

# Remarks on the Non-Relativistic Bohr-Sommerfeld Model with Corrected Orbital Angular Momentum

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## Abstract

Since the 1930s, a correction to the non-relativistic Bohr-Sommerfeld model of hydrogen-like atoms has been known, which aligns many predictions of this model with the predictions of the non-relativistic spinless Schrödinger equation for hydrogen-like atoms. In particular, the orbital angular momentum of electronic orbits in the corrected Bohr-Sommerfeld model match the orbital angular momentum of the corresponding single-electron wave functions that solve the mentioned Schrödinger equation. However, the only known justification for this correction has been the improvement of the predictions of the Bohr-Sommerfeld model. The present remarks revisit this correction in the context of a recently proposed vibrational interpretation of the Schrödinger equation, which provides a physical reason for this correction. Furthermore, numerical results are presented that support a previously hypothesized relation between nodal patterns in single-electron wave functions and corresponding de Broglie-wave-analyzed orbits provided that the mentioned correction is employed.

## 1 Introduction

While the Bohr model of hydrogen-like atoms is limited to circular electronic orbits of quantized radii corresponding to integer values of the principal quantum number  $n$ , the Bohr-Sommerfeld model includes elliptic electronic orbits of quantized eccentricity corresponding to integer values of the azimuthal quantum number  $l$  for each value of the principal quantum number  $n$ . The original Bohr-Sommerfeld model assumes that the allowed integer values of  $l$  are 1, 2, ...,  $n$ .

While the “old quantum theory” in the form of the Bohr-Sommerfeld model was the first theory to describe the fine structure of energy levels of hydrogen-like atoms, it failed to describe many other features of hydrogen-like atoms and more complex atoms. These shortcomings contributed to the rapid rise of quantum mechanics. In the quantum mechanical model of a non-relativistic electron in a hydrogen-like atom, the allowed integer values of  $l$  are 0, 1, ...,  $n - 1$  and the eigenvalues of the squared orbital angular momentum operator are  $l(l + 1)\hbar^2$ . Employing these values of  $l$  and setting the orbital angular momentum to  $\sqrt{l(l + 1)}\hbar$  in the Bohr-Sommerfeld model not only brings its predictions closer to quantum mechanical predictions but also closer to experimental results [PW35, Pau60, Buc08]. In this sense, the orbital angular momentum of orbits in the Bohr-Sommerfeld model are “corrected” by these modifications. However, the old quantum theory could not provide a justification for these modifications in terms of a physical model. More recently, Bucher hypothesized that a more careful quantization of the three-dimensional hydrogen problem could explain the need for the  $\sqrt{l(l + 1)}$  correction, but he was not able to achieve it [Buc08, Buc14].

While a different quantization might resolve the issue, Section 2 shows how the  $\sqrt{l(l + 1)}$  correction follows naturally from the role that electronic orbits of the non-relativistic Bohr-Sommerfeld model play in a recently proposed vibrational interpretation of the Schrödinger equation [Kra26b]. One consequence of this role is a close relation between de Broglie-wave-analyzed electronic orbits and nodal patterns of single-electron wave functions, which was hypothesized previously by Littman et al. [LGZA23]. Section 3 evaluates this relation numerically and finds that the  $\sqrt{l(l + 1)}$  correction improves the relation for all analyzed orbits. Section 4 presents concluding remarks.

## 2 Orbits of Electrons in a Vibrational Interpretation of the Schrödinger Equation

One vibrational interpretation of the non-relativistic spinless Schrödinger equation for hydrogen-like atoms posits that stationary wave functions describe high-frequency stationary vibrational modes of the electromagnetic field around the nucleus, which are driven by the internal clock of the orbiting electron [Kra26b]. In this interpretation, Einstein's quantization condition [Ein17] implies that after each orbital period, the oscillating electromagnetic field and the electron's internal clock return to the same phase difference [Kra25]. However, this quantization condition does not determine the orbital angular momentum of the electron for a given energy level. Instead, the orbital angular momenta of the stationary vibrational modes that solve the Schrödinger equation in three dimensions are constrained to  $\sqrt{l(l+1)}\hbar$  with positive integer values of  $l < n$  as is well known in quantum mechanics. (The situation is different in two dimensions [Kra26a].)

If an electron starts with a different orbital angular momentum, it causes a non-stationary vibration of the electromagnetic field. Once this vibration settles into a stationary vibrational mode, its orbital angular momentum is equal to  $\sqrt{l(l+1)}\hbar$  for a positive integer value  $l$ , and the orbiting electron has to have the same orbital angular momentum; otherwise the electromagnetic field could not have settled into a stationary vibrational mode.

