# Special Relativistic Projectile Motion: Constant Laboratory Force versus Constant Proper Acceleration

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**Abstract**

We formulate the special relativistic generalization of Newtonian projectile motion in three spatial dimensions. Two models are considered: (i) a constant laboratory force, and (ii) a constant proper acceleration, i.e. hyperbolic motion. For the constant-force model we derive closed-form expressions for the coordinates, velocity, four-momentum, energy, and four-force as functions of the laboratory coordinate time. For the constant-proper-acceleration model we derive the worldline parametrized by proper time, together with the four-velocity, four-acceleration, four-momentum, energy, and four-force. We then establish the precise relation between the two models (Sec. 3.6): both have constant proper acceleration, but they describe the same worldline only for one-dimensional motion with matched initial conditions; in general the constant-force model has proper acceleration $\alpha = g\sqrt{1 + p_\perp^2/(m^2c^2)}$ and a rotating four-acceleration direction. We develop the four-dimensional Lorentz transformation in full generality, including arbitrary-direction boosts, the transformation of time and spatial coordinates, and the transformation of four-vectors. Finally, we demonstrate the Lorentz covariance of Maxwell's equations and of the four-dimensional Lorentz force in manifestly tensor form. In both projectile models the nonrelativistic limit recovers the standard parabolic trajectory. We discuss the relevant conservation laws and emphasize that a true gravitational field requires general relativity, while the present treatment corresponds to a uniform external force or a local uniform-field approximation.

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# 1 Introduction

Newtonian projectile motion under a uniform gravitational field is described by the familiar parabolic trajectory

$$x(t) = x_0 + v_{0x}t, \qquad y(t) = y_0 + v_{0y}t - \tfrac{1}{2}gt^2, \tag{1}$$

where $g$ is the constant acceleration of gravity. In special relativity, however, the velocity of a material particle cannot exceed the speed of light, and the relation between force and acceleration is modified. A naive replacement $v_y = v_{0y} - gt$ can lead to superluminal speeds for sufficiently large $t$ and is therefore inconsistent with special relativity.

In this paper we study two consistent relativistic generalizations of (1). The first is a *constant laboratory force* model, in which the force $\mathbf{F} = (0, -mg, 0)$ is constant in a given inertial frame. The second is a *constant proper acceleration* model, in which the magnitude of the four-acceleration is constant. The latter is the standard special relativistic uniformly accelerated motion, often called hyperbolic motion.

We derive the coordinate trajectories, coordinate velocities, four-velocities, four-momenta, energies, and four-forces for both models. We then develop the full four-dimensional Lorentz transformation, including arbitrary-direction boosts and the transformations of time, coordinates, and four-vectors. We also demonstrate the Lorentz covariance of Maxwell's equations and of the four-dimensional Lorentz force. Throughout, we use the metric convention

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \tag{2}$$

and units in which the speed of light is denoted by $c$. The vertical direction is chosen as $-y$; the horizontal directions are $x$ and $z$.

# 2 Constant laboratory force model

## 2.1 Equations of motion

Consider a particle of mass $m$ subject to the constant laboratory force

$$\mathbf{F} = (0, -mg, 0). \tag{3}$$

The relativistic equation of motion is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \qquad \mathbf{p} = \gamma m\mathbf{v}, \qquad \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}. \tag{4}$$

In components,

$$\frac{d}{dt}(\gamma m v_x) = 0, \qquad \frac{d}{dt}(\gamma m v_y) = -mg, \qquad \frac{d}{dt}(\gamma m v_z) = 0. \tag{5}$$

Thus

$$p_x = mA = \text{const}, \qquad p_y = m(B - gt), \qquad p_z = mC = \text{const}, \tag{6}$$

where the initial data are encoded by

$$A = \gamma_0 v_{0x}, \qquad B = \gamma_0 v_{0y}, \qquad C = \gamma_0 v_{0z}, \tag{7}$$

with

$$\gamma_0 = \frac{1}{\sqrt{1 - (v_{0x}^2 + v_{0y}^2 + v_{0z}^2)/c^2}}. \tag{8}$$

The energy is

$$E(t) = mc^2\sqrt{1 + \frac{A^2 + (B - gt)^2 + C^2}{c^2}}. \tag{9}$$

### 2.2 Coordinates

Introducing

$$D = c^2 + A^2 + C^2, \tag{10}$$

the worldline as a function of the laboratory time $t$ is

$$x(t) = x_0 + \frac{Ac}{g}\left[\operatorname{arsinh}\frac{B}{\sqrt{D}} - \operatorname{arsinh}\frac{B - gt}{\sqrt{D}}\right], \tag{11}$$

$$y(t) = y_0 + \frac{c}{g}\left[\sqrt{D + B^2} - \sqrt{D + (B - gt)^2}\right], \tag{12}$$

$$z(t) = z_0 + \frac{Cc}{g}\left[\operatorname{arsinh}\frac{B}{\sqrt{D}} - \operatorname{arsinh}\frac{B - gt}{\sqrt{D}}\right]. \tag{13}$$

Here $t$ is the laboratory coordinate time, not the proper time.

