AN ANTICIPATED SOLUTION OF THE RIEMANN HYPOTHESIS

ABDELMAJID BEN HADJ SALEM

Abstract. In 1859, Georg Friedrich Bernhard Riemann announced the following conjecture, called the Riemann hypothesis : *The nontrivial roots (zeros) of the zeta function, have real part* $\Re(s) = 1/2$.
In this paper, I will provide the proof using an equivalent statement of the Riemann hypothesis: the Dirichlet eta function.

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To the memory of my Father who taught me arithmetic
To my wife, my daughter, my son and my granddaughter Rayhane
To the memory of the Prof. Mohamed AMARA, former Dean of the Faculty of Sciences of Tunis

1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1] known Riemann hypothesis:

Conjecture 1.1. *Let* $\zeta(s)$ *be the complex function of the complex variable* $s = \sigma + it$ *defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1 \tag{1}$$

over the whole complex plane, with the exception of $s = 1$. *Then the nontrivial zeros of* $\zeta(s) = 0$ *are written as:* $s = \frac{1}{2} + it$.

In this paper, our idea is to start from an equivalent statement of the Riemann hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real-valued sequences and Abel summation so we obtain a contradiction and we will provide the proof that $\Re(s) = \sigma = \frac{1}{2}$.

1.1. The function zeta(s).

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem ([2], p.15):

Theorem 1.2. *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line. The critical band is $0 < \Re(s) < 1$, it is a classical result, equivalent to the prime number theorem, that *zeta* has no zeros on $\Re(s) = 0$ or $\Re(s) = 1$. We know also that the zeros of the function zeta inside the critical band are all complex numbers $\neq 0$ (see p.30, [3]).

The Riemann Hypothesis is formulated as:

Conjecture 1.3. *(The Riemann Hypothesis,*[2]*, p.55) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

So, instead of using one detailed formula of the zeta function, we will use for our proof the function presented by G.H. Hardy ([3], p.16) namely the Dirichlet eta function ([2], p.56):

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$.

1.2. **A Equivalent statement to the Riemann Hypothesis.** Among the equivalent statements to the Riemann hypothesis is that of the Dirichlet eta function which is stated as follows ([2], p.49):

Equivalence 1.4. *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

The series (2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pp. 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{3}$$

As we know this proposition ([4], p. 4):

Proposition 1.5. *For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.*

We consider in the following $s = \sigma + i.t$ with $0 < \sigma < 1$ and $t > 0$. $\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = l.e^{i\alpha} \Longrightarrow l^2 = |\eta(s)|^2 = \eta(s).\overline{\eta(s)} \tag{4}$$

and $\eta(s) = 0 \Longleftrightarrow l = 0$.

2. Preliminaries of the proof that the zeros of the function eta(s) are on the critical line

Proof. We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

$$\boxed{\textit{s, is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of}}$$
$$\boxed{\zeta(s) \text{ in the critical strip}} \tag{5}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{\textit{s, is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of}}$$
$$\boxed{\eta(s) \text{ in the critical strip}} \tag{6}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let $s = \sigma + it$ with $0 < \sigma < 1$ be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s)=0=\eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^{\sigma}}=0$$

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^{\sigma}}=0$$

Using the definition of the limit of a sequence, we can write:

$$\forall\epsilon_1>0\,\exists n_r,\forall N>n_r,\mid\Re(\eta(s)_N)\mid<\epsilon_1\Longrightarrow\Re^2(\eta(s)_N)<{\epsilon_1}^2 \tag{7}$$

$$\forall\epsilon_2>0\,\exists n_i,\forall N>n_i,\mid\Im(\eta(s)_N)\mid<\epsilon_2\Longrightarrow\Im^2(\eta(s)_N)<{\epsilon_2}^2 \tag{8}$$

Then:

$$0<\sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}}+\sum_{k,k'=1;k\neq k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^{\sigma}k'^{\sigma}}<\epsilon_1^2 \tag{9}$$

$$0<\sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}}+\sum_{k,k'=1;k\neq k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^{\sigma}k'^{\sigma}}<\epsilon_2^2 \tag{10}$$

Taking $\epsilon=\epsilon_1=\epsilon_2$ and $N>max(n_r,n_i)$, we get by making the sum member to member of the last two inequalities:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+\sum_{k,k'=1;k\neq k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}<2\epsilon^2 \tag{11}$$

