΔS_k can be made substantially faster, then the practical speed of DPSE will also improve. In a broader sense, the same methodology may be useful not only for mathematical constants, but also for high-precision engineering observation, calibration, and measurement problems in which many correlated observations must be fused into a final estimate.

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