### 2.3 Velocities

Differentiating (11)–(13) with respect to $t$, or using $v_i = p_i c^2/E$, gives

$$v_x(t) = \frac{Ac}{\sqrt{D + (B - gt)^2}}, \tag{14}$$

$$v_y(t) = \frac{(B - gt)c}{\sqrt{D + (B - gt)^2}}, \tag{15}$$

$$v_z(t) = \frac{Cc}{\sqrt{D + (B - gt)^2}}. \tag{16}$$

At $t = 0$ these reduce to $v_x(0) = v_{0x}$, $v_y(0) = v_{0y}$, and $v_z(0) = v_{0z}$. Moreover, for all $t$,

$$v_x^2 + v_y^2 + v_z^2 < c^2. \tag{17}$$

### 2.4 Four-momentum and four-force

The four-velocity is

$$u^\mu = \gamma(c, v_x, v_y, v_z), \tag{18}$$

and the four-momentum is $p^\mu = mu^\mu$. Using (6) and (9), we obtain

$$p^\mu(t) = m\left(\sqrt{c^2 + A^2 + (B - gt)^2 + C^2},\, A,\, B - gt,\, C\right). \tag{19}$$

Equivalently,

$$p^0 = \frac{E}{c}, \qquad p^1 = p_x = mA, \qquad p^2 = p_y = m(B - gt), \qquad p^3 = p_z = mC. \tag{20}$$

The four-force is

$$K^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt}, \tag{21}$$

where $\gamma = E/(mc^2)$. A short calculation gives

$$K^\mu = \gamma\left(-\frac{mgv_y}{c},\, 0,\, -mg,\, 0\right). \tag{22}$$

### 2.5 Conservation laws

Since there is no force in the $x$ and $z$ directions,

$$p_x = \text{const}, \qquad p_z = \text{const}. \tag{23}$$

The energy is not conserved by itself because the external force does work. However,

$$\frac{dE}{dt} = -mgv_y = -mg\frac{dy}{dt}, \tag{24}$$

so the quantity

$$E(t) + mgy(t) = \text{const} \tag{25}$$

is conserved. This is the relativistic analogue of the sum of kinetic and potential energy in a uniform field.

## 3 Constant proper acceleration model

### 3.1 Four-acceleration and hyperbolic motion

We now consider the special relativistic uniformly accelerated motion, defined by a constant proper acceleration $a_0 > 0$. The four-velocity $u^\mu$ and four-acceleration $A^\mu$ satisfy

$$u^\mu u_\mu = c^2, \qquad A^\mu u_\mu = 0, \qquad A^\mu A_\mu = -a_0^2. \tag{26}$$

Let the initial four-velocity be

$$u_0^\mu = \gamma_0(c, v_{0x}, v_{0y}, v_{0z}), \tag{27}$$

with $\gamma_0$ defined as before. Let $e^\mu$ be a unit spacelike four-vector orthogonal to $u_0^\mu$,

$$e^\mu e_\mu = -1, \qquad e^\mu u_{0\mu} = 0, \tag{28}$$

chosen so that the proper acceleration is in the $-y$ direction. Then the solution of (26) is

$$u^\mu(\tau) = u_0^\mu \cosh\frac{a_0\tau}{c} + c\, e^\mu \sinh\frac{a_0\tau}{c}, \tag{29}$$

$$X^\mu(\tau) = X_0^\mu + \frac{c}{a_0}u_0^\mu \sinh\frac{a_0\tau}{c} + \frac{c^2}{a_0}e^\mu\left(\cosh\frac{a_0\tau}{c} - 1\right), \tag{30}$$

where $\tau$ is the proper time.

### 3.2 The spacelike vector $e^\mu$

For $v_0 \neq 0$, the components of $e^\mu$ are

$$e^0 = -\frac{\gamma_0 v_{0y}}{c}, \tag{31}$$

$$e_x = -(\gamma_0 - 1)\frac{v_{0y}v_{0x}}{v_0^2}, \tag{32}$$

$$e_y = -1 - (\gamma_0 - 1)\frac{v_{0y}^2}{v_0^2}, \tag{33}$$

$$e_z = -(\gamma_0 - 1)\frac{v_{0y}v_{0z}}{v_0^2}, \tag{34}$$

where $v_0^2 = v_{0x}^2 + v_{0y}^2 + v_{0z}^2$. For $v_0 = 0$, one simply has

$$e^\mu = (0, 0, -1, 0). \tag{35}$$

**Notation.** Throughout Secs. 3 and 4 we write the contravariant components of $e^\mu$ as $e^\mu = (e^0, e_x, e_y, e_z)$, i.e. $e_x \equiv e^1$, $e_y \equiv e^2$, $e_z \equiv e^3$. This is a notational shorthand; it is *not* the covariant spatial component. With the metric (2), the covariant components are $e_1 = -e_x$, $e_2 = -e_y$, $e_3 = -e_z$.

The vector $e^\mu$ has a direct kinematic meaning: it is the image, under the boost that takes the laboratory frame $S$ to the initial rest frame $S_0$ of the particle, of the spatial unit vector $(0, 0, -1, 0)$. Equivalently, $e^\mu$ is the $-y$ axis of the initial rest frame expressed in the laboratory frame; it is *not* the laboratory $-y$ direction.

