

A Fourier view of the Riemann hypothesis

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Abstract

An elementary proof of the Riemann hypothesis is provided using Fourier analysis on the alternating zeta function $\eta(s)$. The real series $|\eta(s)|^2$, whose zeros are equivalent to the non-trivial zeros, increases along $\Im(s)$ in the left half of the critical strip $0 < \Re(s) < 1/2$.

1. Introduction

The Riemann zeta function is intimately connected to the distribution of prime numbers through the Euler product:

$$(1.1) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

where p runs over all prime numbers and s is a complex variable. The zeta function is extended to the complex plane by analytical continuation, except for its simple pole at $s=1$. Its logarithmic derivative can be expressed as the Dirichlet series:

$$(1.2) \quad \varphi(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n)$ denotes the von Mangoldt function, which is nonzero precisely when n is a prime power. In contrast, the prime zeta function $P(s)$ is defined directly in terms of the primes:

$$(1.3) \quad P(s) = \sum_{\text{prime } p} \frac{1}{p^s}$$

For $0 < \Re(s) < 1$, the Zeta-function can be evaluated through the alternating zeta function $\eta(s)$, according to:

$$(1.4) \quad \eta(s) = (1-2^{1-s})\zeta(s)$$

This representation provides a convergent formulation in the critical strip. The zeta function is also related to $\zeta(1-s)$ through its functional equation:

$$(1.5) \quad \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

The non-trivial zeros of the zeta-function, which lie in the critical strip, $0 < \Re(s) < 1$, are believed to be deeply connected to the distribution of prime numbers. The Riemann hypothesis, first formulated by Bernhard Riemann in 1859[1], states that all non-trivial zeros lie on the critical line $\Re(s) = 1/2$. Despite extensive efforts to prove or disprove the hypothesis, it remains one of the most important open problems in mathematics [2–6].

2. Theorem

Lemma 2.1 The Riemann zeta function represents the Fourier transform of a one-dimensional weighted logarithmic series of all natural numbers.

Proof:

Let $f(x)$ represent a series of point mass at position set $\{x_i = \ln(n_i)\}$ with weight set $\{w_i = n_i^{-\beta}\}$ in one dimensional (1D) direct space $\mathbf{x}$ (β is a tuning parameter to adjust the weight):

$$(2.1) \quad f(x) = \sum_{n=1}^{\infty} n^{-\beta} \delta(x - \ln(n))$$

where δ is the Dirac delta function.

The Fourier transform of $f(x)$ is:

$$(2.2) \quad \begin{aligned} \zeta(k) &= \int_{-\infty}^{+\infty} \sum_{n=1}^{\infty} n^{-\beta} \delta(x - \ln(n)) e^{-2\pi i k x} dx = \sum_{n=1}^{\infty} n^{-\beta} \int_{-\infty}^{+\infty} \delta(x - \ln(n)) e^{-2\pi i k x} dx \\ &= \sum_{n=1}^{\infty} n^{-\beta} e^{-2\pi i k \ln(n)} = \sum_{n=1}^{\infty} n^{-\beta} n^{-2\pi i k} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \end{aligned}$$

where $s = \beta + 2\pi k i$ is a complex number as in (1.1), and k is the frequency variable in 1D k -space.

Property 2.1 Similarly, the alternating zeta function $\eta(k)$ represents the same series as $\zeta(k)$ but with different weight:

$$(2.3) \quad \eta(k) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\beta}} e^{-2\pi i k \ln(n)}$$

Also the Dirichlet series $\varphi(s)$ (1.2) represents the Fourier transform of the logarithmic series of the prime powers $\{x_i = \ln(q_i)\}$ with weight $\{w_i = \Lambda(q_i) q_i^{-\beta}\}$, and the prime zeta function $P(k)$ (1.3) represents the Fourier transform of the logarithmic series of all primes $\{x_i = \ln(p_i)\}$ with weight $\{w_i = p_i^{-\beta}\}$. Since $\eta(k)$ represents $\zeta(k)$ in the critical strip, we'll mainly use it in the following discussion.

