Christoffel symbols (MiniReview)

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I. Introduction

In mathematics and physics, the Christoffel symbols, named for Elwin Bruno Christoffel[1] (1829-1900), are numerical arrays of real numbers that describe, in coordinates, the effects of parallel transport in curved surfaces and, more generally, manifolds. As such, they are coordinate-space expressions for the Levi-Civita connection derived from the metric tensor. In a broader sense, the connection coefficients of an arbitrary (not necessarily metric) affine connection in a coordinate basis are also called Christoffel symbols.[2] The Christoffel symbols may be used for performing practical calculations in differential geometry. For example, the Riemann curvature tensor can be expressed entirely in terms of the Christoffel symbols and their first partial derivatives.

At each point of the underlying n-dimensional manifold, for any local coordinate system, the Christoffel symbol is an array with three dimensions: $n \times n \times n$. Each of the n3 components is a real number.

Under linear coordinate transformations on the manifold, its components transform like those of a tensor, but under general coordinate transformations, they do not. In many practical problems, most components of the Christoffel symbols are equal to zero, provided the coordinate system and the metric tensor possess some common symmetries.

In general relativity, the Christoffel symbol plays the role of the gravitational force field with the corresponding gravitational potential being the metric tensor.

II. Definition

The definitions given below are valid for both Riemannian manifolds and pseudo-Riemannian manifolds, such as those of general relativity, with careful distinction being made between upper and lower indices (contra-variant and co-variant indices). The formulas hold for either sign convention, unless otherwise noted. Einstein summation convention is used in this article. The connection coefficients of the Levi-Civita connection (or pseudo-Riemannian connection) expressed in a coordinate basis are called the Christoffel symbols.

Given a local coordinate system x_i, i = 1, 2, ..., n on a n-manifold M with metric tensor g, the tangent vectors

$$\mathrm{e}_i = \frac{\partial}{\partial x^i} = \partial_i, \quad i = 1, 2, \ldots, n \tag{1}$$

define a local coordinate basis of the tangent space to M at each point of its domain.

Christoffel symbols of the first kind

The Christoffel symbols of the first kind can be derived either from the Christoffel symbols of the second kind and the metric,[3]

$$\Gamma_{cab} = g_{cd}\Gamma^{d}{}_{ab}\,, \tag{2}$$

or from the metric alone,[3]

$$2\Gamma_{cab} = \left(\frac{\partial g_{ca}}{\partial x^b} + \frac{\partial g_{cb}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^c}\right) =$$

$$= (g_{ca,b} + g_{cb,a} - g_{ab,c}) = (\partial_b g_{ca} + \partial_a g_{cb} - \partial_c g_{ab})\,. \tag{3}$$

As an alternative notation one also finds[1][4][5]

$$\Gamma_{cab} = [ab, c]. \tag{4}$$

It is worth noting that [ab, c] = [ba, c].[6]

Christoffel symbols of the second kind (symmetric definition)

The Christoffel symbols of the second kind are the connection coefficients—in a coordinate basis—of the Levi-Civita connection, and since this connection has zero torsion, then in this basis the connection coefficients are symmetric, i.e.,
$\Gamma^{k}{}_{ij} = \Gamma^{k}{}_{ji}$,.[4] For this reason a torsion-free connection is often called 'symmetric'.

In other words, the Christoffel symbols of the second kind[7, 8][4]

$$\Gamma^{k}{}_{ij} \tag{5}$$

sometimes

$$\Gamma^k_{ij} \tag{6}$$

or

$$\left\{ {k \atop ij} \right\} \tag{7}$$

[1][4] are defined as the unique coefficients such that the equation

$$\nabla_i \mathrm{e}_j = \Gamma^k{}_{ij} \mathrm{e}_k \tag{8}$$

holds, where ∇_i is the Levi-Civita connection on M taken in the coordinate direction e_i, i.e., $\nabla_i \equiv \nabla_{\mathrm{e}_i}$ and where $\mathrm{e}_i = \partial_i$ is a local coordinate (holonomic) basis.