### 3.3 Coordinates

Let

$$\theta = \frac{a_0 \tau}{c}. \tag{36}$$

Writing $u_0^0 = \gamma_0 c$, $u_0^x = \gamma_0 v_{0x}$, $u_0^y = \gamma_0 v_{0y}$, and $u_0^z = \gamma_0 v_{0z}$, the worldline (30) becomes

$$t(\tau) = t_0 + \frac{u_0^0}{a_0}\sinh\theta + \frac{c}{a_0}e^0(\cosh\theta - 1), \tag{37}$$

$$x(\tau) = x_0 + \frac{c}{a_0}u_0^x \sinh\theta + \frac{c^2}{a_0}e_x(\cosh\theta - 1), \tag{38}$$

$$y(\tau) = y_0 + \frac{c}{a_0}u_0^y \sinh\theta + \frac{c^2}{a_0}e_y(\cosh\theta - 1), \tag{39}$$

$$z(\tau) = z_0 + \frac{c}{a_0}u_0^z \sinh\theta + \frac{c^2}{a_0}e_z(\cosh\theta - 1). \tag{40}$$

Unlike the constant-force model, the coordinates are naturally expressed as functions of the proper time $\tau$.

### 3.4 Velocities and four-momentum

From (29), the four-velocity is

$$u^\mu(\tau) = u_0^\mu \cosh\theta + c\, e^\mu \sinh\theta. \tag{41}$$

Therefore the four-momentum is

$$p^\mu(\tau) = m\left(u_0^\mu \cosh\theta + c\, e^\mu \sinh\theta\right). \tag{42}$$

The energy is

$$E(\tau) = mc\left(u_0^0 \cosh\theta + c\, e^0 \sinh\theta\right). \tag{43}$$

The four-acceleration is

$$A^\mu(\tau) = \frac{a_0}{c}u_0^\mu \sinh\theta + a_0 e^\mu \cosh\theta, \tag{44}$$

and the four-force is

$$K^\mu = mA^\mu = m\left(\frac{a_0}{c}u_0^\mu \sinh\theta + a_0 e^\mu \cosh\theta\right). \tag{45}$$

Its magnitude is constant:

$$K^\mu K_\mu = -m^2 a_0^2. \tag{46}$$

### 3.5 One-dimensional limit

If the initial velocity is purely vertical, $v_{0x} = v_{0z} = 0$, and $v_{0y} = v_0$, then

$$e^\mu = \left(-\frac{\gamma_0 v_0}{c},\, 0,\, -\gamma_0,\, 0\right). \tag{47}$$

The worldline reduces to the standard hyperbolic motion

$$ct = ct_0 + \frac{\gamma_0 c^2}{a_0}\sinh\theta - \frac{\gamma_0 c v_0}{a_0}(\cosh\theta - 1), \tag{48}$$

$$y = y_0 + \frac{\gamma_0 c v_0}{a_0}\sinh\theta - \frac{\gamma_0 c^2}{a_0}(\cosh\theta - 1). \tag{49}$$

In this case the coordinate acceleration is not constant:

$$a_y(t) = -\frac{a_0}{\gamma(t)^3}. \tag{50}$$

Thus, although the proper acceleration is constant, the acceleration measured in the laboratory frame decreases as the speed increases.

### 3.6 Relation to the constant-force model

The two models are related as follows. Before comparing them, recall the definitions

$$D = c^2 + A^2 + C^2, \qquad p_\perp^2 = m^2(A^2 + C^2), \tag{51}$$

so that

$$D = c^2 + \frac{p_\perp^2}{m^2}, \qquad p_\perp = \gamma_0\, m\, v_\perp, \tag{52}$$

with $v_\perp^2 = v_{0x}^2 + v_{0z}^2$. The quantity $p_\perp$ is the transverse momentum in the laboratory frame, and it includes the factor $\gamma_0$.

The four-force (22) of the constant-force model has the time component $K^0 = -\gamma m g v_y/c$, which is the energy balance (24) written covariantly. Its invariant norm is

$$K^\mu K_\mu = \gamma^2 m^2 g^2\left(\frac{v_y^2}{c^2} - 1\right) = -\,m^2 g^2 \gamma^2\left(1 - \frac{v_y^2}{c^2}\right). \tag{53}$$

From (9) and (15),

$$\gamma = \frac{\sqrt{D + (B - gt)^2}}{c}, \qquad v_y = \frac{c(B - gt)}{\sqrt{D + (B - gt)^2}}, \tag{54}$$

so

$$\gamma^2\left(1 - \frac{v_y^2}{c^2}\right) = \frac{D}{c^2}. \tag{55}$$

Hence $K^\mu K_\mu = -m^2 g^2 D/c^2$ is independent of time. Since $K^\mu = mA^\mu$, the constant-force model has constant proper acceleration

$$\alpha = g\sqrt{\frac{D}{c^2}} = g\sqrt{1 + \frac{p_\perp^2}{m^2 c^2}} = g\sqrt{1 + \frac{\gamma_0^2 v_\perp^2}{c^2}}, \tag{56}$$

where $p_\perp^2 = m^2(A^2 + C^2)$ by (6)–(7). Thus $\alpha = g$ if and only if $p_\perp = 0$.