Property 2.2 The frequency component set $\{k_i\}$ of $\zeta(s)$ in k-space is determined by the position set $\{x_i\}$ and is independent of the weighting parameter β . It is expected that $\zeta(k)$, $\eta(k)$, $P(k)$ and $\varphi(k)$ all share the same component set $\{k_i\}$ because they are all logarithmic series of the natural numbers and the logarithm of every natural number is a finite linear combination of the logarithms of the primes. Figure 1 shows 4 calculated Fourier transforms. It can be seen that $\zeta(k)$ and $\eta(k)$ have minima at the non-trivial zero positions marked by green dotted lines with $\{\gamma_i = 2\pi k_i = 14.13, 21.02, 25.01, \dots\}$, where $\varphi(k)$ and $P(k)$ have maxima at these positions. Therefore, the imaginary parts of the non-trivial zeros are the frequency components of $\zeta(k)$ and their positions can be determined from the minima of $\zeta(k)$ and $\eta(k)$, or the maxima of $P(k)$ and $\varphi(k)$ with different accuracy.

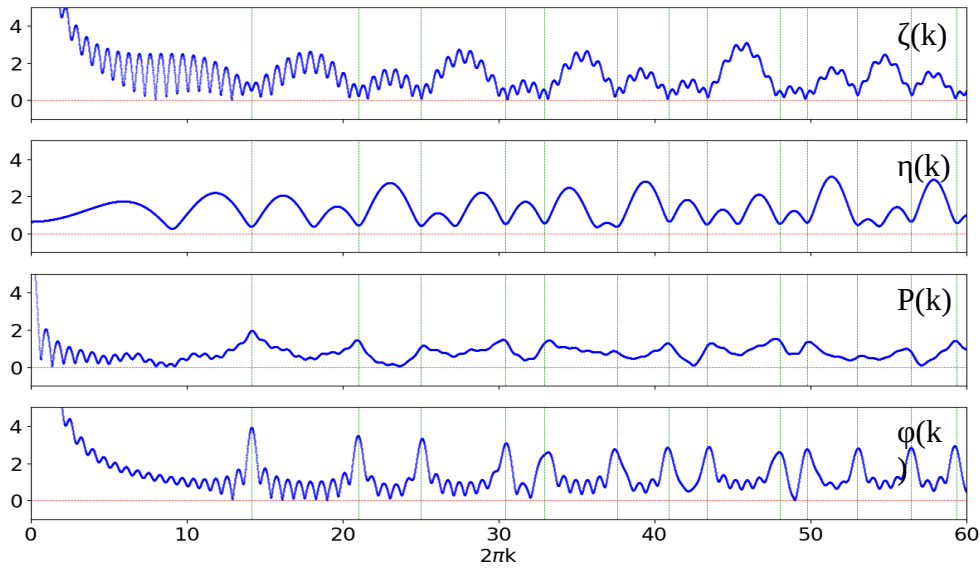


Figure 1. Fourier transforms of four logarithmic series of numbers with $\beta=0.75$. The frequency positions of the non-trivial zeros are marked with green lines. Vertical axis: amplitude of the spectrum.

Property 2.3 The weight set $\{w_i\}$ modulates the appearance or amplitudes of $\{k_i\}$ in k -space through the tuning parameter β . $\eta(k)$ at a different β represents a different view of the same component set $\{k_i\}$; therefore, the Fourier transform can be expressed as $\eta(k)|_\beta$ to reflect the role of β . For convenience, we use $\eta(s)$ in the following proof. The variation of β spans the 1D k -space into a complex plane with k running along the vertical imaginary axis and β on the real axis.

Theorem 2.1
$$\eta(s) = \sum_{k=1}^{\infty} \frac{(s)_k}{2^{s+k} k!} \zeta(s+k)$$

Proof:

The difference d_n between the odd-even pairs $(2n-1)^{-s}$, $(2n)^{-s}$ of $\eta(s)$ (2.3) is

$$\begin{aligned} d_n &= (2n-1)^{-s} - (2n)^{-s} = -(2n)^{-s} \left(1 - \left(1 - \frac{1}{2n}\right)^{-s}\right) \\ (2.4) \quad &= -\frac{1}{(2n)^s} \left(1 - \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \frac{1}{(2n)^k}\right) = \sum_{k=1}^{\infty} \frac{(s)_k}{k!} \frac{1}{(2n)^{s+k}} \end{aligned}$$

where binomial expansion is used and $(s)_k = s(s+1)\dots(s+k-1)$ with $(s)_0 = 1$. Then,

$$(2.5) \quad \eta(s) = \sum_{n=1}^{\infty} d_n = \sum_{k=1}^{\infty} \frac{(s)_k}{2^{s+k} k!} \zeta(s+k)$$

Theorem 2.2 Real series $|\eta(s)|^2$ increases with $\Im(s)$ in the left half of the critical strip $0 < \Re(s) < 1/2$.