The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor g_{ik} :

$$0 = \nabla_\ell g_{ik} = \frac{\partial g_{ik}}{\partial x^\ell} - g_{mk}\Gamma^m{}_{i\ell} - g_{im}\Gamma^m{}_{k\ell} = \\ = \frac{\partial g_{ik}}{\partial x^\ell} - 2g_{m(k}\Gamma^m{}_{i)\ell}. \tag{9}$$

As a shorthand notation, the nabla symbol and the partial derivative symbols are frequently dropped, and instead a semi-colon and a comma are used to set off the index that is being used for the derivative. Thus, the above is sometimes written as

$$0 = g_{ik;\ell} = g_{ik,\ell} - g_{mk}\Gamma^m{}_{i\ell} - g_{im}\Gamma^m{}_{k\ell}. \tag{10}$$

Using that the symbols are symmetric in the lower two indices, one can solve explicitly for the Christoffel symbols as a function of the metric tensor by permuting the indices and resumming:[6]

$$\Gamma^i{}_{k\ell} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m}\right) = \\ = \frac{1}{2}g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}), \tag{11}$$

where (g^{jk}) is the inverse of the matrix (g_{jk}), defined as (using the Kronecker delta, and Einstein notation for summation) $g^{ji}\ g_{ik} = \delta^j{}_k$. Although the Christoffel symbols are written in the same notation as tensors with index notation, they are not tensors,[8] since they do not transform like tensors under a change of coordinates; see below.

Connection coefficients in a non holonomic basis

The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. In other words, the name Christoffel symbols is reserved only for coordinate (i.e., holonomic) frames. However, the connection coefficients can also be defined in an arbitrary (i.e., non holonomic) basis of tangent vectors $\mathbf{u}_i$ by

$$\nabla_{\mathbf{u}_i}\mathbf{u}_j = \omega^k{}_{ij}\mathbf{u}_k. \tag{12}$$

Explicitly, in terms of the metric tensor, this is[7]

$$\omega^i{}_{k\ell} = \frac{1}{2}g^{im}\left(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m} + c_{mk\ell} + c_{m\ell k} - c_{k\ell m}\right), \tag{13}$$

where $c_{k\ell m} = g_{mp}c_{k\ell}{}^p$ are the commutation coefficients of the basis; that is,

$$[\mathbf{u}_k, \mathbf{u}_\ell] = c_{k\ell}{}^m\mathbf{u}_m \tag{14}$$

where $\mathbf{u}_k$ are the basis vectors and $[\,,]$ is the Lie bracket. The standard unit vectors in spherical and cylindrical coordinates furnish an example of a basis with non-vanishing commutation coefficients.

Ricci rotation coefficients (asymmetric definition)

When we choose the basis $\mathrm{X}_i \equiv \mathbf{u}_i$ orthonormal: $g_{ab} \equiv \eta_{ab} = \langle X_a, X_b\rangle then g_{mk,\ell} \equiv \eta_{mk,\ell} = 0$. This implies that

$$\omega^i{}_{k\ell} = \frac{1}{2}\eta^{im}\left(c_{mk\ell} + c_{m\ell k} - c_{k\ell m}\right) \tag{15}$$

and the connection coefficients become antisymmetric in the first two indices:

$\omega_{abc} = -\omega_{bac}$,
where $\omega_{abc} = \eta_{ad}\omega^d{}_{bc}$.
In this case, the connection coefficients $\omega^a{}_{bc}$ are called the Ricci rotation coefficients.[9][10]

Equivalently, one can define Ricci rotation coefficients as follows:[7]

$$\omega^k{}_{ij} := \mathbf{u}^k \cdot (\nabla_j \mathbf{u}_i)\ , \tag{16}$$

where $\mathbf{u}_i$ is an orthonormal non holonomic basis and $\mathbf{u}^k = \eta^{k\ell}\mathbf{u}_\ell$ its co-basis.