**The two models coincide only when the motion is one-dimensional and the initial data are matched.** Specifically, they describe the same worldline only if $p_\perp = 0$, $a_0 = g$, and

the initial four-velocity $u_0^\mu$ is the same in both models. In that case, for $v_{0x} = v_{0z} = 0$, Eqs. (11)–(13) reduce to

$$y(t) = y_0 + \frac{c}{g}\left[\sqrt{c^2 + B^2} - \sqrt{c^2 + (B - gt)^2}\right], \tag{57}$$

which is the hyperbolic worldline (48)–(49) with $a_0 = g$.

For $p_\perp \neq 0$ the two models differ in two respects.

1. $\alpha$ in (56) exceeds $g$.
2. The direction of the four-acceleration rotates in the constant-force model—the worldline is not confined to a two-plane in spacetime—whereas in the hyperbolic model (29)–(30) the four-acceleration stays in the plane spanned by $u_0^\mu$ and $e^\mu$.

Consequently, for $v_{0x} \neq 0$ (or $v_{0z} \neq 0$) the hyperbolic solution with $a_0 = g$ is *not* a constant-force trajectory: its laboratory three-force is not $(0, -mg, 0)$.

# 4 Four-dimensional Lorentz transformations

## 4.1 General Poincaré transformation

Let $X^\mu = (ct, x, y, z)$ denote the coordinates of an event in an inertial frame $S$. A Poincaré transformation to another inertial frame $S'$ is

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu, \tag{58}$$

where $a^\mu$ is a constant four-vector (translation) and $\Lambda^\mu{}_\nu$ is a Lorentz transformation matrix satisfying

$$\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}, \qquad \eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{59}$$

Equivalently, in matrix notation,

$$\Lambda^T \eta \Lambda = \eta. \tag{60}$$

The condition (59) ensures that the interval

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{61}$$

is preserved:

$$\eta_{\mu\nu} X'^\mu X'^\nu = \eta_{\mu\nu} X^\mu X^\nu. \tag{62}$$

Throughout this section we set $a^\mu = 0$ unless stated otherwise, so that (58) reduces to a pure Lorentz transformation

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu. \tag{63}$$

## 4.2 Boost along an arbitrary direction

A Lorentz boost with velocity $\mathbf{V} = (V_x, V_y, V_z)$, where $V = |\mathbf{V}| < c$, can be written in a manifestly vectorial form. Define

$$\boldsymbol{\beta} = \frac{\mathbf{V}}{c}, \qquad \beta = |\boldsymbol{\beta}|, \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{64}$$

Then the boost acts on the time and space components as

$$ct' = \gamma \left(ct - \boldsymbol{\beta} \cdot \mathbf{r}\right), \tag{65}$$

$$\mathbf{r}' = \mathbf{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{r})\boldsymbol{\beta} - \gamma ct\, \boldsymbol{\beta}, \tag{66}$$

where $\mathbf{r} = (x, y, z)$ and $\boldsymbol{\beta} \cdot \mathbf{r} = \beta_x x + \beta_y y + \beta_z z$. In components,

$$ct' = \gamma(ct - \beta_x x - \beta_y y - \beta_z z), \tag{67}$$

$$x' = x + \frac{\gamma - 1}{\beta^2}(\beta_x x + \beta_y y + \beta_z z)\beta_x - \gamma ct\, \beta_x, \tag{68}$$

$$y' = y + \frac{\gamma - 1}{\beta^2}(\beta_x x + \beta_y y + \beta_z z)\beta_y - \gamma ct\, \beta_y, \tag{69}$$

$$z' = z + \frac{\gamma - 1}{\beta^2}(\beta_x x + \beta_y y + \beta_z z)\beta_z - \gamma ct\, \beta_z. \tag{70}$$

### 4.3 Time transformation

Equation (65) is the general time transformation between inertial frames:

$$t' = \gamma \left( t - \frac{\mathbf{V} \cdot \mathbf{r}}{c^2} \right). \tag{71}$$

Two features are worth emphasizing.

First, time is no longer absolute. Simultaneity is frame-dependent: two events that are simultaneous in $S$ ($\Delta t = 0$) are generally not simultaneous in $S'$ unless they are also spatially coincident along the boost direction:

$$\Delta t' = -\gamma \frac{\mathbf{V} \cdot \Delta \mathbf{r}}{c^2}. \tag{72}$$

This is the relativity of simultaneity.

Second, for a clock at rest in $S$ ($\Delta \mathbf{r} = 0$), the proper time interval is related to the coordinate time interval by

$$\Delta t' = \gamma \Delta t, \tag{73}$$

so that a moving clock runs slow (time dilation). Conversely, the proper time along an arbitrary worldline $X^\mu(\tau)$ is

$$d\tau = \frac{1}{\gamma} dt = \sqrt{1 - \mathbf{v}^2/c^2}\, dt, \tag{74}$$

which is invariant under Lorentz transformations.

In the constant-force model of Sec. 2, $t$ is the laboratory coordinate time; the proper time is obtained by integrating (74) along the worldline. In the constant-proper-acceleration model of Sec. 3, the worldline is naturally parametrized by $\tau$, and the laboratory time $t(\tau)$ is given by (37).