Proof:

The intensity of $\eta(s=\beta+i\gamma)$ can be expressed as:

$$\begin{aligned} I(s) &= |\eta(s)|^2 \\ (2.6) \quad &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{(-1)^{n_1+n_2}}{(n_1 n_2)^\beta} \cos(\gamma \ln(\frac{n_2}{n_1})) \end{aligned}$$

$$(2.7) \quad = \sum_{n=1}^{\infty} \frac{1}{n^{2\beta}} + 2 \sum_{n_1=1}^{\infty} \sum_{n_2=n_1+1}^{\infty} \frac{(-1)^{n_1+n_2}}{(n_1 n_2)^\beta} \cos(\gamma \ln(\frac{n_2}{n_1}))$$

$$(2.8) \quad = I_e - I_o$$

where I_o is the sum of odd terms (n_1+n_2 is odd) and I_e is the sum of even terms including the diagonal sum in (2.7). Since $I(s)$ is non-negative, $I_e \geq I_o$.

A non-trivial zero occurs if and only if $I(s)=0$, so the even sum should equal to the odd sum:

$$(2.9) \quad \lim_{N \rightarrow \infty} \sum_{k=1}^{2k < N} \sum_{n=1}^{N-2k} \frac{\cos(\gamma \ln(\frac{n+2k}{n}))}{(n(n+2k))^\beta} + \frac{1}{2} \sum_{n=1}^N \frac{1}{n^{2\beta}} = \lim_{N \rightarrow \infty} \sum_{k=1}^{2k-1 < N} \sum_{n=1}^{N-2k+1} \frac{\cos(\gamma \ln(\frac{n+2k-1}{n}))}{(n(n+2k-1))^\beta}$$

Since the frequency components $\{\gamma_i\}$ exist for all β values in the critical strip $(0, 1/2]$ and they can be determined independent of β , this constrain is mainly on β and it determines the optimal β value that turns the normal minima at positions $\{\gamma_i\}$ to zeros.

Figure 2 shows 6 $I(s)$ spectra at $\beta=0.0, 0.1, 0.2, 0.3, 0.4$ and 0.5 . It is apparent that for $\beta < 1/2$, the gap $\Delta = I_e - I_o$ increases with γ and attenuates with β , and it oscillates in between two envelopes.