Relationship to index-free notation

Let X and Y be vector fields with components X^i and Y^k . Then the kth component of the covariant derivative of Y with respect to X is given by

$$(\nabla_X Y)^k = X^i(\nabla_i Y)^k = X^i\left(\frac{\partial Y^k}{\partial x^i} + \Gamma^k{}_{im}Y^m\right). \tag{17}$$

Here, the Einstein notation is used, so repeated indices indicate summation over indices and contraction with the metric tensor serves to raise and lower indices:

$$g(X,Y) = X^iY_i = g_{ik}X^iY^k = g^{ik}X_iY_k. \tag{18}$$

Keep in mind that

$$g_{ik} \neq g^{ik} \tag{19}$$

and that

$$g^i{}_k = \delta^i{}_k \tag{20}$$

the Kronecker delta. The convention is that the metric tensor is the one with the lower indices; the correct way to obtain g^{ik} from g_{ik} is to solve the linear equations

$$g^{ij}g_{jk} = \delta^i{}_k. \tag{21}$$

The statement that the connection is torsion-free, namely that

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{22}$$

is equivalent to the statement that—in a coordinate basis—the Christoffel symbol is symmetric in the lower two indices:

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj}. \tag{23}$$

The index-less transformation properties of a tensor are given by pullbacks for covariant indices, and pushforwards for contravariant indices. The article on covariant derivatives provides additional discussion of the correspondence between index-free notation and indexed notation.

A. *Covariant derivatives of tensors*

The covariant derivative of a vector field Vm is

$$\nabla_\ell V^m = \frac{\partial V^m}{\partial x^\ell} + \Gamma^m{}_{k\ell} V^k. \tag{24}$$

The covariant derivative of a scalar field φ is just

$$\nabla_i \varphi = \frac{\partial \varphi}{\partial x^i} \tag{25}$$

and the covariant derivative of a covector field ω_m is

$$\nabla_\ell \omega_m = \frac{\partial \omega_m}{\partial x^\ell} - \Gamma^k{}_{m\ell} \omega_k. \tag{26}$$

The symmetry of the Christoffel symbol now implies

$$\nabla_i \nabla_j \varphi = \nabla_j \nabla_i \varphi \tag{27}$$

for any scalar field, but in general the covariant derivatives of higher order tensor fields do not commute (see curvature tensor).

The covariant derivative of a type (2,0) tensor field A^{ik} is

$$\nabla_\ell A^{ik} = \frac{\partial A^{ik}}{\partial x^\ell} + \Gamma^i{}_{m\ell} A^{mk} + \Gamma^k{}_{m\ell} A^{im}, \tag{28}$$

that is,

$$A^{ik}{}_{;\ell} = A^{ik}{}_{,\ell} + A^{mk} \Gamma^i{}_{m\ell} + A^{im} \Gamma^k{}_{m\ell}. \tag{29}$$

If the tensor field is mixed then its covariant derivative is

$$A^i{}_{k;\ell} = A^i{}_{k,\ell} + A^m{}_k \Gamma^i{}_{m\ell} - A^i{}_m \Gamma^m{}_{k\ell}, \tag{30}$$

and if the tensor field is of type (0,2) then its covariant derivative is

$$A_{ik;\ell} = A_{ik,\ell} - A_{mk} \Gamma^m{}_{i\ell} - A_{im} \Gamma^m{}_{k\ell}. \tag{31}$$

Change of variable

Under a change of variable from

$$(y^1, \ldots, y^n) \; to (x^1, \ldots, x^n) \; , vectorstransformas \tag{32}$$

$$\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i} \frac{\partial}{\partial x^k} \tag{33}$$

and so

$$\bar{\Gamma}^k{}_{ij} = \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} \Gamma^r{}_{pq} \frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^m} \frac{\partial^2 x^m}{\partial y^i \partial y^j} \tag{34}$$

where the overline denotes the Christoffel symbols in the y coordinate system. Note that the Christoffel symbol does not transform as a tensor, but rather as an object in the jet bundle. More precisely, the Christoffel symbols can be considered as functions on the jet bundle of the frame bundle of M, independent of any local coordinate system. Choosing a local coordinate system determines a local section of this bundle, which can then be used to pull back the Christoffel symbols to functions on M, though of course these functions then depend on the choice of local coordinate system.

At each point, there exist coordinate systems in which the Christoffel symbols vanish at the point.[11] These are called (geodesic) normal coordinates, and are often used in Riemannian geometry.