### 4.4 Vector coordinate transformation

Under a Lorentz transformation, any four-vector $A^\mu = (A^0, \mathbf{A})$ transforms as

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{75}$$

For a boost with velocity $\mathbf{V}$, this gives

$$A'^0 = \gamma \left( A^0 - \boldsymbol{\beta} \cdot \mathbf{A} \right), \tag{76}$$

$$\mathbf{A}' = \mathbf{A} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{A})\boldsymbol{\beta} - \gamma A^0 \boldsymbol{\beta}. \tag{77}$$

In particular, the four-velocity $u^\mu = \gamma(c, \mathbf{v})$ and the four-momentum $p^\mu = (E/c, \mathbf{p})$ transform in this way:

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \boldsymbol{\beta}\cdot\mathbf{p}\right), \tag{78}$$

$$\mathbf{p}' = \mathbf{p} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{p})\boldsymbol{\beta} - \gamma\frac{E}{c}\boldsymbol{\beta}. \tag{79}$$

These are the relativistic generalizations of the Galilean transformations for energy and momentum. For a boost along the $x$-axis, (79) reduces to

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta p_x\right), \tag{80}$$

$$p'_x = \gamma\left(p_x - \beta\frac{E}{c}\right), \tag{81}$$

$$p'_y = p_y, \qquad p'_z = p_z. \tag{82}$$

The invariant mass-shell condition

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2 \tag{83}$$

is preserved, as required by (59).

A general Lorentz transformation can also include spatial rotations. A rotation $R \in SO(3)$ acts on the spatial components as

$$\mathbf{r}' = R\mathbf{r}, \qquad \mathbf{p}' = R\mathbf{p}, \tag{84}$$

while leaving $ct$ and $E/c$ unchanged. Any proper orthochronous Lorentz transformation can be decomposed as

$$\Lambda = R(\hat{n}, \phi)\, B(\mathbf{V}), \tag{85}$$

where $B(\mathbf{V})$ is a boost and $R(\hat{n}, \phi)$ is a rotation by angle $\phi$ about axis $\hat{n}$.

### 4.5 Transformation of the equations of motion

The relativistic equation of motion

$$\frac{dp^\mu}{d\tau} = K^\mu \tag{86}$$

is Lorentz covariant: under (58),

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu, \qquad K'^\mu = \Lambda^\mu{}_\nu K^\nu, \qquad \tau' = \tau, \tag{87}$$

so the equation retains the same form in every inertial frame:

$$\frac{dp'^\mu}{d\tau} = K'^\mu. \tag{88}$$

The proper time $\tau$ is invariant, and the four-force transforms as a four-vector. This is the precise sense in which the constant-proper-acceleration model of Sec. 3 is Lorentz covariant.

By contrast, the Newtonian equation (1) is not Lorentz covariant. It is covariant only under Galilean transformations

$$t' = t, \qquad \mathbf{r}' = \mathbf{r} - \mathbf{V}t, \tag{89}$$

which are recovered from (65)–(66) only in the limit $V \ll c$. Under a genuine Lorentz transformation, the Newtonian parabolic trajectory does not retain the form (1); this is why a consistent relativistic projectile theory requires the four-dimensional formulation presented in Secs. 2 and 3.

### 4.6 Application to the constant-force model

In the constant-force model of Sec. 2, the force $\mathbf{F} = (0, -mg, 0)$ is defined in the laboratory frame $S$. Under a boost to $S'$, the four-force transforms according to (77). Using (22),

$$K^\mu = \gamma \left( -\frac{mgv_y}{c},\, 0,\, -mg,\, 0 \right), \tag{90}$$

one finds that in $S'$ the force is generally no longer purely spatial and no longer purely along $-y$. Consequently, the simple form $\mathbf{F}' = (0, -mg, 0)$ is not preserved. This is expected: a constant laboratory force is a frame-dependent concept, and only the four-force $K^\mu$ has a covariant meaning. The constant-force model is therefore naturally interpreted as a local uniform-field approximation valid in a single inertial frame.

To make this explicit, consider a boost along the $x$-axis with velocity $V = \beta c$. Using (90) and the transformation law (77), one finds for the $x$-component of the three-force in the primed frame

$$F'_x = \frac{mg\,\beta\,(B - gt)}{s - \beta A}, \qquad s \equiv \sqrt{D + (B - gt)^2} = c\gamma. \tag{91}$$

This component is time-dependent and vanishes only at the instant $B - gt = 0$. By contrast, for a boost along the force direction (the $y$-axis) one finds $F'_x = F'_z = 0$ and

$$F'_y = -mg, \tag{92}$$

so that the three-force $(0, -mg, 0)$ is preserved in form; the boost merely changes the initial rapidity, i.e. the initial condition encoded in $B$. Hence the three-force $(0, -mg, 0)$ is preserved only under boosts along its own direction, whereas the four-force $K^\mu$ transforms as a four-vector under *every* boost and the proper acceleration $\alpha$ is invariant.