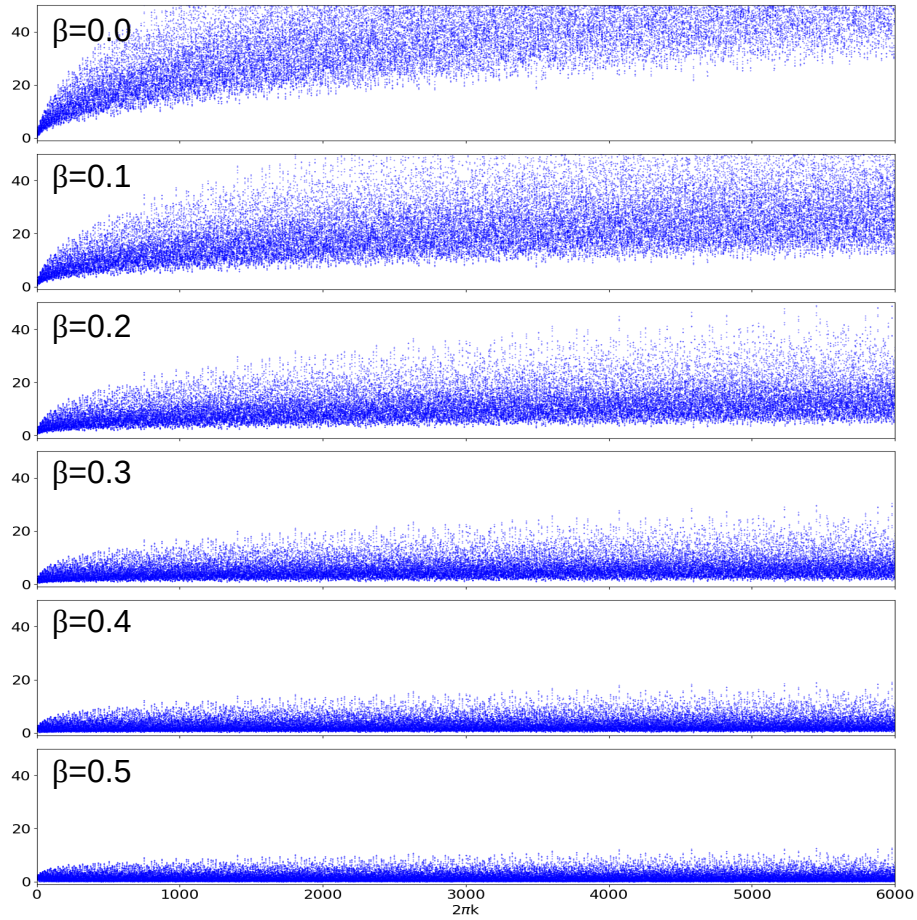


Figure 2. Amplitude spectra of $\eta(s=\beta+i\gamma)$ show the increase of $I_e - I_o$ along imaginary axis ($\gamma=2\pi k$) and attenuation with β . Vertical axis: amplitude $\sqrt{I(s)}$. The upper and lower envelopes of the spectra both appear linear in $I(s)$.

We also calculated spectra with β close to 0.5. The mean density of the spectra (partial sum with $N=500,000$) for β (0, 0.6) in three intervals of γ are listed in the following table.

$\gamma \setminus \beta$	0.0	0.1	0.2	0.3	0.4	0.49	0.5	0.51	0.6
[2000, 2100]	812.43	277.71	99.01	37.75	15.96	8.31	7.79	7.31	4.49
[4000,4100]	1623.24	482.45	149.77	50.01	18.8	9.02	8.4	7.84	4.6
[6000,6100]	2457.72	683.01	199.29	63	22.62	10.49	9.73	9.04	5.16

Both envelopes appear linear in the intensity spectra $I(s)$. The lower envelope can be estimated from the minima of the intensity spectra in each interval $\Delta\gamma=10$ of γ from 10. Assuming linearity and zero intercept, the estimated slopes are listed below and the lowest is $8.3405e^{-7}$ at $\beta=0.5$. For every γ ,

β	0.0	0.1	0.2	0.3	0.4	0.49	0.5	0.51	0.6
slope	0.1298	0.021	$2.7e^{-3}$	$2.47e^{-4}$	$1.01e^{-5}$	$8.59e^{-7}$	$8.34e^{-7}$	$8.56e^{-7}$	$2.81e^{-6}$

the intensity at $\beta=0.5$ is the lowest, so there is no zero detected for $\beta<1/2$ for γ values up to 6100.

The above direct evidence perfectly supports the Riemann hypothesis and is a result of Theorem

2.1. Each term of $\eta(s)$ in (2.5) generates a spectrum of rising background along $\Im(s)=\gamma$:

$$(2.10) \quad \eta(s) = \frac{\beta+i\gamma}{2^{s+1}} [\zeta(s+1) + \frac{s+1}{4}\zeta(s+2) + \frac{(s+1)(s+2)}{24}\zeta(s+3) + \dots]$$

For example, the first term $\zeta(s+1)$ is convergent and has no zero for $0<\beta=\Re(s)$ [2], the intensity of $\zeta(s+1)$ at all γ positions including the non-zero minima is magnified by $\sim\gamma^2$:

$$(2.11) \quad r_1 = \left| \frac{\beta+i\gamma}{2^{s+1}} \right|^2 \approx \frac{\gamma^2}{2^{2\beta+2}}$$

which results in a rising background. In fact, term $\zeta(s+k)$ is convergent and zero-free for $1-k<\beta$. Non-trivial zeros occur only when all these spectra become coherent for a specific β . Therefore, $I(s)$ has a generally increasing background along γ . For a given $\beta \in (0, \frac{1}{2})$, as long as $\eta(s)$ is zero-free in a lower range of γ , for example, $10 < \gamma \leq \gamma_0 = 100$ (which covers the first 29 frequency components), it will be zero-free forward for $\gamma > \gamma_0$.