Thus, the mentioned vibrational interpretation suggests a physical reason why the orbital angular momentum of an electronic orbit is given by the orbital angular momentum of the corresponding single-electron wave function, which justifies the  $\sqrt{l(l+1)}$  correction. The next section presents numerical evidence that the de Broglie wavelength of orbiting electrons is closely related to nodal patterns of certain wave functions if the  $\sqrt{l(l+1)}$  correction is applied.

## 3 De Broglie-Wave-Analyzed Orbits of Electrons and Nodal Patterns of Wave Functions

One effect of the  $\sqrt{l(l+1)}$  correction is a change of the electron's linear momentum  $p$  and, therefore, of its de Broglie wavelength. Littman et al. have hypothesized that the de Broglie wavelength is closely related to the nodal pattern of single-electron wave functions [LGZA23]. If this is true, then the  $\sqrt{l(l+1)}$  correction should strengthen this relation.

In general, the structure of single-electron wave functions is a superposition of phase waves on infinitely many elliptic electronic orbits [Kra26a]. Therefore, a few very special cases were chosen, which can be analyzed more easily than the general case. Specifically, single-electron wave functions of the non-relativistic spinless hydrogen problem for azimuthal quantum number  $l > 0$  and magnetic quantum number  $m = 0$  were chosen because they represent standing waves with non-zero orbital angular momentum. For a fixed orbital plane parallel to the quantization axis, i.e. the  $z$ -axis, a standing wave on this plane can be considered a superposition of two two-dimensional wave functions of opposite orbital angular momentum [Kra26a]. Each of these two-dimensional wave functions can be considered a superposition of infinitely many phase waves (of the same orbital angular momentum) on elliptic orbits in the same orbital plane that are generated by apsidal precession [Kra26a]. However, at the periapsis of any of these orbits, the main contribution to the wave function is provided by the phase wave of a single orbit since none of the other orbits passes through the same periapsis. In a first approximation, we assume that it is sufficient to consider this single orbit and the orbit with the opposite orbital angular momentum for all points close to the periapsis (specifically within a quarter of the local wavelength of the standing wave). For convenience, we choose the orbits with the periapsis on the negative  $z$ -axis and the apoapsis on the positive  $z$ -axis. As mentioned, the two phase waves of opposite orbital angular momentum form a standing wave. In the standard form of hydrogen wave functions, there is an anti-node of the wave function on the negative  $z$ -axis (i.e. at the periapsis) and two nodes to either side on the orbits. Since these nodes are part of a standing wave, the arc length between them is assumed to correspond to half the de Broglie wavelength. This is our estimate of the de Broglie wavelength based on the nodal patterns of hydrogen wave functions.

Usually, the de Broglie wavelength is computed as  $h/p$  with the Planck constant  $h$  [dB25]. However, the electron's linear momentum  $p$  is rapidly changing at the periapsis of an elliptic orbit, which complicates the computation of an effective de Broglie wavelength forming the nodal pattern of a

standing wave. Alternatively, the de Broglie wavelength can be numerically computed from the integrated phase difference between the time-dilated internal clock of the electron and an electromagnetic field oscillating at an averaged Compton frequency with relativistically increased mass [Kra25, eq. (1)]. This computation was employed to estimate the effective de Broglie wavelength between the two mentioned nodes next to the periapsis based on the relativistic effects that cause the de Broglie wavelength [dB25].

Results for electronic orbits with principal quantum number  $n \leq 5$ , azimuthal quantum number  $l > 0$  and magnetic quantum number  $m = 0$  show that in some cases ( $n = 2, l = 1, m = 0$  and  $n = 3, l = 2, m = 0$ ) the  $\sqrt{l(l+1)}$  correction leads to a very good match between the two estimates of the de Broglie wavelength (deviations of less than 0.01%), while for all cases with  $n \leq 5, 0 < l \leq 4$  and  $m = 0$ , the  $\sqrt{l(l+1)}$  correction reduces the difference between the two estimates (compared to computations without the  $\sqrt{l(l+1)}$  correction) with deviations of less than 15%.

These results support the relation between the de Broglie wavelength and nodal patterns of wave functions hypothesized by Littman et al. [LGZA23]. They also show that the  $\sqrt{l(l+1)}$  correction improves electronic orbits of the Bohr-Sommerfeld model in a way that has not been discussed in the literature so far [PW35, Pau60, Buc08]. What these results do not show is a physical reason why the  $\sqrt{l(l+1)}$  correction might work, which is the topic of Section 2.

## 4 Conclusion

As discussed by Bucher [Buc08], the old quantum theory probably has not received as much attention as it deserves. Given this historic context, it is not surprising that even after more than 100 years, there are still new features to be discovered. With 100 years of new experimental results, new quantum theories, and new computational techniques, there probably has never been a better time to research the old quantum theory.

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## A Revisions

- Original version submitted to vixra.org on October 25, 2026.