### 4.7 Application to the constant-proper-acceleration model

In the constant-proper-acceleration model of Sec. 3, the four-acceleration satisfies

$$A^\mu A_\mu = -a_0^2, \tag{93}$$

which is a Lorentz scalar. Under (58),

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu, \tag{94}$$

and therefore

$$A'^\mu A'_\mu = A^\mu A_\mu = -a_0^2. \tag{95}$$

Thus the magnitude of the proper acceleration is the same in every inertial frame. The worldline (30) transforms as a four-vector, and the hyperbolic form of the motion is preserved. It should be emphasized that the hyperbolic form is preserved in the following sense: in any inertial frame $S'$, the worldline can be written as (30) with the transformed vectors $u'^\mu_0 = \Lambda^\mu{}_\nu u^\nu_0$ and $e'^\mu = \Lambda^\mu{}_\nu e^\nu$, the same constant $a_0$, and the same proper time $\tau$. The proper time is a Lorentz scalar, $d\tau' = d\tau$, so the parametrization by $\tau$ is frame-independent. This is the sense in which constant proper acceleration is a genuinely covariant concept, in contrast with constant laboratory force.

### 4.8 Summary

The four-dimensional Lorentz transformation (58) acts simultaneously on time and space. The time transformation (71) encodes time dilation and the relativity of simultaneity, while the vector transformation (77) governs four-velocities, four-momenta, and four-forces. The equations of motion written in terms of four-vectors are Lorentz covariant, whereas the Newtonian parabolic trajectory is only Galilean covariant. Among the two models considered in this paper, the constant-proper-acceleration model is fully Lorentz covariant, while the constant-laboratory-force model is covariant only in the sense that its four-force transforms as a four-vector, with the force field itself being frame-dependent.

# 5 Lorentz covariance of Maxwell's equations and the Lorentz force

## 5.1 Four-potential and field-strength tensor

The electromagnetic field is described by the four-potential

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right), \tag{96}$$

and the field-strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad \partial^\mu = \left(\frac{1}{c}\partial_t, -\boldsymbol{\nabla}\right). \tag{97}$$

It is manifestly antisymmetric, $F^{\mu\nu} = -F^{\nu\mu}$, and in matrix form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{98}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The dual tensor is

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{99}$$

with $\epsilon^{0123} = +1$. The homogeneous Maxwell equations will be written using $\widetilde{F}^{\mu\nu}$.

Under a gauge transformation $A^\mu \to A^\mu + \partial^\mu\chi$, the field tensor is unchanged:

$$F^{\mu\nu} \to F^{\mu\nu} + \partial^\mu\partial^\nu\chi - \partial^\nu\partial^\mu\chi = F^{\mu\nu}. \tag{100}$$

Thus $F^{\mu\nu}$ is gauge invariant.

## 5.2 Maxwell's equations in four-dimensional form

The inhomogeneous Maxwell equations,

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} = \mu_0\mathbf{j}, \tag{101}$$

combine into the single four-dimensional equation

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \tag{102}$$

with the four-current

$$J^\mu = (c\rho, \mathbf{j}). \tag{103}$$

Let us verify the components. For $\nu = 0$,

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \frac{1}{c}\partial_i E^i = \frac{1}{c}\nabla\cdot\mathbf{E} = \mu_0 J^0 = \mu_0 c\rho, \tag{104}$$

which gives $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$. For $\nu = j$,

$$\partial_\mu F^{\mu j} = \partial_0 F^{0j} + \partial_i F^{ij} = -\frac{1}{c^2}\partial_t E^j + (\nabla\times\mathbf{B})^j = \mu_0 j^j, \tag{105}$$

which is Ampère–Maxwell's law. The homogeneous Maxwell equations,

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0, \tag{106}$$

combine into

$$\boxed{\partial_\mu\widetilde{F}^{\mu\nu} = 0.} \tag{107}$$

Equations (102)–(107) are the full set of Maxwell's equations written in manifestly four-dimensional form. Their Lorentz covariance will be established below.

### 5.3 Lorentz covariance of Maxwell's equations

Consider a Lorentz transformation $X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu}$. The four-potential transforms as a four-vector,

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}. \tag{108}$$

Since the derivative $\partial_{\mu}$ transforms as a covariant vector,

$$\partial'_{\mu} = (\Lambda^{-1})^{\alpha}{}_{\mu}\, \partial_{\alpha}, \tag{109}$$

the field tensor transforms as a rank-two tensor:

$$F'^{\mu\nu} = \partial'^{\mu} A'^{\nu} - \partial'^{\nu} A'^{\mu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}. \tag{110}$$

The four-current also transforms as a four-vector:

$$J'^{\nu} = \Lambda^{\nu}{}_{\beta} J^{\beta}. \tag{111}$$

Now evaluate the left-hand side of (102) in the primed frame:

$$\begin{aligned} \partial'_{\mu} F'^{\mu\nu} &= (\Lambda^{-1})^{\alpha}{}_{\mu} \partial_{\alpha} \left( \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} F^{\rho\sigma} \right) \\ &= (\Lambda^{-1})^{\alpha}{}_{\mu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \partial_{\alpha} F^{\rho\sigma} \\ &= \delta^{\alpha}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \partial_{\alpha} F^{\rho\sigma}. \end{aligned} \tag{112}$$