B. *Applications to general relativity*

The Christoffel symbols find frequent use in Einstein's theory of general relativity, where spacetime is represented by a curved 4-dimensional Lorentz manifold with a Levi-Civita connection. The Einstein field equations—which determine the geometry of spacetime in the presence of matter—contain the Ricci tensor, and so calculating the Christoffel symbols is essential. Once the geometry is determined, the paths of particles and light beams are calculated by solving the geodesic equations in which the Christoffel symbols explicitly appear.

III. MECHANICS

Development ...

$$\bar{\Gamma}^k{}_{ij} = \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} \Gamma^r{}_{pq} \frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^m} \frac{\partial^2 x^m}{\partial y^i \partial y^j} \tag{35}$$

$$x = rcos\theta e_x + rsin\theta e_y \tag{36}$$

$$g_r = \frac{\partial x}{\partial r} = cos\theta e_x + sin\theta e_y \tag{37}$$

$$g_\theta = \frac{\partial x}{\partial \theta} = -rsin\theta e_x + rcos\theta e_y \tag{38}$$

where

$$g_i = \frac{\partial x}{\partial \mu^i} = \left(\frac{\partial x^k}{\partial \mu^i} \right) e_k \tag{39}$$

$$v = \dot{x} = \frac{\partial x}{\partial r} \dot{r} + \frac{\partial x}{\partial \theta} \dot{\theta} \tag{40}$$

$$n_j(x) := \prod_{i=0}^{j-1} (x - x_i) \qquad j = 0, \ldots, k. \tag{41}$$

where

$$v^r = \dot{r} = \frac{\partial r}{\partial t} \tag{42}$$

$$v = v^r g_r + v^\theta g_\theta \tag{43}$$

$$a = \dot{v} = \dot{v}^r g_r + v^r \dot{g}_r + \dot{v}^\theta g_\theta + v^\theta \dot{g}_\theta \tag{44}$$

$$a = \dot{v}^i g_i + v^i \dot{g}_i \tag{45}$$

$$\dot{g}_r = \left(\frac{\partial g_r}{\partial r}\right)\dot{r} + \left(\frac{\partial g_r}{\partial \theta}\right)\dot{\theta} \tag{46}$$

$$\dot{g}_r = v^r g_{r,r} + v^\theta g_{r,\theta} \tag{47}$$

$$\dot{g}_\theta = \left(\frac{\partial g_\theta}{\partial r}\right)\dot{r} + \left(\frac{\partial g_\theta}{\partial \theta}\right)\dot{\theta} \tag{48}$$

$$\dot{g}_\theta = v^r g_{\theta,r} + v^\theta g_{\theta,\theta} \tag{49}$$

$$\dot{g}_i = \left(\frac{\partial g_i}{\partial u^j}\right) u^j \tag{50}$$

$$\dot{g}_i = v^i g_{i,j} \tag{51}$$

$$g_{r,r} = \Gamma^r_{rr} g_r + \Gamma^\theta_{rr} g^\theta \tag{52}$$

$$g_{r,\theta} = \Gamma^r_{r\theta} g_r + \Gamma^\theta_{r\theta} g^\theta \tag{53}$$

$$g_{\theta,r} = \Gamma^r_{\theta r} g_r + \Gamma^\theta_{\theta r} g^\theta \tag{54}$$

$$g_{\theta,\theta} = \Gamma^r_{\theta\theta} g_r + \Gamma^\theta_{\theta\theta} g^\theta \tag{55}$$

$$g_{i,j} = \Gamma^k_{ij} g^k \tag{56}$$

$$g_{i,j} = g_{j,i} = \Gamma^k_{ij} g^k \tag{57}$$

A. *Cylindrical coordinates*

Calculate the Christoffel symbols of the canonical flat connection in $\mathbf{E}^3$ in cylindrical coordinates for $(x = r\cos\phi, y = r\sin\phi, z = h)$

In cylindrical coordinates (r, ϕ, h) have

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = h \end{cases} \quad and \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan\frac{y}{z} \\ h = z \end{cases}$$

Know in Cartesian coordinates all Christoffel symbols vanish. Hence:

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial r}{\partial z} = 0, \tag{58}$$

$$\Gamma^r_{r\phi} = \Gamma^r_{\phi r} = \frac{\partial^2 x}{\partial r\partial\phi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\phi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\phi}\frac{\partial r}{\partial z} = \\ = -\sin\phi\cos\phi + \sin\phi\cos\phi = 0. \tag{59}$$

$$\Gamma^r_{\phi\phi} = \frac{\partial^2 x}{\partial^2\phi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2\phi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2\phi}\frac{\partial r}{\partial z} = \\ = -x\frac{x}{r} - y\frac{y}{r} = -r. \tag{60}$$

$$\Gamma^\phi_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial\phi}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial\phi}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial\phi}{\partial z} = 0. \tag{61}$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{\partial^2 x}{\partial r\partial\phi}\frac{\partial\phi}{\partial x} + \frac{\partial^2 y}{\partial r\partial\phi}\frac{\partial\phi}{\partial y} + \frac{\partial^2 z}{\partial r\partial\phi}\frac{\partial\phi}{\partial z} = \\ = -\sin\phi\frac{-y}{r^2} + \cos\phi\frac{x}{r^2} = \frac{1}{r}, \tag{62}$$

$$\Gamma^\phi_{\phi\phi} = \frac{\partial^2 x}{\partial^2\phi}\frac{\partial\phi}{\partial x} + \frac{\partial^2 y}{\partial^2\phi}\frac{\partial\phi}{\partial y} + \frac{\partial^2 z}{\partial^2\phi}\frac{\partial\phi}{\partial z} = \\ = -x\frac{x}{r^2} - y\frac{y}{r^2} = 0. \tag{63}$$

All symbols $\Gamma^{'}_{,h}, \Gamma^{'}_{h,}$ vanish

$$\Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{hh} = \Gamma^r_{\phi h} = \Gamma^r_{h\phi} = \Gamma^\phi_{hr} = \cdots = 0 \tag{64}$$

since

$$\frac{\partial^2 z}{\partial h\partial\ldots} = \frac{\partial^2 y}{\partial h\partial\ldots} = \frac{\partial^2 x}{\partial h\partial\ldots} \tag{65}$$

For all symbols $\Gamma^h_{\cdot\cdot}\Gamma^h_{\cdot\cdot} = \frac{\partial^2 z}{\partial.\partial.}$
since
$\frac{\partial h}{\partial z} = \frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial y} = 1$. On the other hand all $\frac{\partial^2 z}{\partial.\partial.}$ vanish. hence all symbols $\Gamma^h_{\cdot\cdot}$, vanish.

B. *Spherical coordenates*

$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases} \quad and \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = arc\cos\frac{z}{x^2+y^2+z^2} \\ \phi = \arctan\frac{y}{x} \end{cases}$$

as
$\Gamma^r_{rr} = 0$ since $\frac{\partial^2 x'}{\partial^2 r} = 0$

$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{\partial^2 x}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\theta}\frac{\partial r}{\partial z} = \\ = \cos\theta\cos\phi\frac{x}{r} + \cos\theta\sin\phi\frac{y}{r} - \sin\theta\frac{z}{r} = 0. \tag{66}$$

$$\Gamma^r_{\theta\theta} = \frac{\partial^2 x}{\partial^2\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2\theta}\frac{\partial r}{\partial z} = \\ = -r\sin\theta\cos\phi\frac{x}{r} - r\sin\theta\sin\phi\frac{y}{r} - r\cos\theta\frac{z}{r} = -r. \tag{67}$$

$$\Gamma^r_{r\phi} = \Gamma^r_{\phi r} = \frac{\partial^2 x}{\partial r\partial\phi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\phi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\phi}\frac{\partial r}{\partial z} = \\ = -\sin\theta\cos\phi\frac{x}{r} + \sin\theta\cos\phi\frac{y}{r} = 0. \tag{68}$$

and so on ...

NOTES

[1] Christoffel, E.B. (1869), "Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades", Jour. für die reine und angewandte Mathematik, B. 70: 46–70

[2] See, for instance, (Spivak 1999) and (Choquet-Bruhat & DeWitt-Morette 1977)

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[6] Bishop, R.L.; Goldberg (1968), Tensor Analysis on Manifolds, p. 241 http://mathworld.wolfram.com/ChristoffelSymboloftheSecondKind.html.

[7] See, for example, (Kreyszig 1991), page 141

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