Because of the increase of $I_e - I_o$ background along γ for $0 < \beta < 1/2$, and considering the fact that there is no zero ($I_e - I_o > 0$) below $\gamma = (6000, 6100)$, we can conclude that there is no non-trivial zero forward for $\gamma > 6100$ in the left half of the critical strip $0 < \beta < 1/2$; and according to the functional equation (1.5), no zero can exist in the other half of the critical strip $1/2 < \Re(s) < 1$. Therefore, the non-trivial zeros have to lie on the critical line $\Re(s) = 1/2$. Figure 3 shows how precisely the non-trivial zeros land on the critical line.

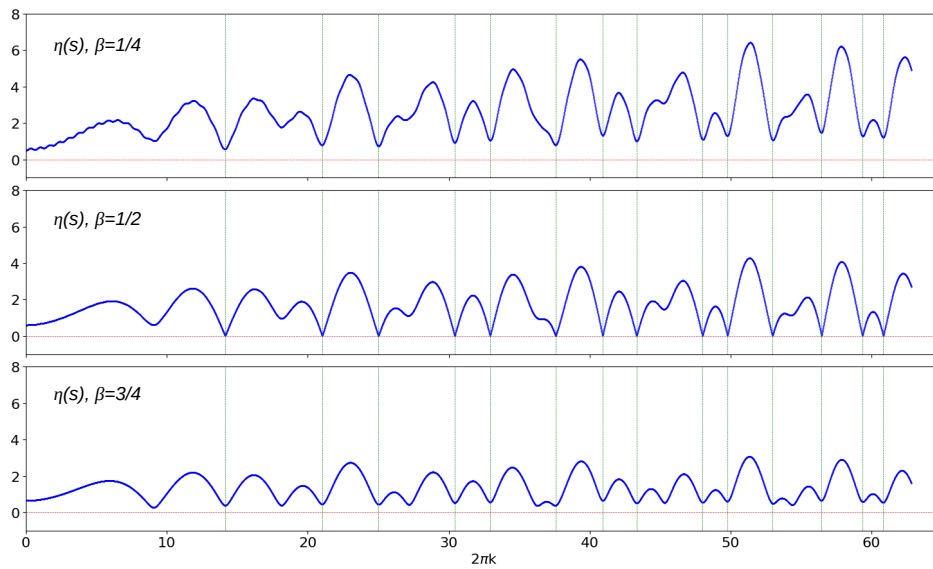


Figure 3. Amplitude spectra of $\eta(s)$ at $\beta = 1/4, 1/2$ and $3/4$. All non-trivial zeros have a sharp ending on the critical line.

Note 2.1 The non-trivial zero set $\{\gamma_i = 2\pi k_i\}$ represents the characteristic frequencies of the logarithmic series of the natural number system and $\{\lambda_i = 2\pi/\gamma_i\}$ are the wavelengths that represent the intrinsic gaps of the series. The first 5 wavelengths $\{0.44452122, 0.29888562, 0.25121831, 0.20651474, 0.19077497\}$ and the first 5 gaps of the prime logs $\{0.40546511, 0.51082562, 0.33647224, 0.45198512, 0.16705408\}$ are on the same scale, where the largest gap 0.51082562 is from the first twin primes 3 and 5.

References

- [1] Riemann, B. (1859). *Über die Anzahl der Primzahlen unter einer gegebenen Größe*. Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 671–680.
- [2] E. C. Titchmarsh and D. R. Heath-Brown(1988). The Theory of the Riemann Zeta-function, Oxford University Press.
- [3] H. M. Edwards (1974). Riemann's Zeta Function, Academic Press.
- [4] Spigler, R. (2025). A Brief Survey on the Riemann Hypothesis and Some Attempts to Prove It. *Symmetry*, 17(2), 225.
- [5] Jailton C. Ferreira (2019). The non-trivial zeros of Riemann's zeta-function, 2019, arXiv: math/0405531v21.
- [6] Michael Shaughnessy (2026). Quasicrystal Scattering and the Riemann Hypothesis, arXiv:2410.03673v11.