Using (102) in the unprimed frame,

$$\partial_{\rho} F^{\rho\sigma} = \mu_0 J^{\sigma}, \tag{113}$$

so that

$$\partial'_{\mu} F'^{\mu\nu} = \Lambda^{\nu}{}_{\sigma} \mu_0 J^{\sigma} = \mu_0 J'^{\nu}. \tag{114}$$

Therefore, if (102) holds in $S$, it holds in $S'$ with exactly the same form:

$$\boxed{\partial'_{\mu} F'^{\mu\nu} = \mu_0 J'^{\nu}.} \tag{115}$$

The homogeneous equation (107) follows identically because $\widetilde{F}^{\mu\nu}$ transforms in the same way as $F^{\mu\nu}$:

$$\partial'_{\mu} \widetilde{F}'^{\mu\nu} = \Lambda^{\nu}{}_{\sigma} \partial_{\rho} \widetilde{F}^{\rho\sigma} = 0. \tag{116}$$

This establishes that Maxwell's equations are Lorentz covariant: they retain their form in every inertial frame, with the fields and currents transforming as in (110).

### 5.4 Transformation of the electric and magnetic fields

From (110) and (98), one extracts the transformation laws for the fields. Splitting $\mathbf{E}$ and $\mathbf{B}$ into components parallel and perpendicular to the boost velocity $\mathbf{V}$,

$$E'_{\parallel} = E_{\parallel}, \qquad \mathbf{E}'_{\perp} = \gamma \left( \mathbf{E}_{\perp} + \mathbf{V} \times \mathbf{B}_{\perp} \right), \tag{117}$$

$$B'_{\parallel} = B_{\parallel}, \qquad \mathbf{B}'_{\perp} = \gamma \left( \mathbf{B}_{\perp} - \frac{1}{c^2} \mathbf{V} \times \mathbf{E}_{\perp} \right). \tag{118}$$

Thus electric and magnetic fields mix under Lorentz transformations. What is invariant is not $\mathbf{E}$ or $\mathbf{B}$ separately, but the two Lorentz scalars

$$\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} = B^2 - \frac{E^2}{c^2}, \qquad \tfrac{1}{4} F_{\mu\nu} \widetilde{F}^{\mu\nu} = -\frac{1}{c} \mathbf{E} \cdot \mathbf{B}. \tag{119}$$

These scalars are the same in every inertial frame.

### 5.5 The four-dimensional Lorentz force

The worldline of a particle of charge $q$ and mass $m$ in an electromagnetic field satisfies

$$\frac{dp^\mu}{d\tau} = K^\mu, \tag{120}$$

where the four-force is

$$\boxed{K^\mu = qF^{\mu\nu}u_\nu.} \tag{121}$$

Here $u_\nu = \eta_{\nu\rho}u^\rho$. Let us verify that (121) reduces to the familiar three-dimensional Lorentz force. For the time component,

$$K^0 = qF^{0i}u_i = q\left(-\frac{E^i}{c}\right)(-\gamma v_i) = \frac{\gamma q}{c}\mathbf{E}\cdot\mathbf{v}, \tag{122}$$

which equals $\gamma(dE_{\text{mech}}/dt)/c$, the rate of change of the particle's mechanical energy per unit coordinate time. For the spatial components,

$$\begin{aligned} K^i &= q\left(F^{i0}u_0 + F^{ij}u_j\right) = q\left(\frac{E^i}{c}\gamma c + \left(-\epsilon^{ijk}B^k\right)(-\gamma v_j)\right) \\ &= \gamma q\left(E^i + (\mathbf{v}\times\mathbf{B})^i\right). \end{aligned} \tag{123}$$

Thus

$$\mathbf{K} = \gamma q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) = \gamma\,\mathbf{F}_{\text{L}}, \tag{124}$$

where $\mathbf{F}_{\text{L}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ is the ordinary Lorentz force. The four-force (121) is therefore the covariant form of the Lorentz force.

### 5.6 Lorentz covariance of the Lorentz force

Under a Lorentz transformation, the four-velocity transforms as

$$u'^\mu = \Lambda^\mu{}_\alpha u^\alpha, \qquad u'_\nu = (\Lambda^{-1})^\beta{}_\nu u_\beta, \tag{125}$$

and, by (110),

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}. \tag{126}$$

Therefore

$$\begin{aligned} K'^\mu &= qF'^{\mu\nu}u'_\nu = q\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}(\Lambda^{-1})^\rho{}_\nu u_\rho \\ &= q\Lambda^\mu{}_\alpha\delta^\rho{}_\beta F^{\alpha\beta}u_\rho = q\Lambda^\mu{}_\alpha F^{\alpha\beta}u_\beta = \Lambda^\mu{}_\alpha K^\alpha. \end{aligned} \tag{127}$$

Hence $K^\mu$ transforms as a four-vector, and the equation of motion

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \tag{128}$$

retains its form in every inertial frame:

$$\boxed{\frac{dp'^\mu}{d\tau} = qF'^{\mu\nu}u'_\nu.} \tag{129}$$

The proper time $\tau$ is Lorentz invariant, and the charge $q$ is a Lorentz scalar. This establishes the Lorentz covariance of the four-dimensional Lorentz force.

### 5.7 Invariance of the on-shell condition

The four-force (121) automatically preserves the mass-shell condition $p^\mu p_\mu = m^2c^2$. Indeed, since $F^{\mu\nu}$ is antisymmetric,

$$u_\mu K^\mu = qF^{\mu\nu}u_\mu u_\nu = 0, \tag{130}$$

and therefore

$$\frac{d}{d\tau}\,(p^\mu p_\mu) = 2p_\mu \frac{dp^\mu}{d\tau} = 2m\,u_\mu K^\mu = 0. \tag{131}$$

The particle mass is constant, as required by the Lorentz-covariant formulation.

### 5.8 Consistency with the earlier models

The present section shows that a genuine relativistic force must be formulated through a four-vector $K^\mu$ built from a rank-two tensor field $F^{\mu\nu}$. In the constant-laboratory-force model of Sec. 2, the force $K^\mu$ was given directly without reference to a field tensor; under a Lorentz boost the simple form $(0, -mg, 0)$ is not preserved, as we noted. In the constant-proper-acceleration model of Sec. 3, the four-acceleration satisfies $A^\mu A_\mu = -a_0^2$, which is a Lorentz scalar. Both constructions are consistent with the general covariance requirement established here: it is the four-vector character of the force and the rank-two tensor character of the field that make the equations Lorentz covariant. The electromagnetic case is the canonical example in which the field itself is a Lorentz tensor, and thus the covariance of both Maxwell's equations and the Lorentz force follows from the same tensor transformation law (110).

## 6 Nonrelativistic limit

For $v \ll c$ and $a_0\tau/c \ll 1$, the hyperbolic functions may be expanded as

$$\cosh\theta \simeq 1, \qquad \sinh\theta \simeq \theta. \tag{132}$$

In the constant-force model, (11)–(13) reduce to

$$x(t) \simeq x_0 + v_{0x}t, \qquad y(t) \simeq y_0 + v_{0y}t - \tfrac{1}{2}gt^2, \qquad z(t) \simeq z_0 + v_{0z}t. \tag{133}$$

In the constant-proper-acceleration model, (37)–(40) give

$$x(t) \simeq x_0 + v_{0x}t, \qquad y(t) \simeq y_0 + v_{0y}t - \tfrac{1}{2}a_0t^2, \qquad z(t) \simeq z_0 + v_{0z}t. \tag{134}$$

Thus both models recover the Newtonian parabolic trajectory (1) in the appropriate limit.

## 7 Discussion and conclusion

We have presented two special relativistic generalizations of Newtonian projectile motion. In the constant-laboratory-force model, the force is fixed in a given inertial frame, and the trajectory can be written explicitly in terms of the laboratory time $t$. In the constant-proper-acceleration model, the four-acceleration has constant magnitude, and the motion is the standard hyperbolic motion of special relativity, naturally parametrized by proper time.

We have also developed the four-dimensional Lorentz transformation in full generality, including boosts along arbitrary directions, the transformation of time and space coordinates, and the transformation of four-vectors such as four-velocity and four-momentum. We have shown that the relativistic equations of motion are Lorentz covariant when written in four-vector form, whereas the Newtonian parabolic trajectory is only Galilean covariant. Furthermore, we have demonstrated the Lorentz covariance of Maxwell's equations and of the four-dimensional Lorentz force in manifestly tensor form, showing that the field-strength tensor $F^{\mu\nu}$ and the four-current

$J^\mu$ transform in such a way that $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ and $K^\mu = qF^{\mu\nu}u_\nu$ retain the same form in every inertial frame.

The two projectile models coincide in the nonrelativistic limit but differ at relativistic speeds. More precisely, as shown in Sec. 3.6, the constant-force model also has constant proper acceleration, with magnitude $\alpha = g\sqrt{1 + p_\perp^2/(m^2c^2)}$. The two models coincide exactly (same worldline, same initial data) only when the motion is one-dimensional ($p_\perp = 0$) and $a_0 = g$; otherwise they differ both in the magnitude of the proper acceleration and in the orientation of the four-acceleration. In the constant-force model, the coordinate acceleration is not simply $-g$; rather, the momentum changes linearly with time. In the constant-proper-acceleration model, the proper acceleration is constant, while the laboratory acceleration decreases as $\gamma^{-3}$.

Several caveats are in order. First, special relativity does not include gravity. The constant force considered here should be interpreted as an external force or as a local uniform-field approximation. A fully relativistic treatment of gravity requires general relativity. Second, the laboratory time $t$ and the proper time $\tau$ must be carefully distinguished; in the constant-proper-acceleration model the worldline is most naturally expressed in terms of $\tau$. Third, the naive Newtonian relation $v_y = v_{0y} - gt$ cannot be used at relativistic speeds because it can violate the speed limit $v < c$.

The results presented here provide a consistent relativistic description of projectile motion under a uniform external field and illustrate how the Newtonian parabolic trajectory emerges as a low-speed approximation. They also illustrate the central role of four-dimensional Lorentz covariance in relativistic dynamics and electrodynamics.

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