We can write the above equation as :

$$0<\ l_N^2\ <2\epsilon^2 \tag{12}$$

or $l(s)=0$. □

3. The Proof of the trivial zeros of the eta function verifying $\sigma=1/2$

Proof. Let $\eta_N(s)=\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k^s}$ be the Dirichlet eta function truncated to order N. Let $s=\sigma+it$ be a non-trivial zero of the function $\eta(s)$ such that $\sigma\in]0,1/2[$; thus, s is also a non-trivial zero of the zeta function. From point 4 of theorem (1.2), $1-\sigma+it=1-\bar{s}$ is also a zero of the zeta function; consequently, $1-\bar{s}$ is a zero of the Dirichlet eta function, i.e., $\eta(1-\bar{s})=0$. For the zero s, we can write:

$$|\eta_N(s)|^2=D_N(\sigma)+C_N(s),\quad D_N(\sigma)=\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}$$

$$C_N(s)=\sum_{k,k'=1;k\neq k'}^{N}\frac{(-1)^{k+k'}cos(tLogk/k')}{k^{\sigma}k'^{\sigma}} \tag{13}$$

$D_N(\sigma)$ and $C_N(s)$ are the diagonal sum and the off-diagonal sum, respectively. Thus, since $2\sigma<1$, $D_N(\sigma)\to+\infty$, which forces $C_N(s)\to-\infty$. Let us write the

same for the symmetric zero of s (point 4 of theorem (1.2)), namely $1-\bar{s}$; in this case, $1-\sigma > 1/2 \Rightarrow 2(1-\sigma) > 1$, whence:

$$|\eta_N(1-\bar{s})|^2 = D_N(\sigma) + C_N(s), \quad D_N(\sigma) = \sum_{k=1}^{N} \frac{1}{k^{2(1-\sigma)}} \longrightarrow \zeta(2(1-\sigma)) < +\infty,$$

$$et \quad C_N(s) = \sum_{k,k'=1;k\neq k'}^{N} \frac{(-1)^{k+k'} cos(tLogk/k')}{(kk')^{2(1-\sigma)}} \longrightarrow -\zeta(2(1-\sigma)) > -\infty \tag{14}$$

Using the definition of the limit of a sequence, we can write:

$$\forall \epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\eta_N(s)| < \epsilon_1 \Longrightarrow |\eta_N(s)|^2 < \epsilon_1^2 \tag{15}$$

$$\forall \epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\eta_N(1-\bar{s})| < \epsilon_2 \Longrightarrow |\eta_N(1-\bar{s})|^2 < \epsilon_2^2 \tag{16}$$

By taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > \max(n_r, n_i)$, and by summing the last two inequalities term by term, we obtain:

$$0 < |\eta_N(s)|^2 + |\eta_N(1-\bar{s})|^2 < 2\epsilon^2 \tag{17}$$

Consequently, we obtain in detail:

$$0 < \left(\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + \sum_{k,k'=1;k\neq k'}^{N} \frac{(-1)^{k+k'} cos(tLogk/k')}{(kk')^{\sigma}} \right) +$$

$$\left(\sum_{k=1}^{N} \frac{1}{k^{2(1-\sigma)}} + \sum_{k,k'=1;k\neq k'}^{N} \frac{(-1)^{k+k'} cos(tLogk/k')}{(kk')^{2(1-\sigma)}} \right) < 2\epsilon^2 \Longrightarrow$$

$$0 < \sum_{k=1}^{N} \left(\frac{1}{k^{2\sigma}} + \frac{1}{k^{2(1-\sigma)}} \right) + \sum_{k,k'=1;k\neq k'}^{N} (-1)^{k+k'} cos(tLog(k/k')) \left[\frac{1}{(kk')^{\sigma}} + \right.$$

$$\left. \frac{1}{(kk')^{2(1-\sigma)}} \right] < 2\epsilon^2 \tag{18}$$

Which can be written in the form:

$$0 < \underbrace{\sum_{k=1}^{N} \left(\frac{1}{k^{2\sigma}} + \frac{1}{k^{2(1-\sigma)}} \right)}_{\text{Block A : Diverges positively towards } +\infty} + \tag{19}$$

$$\underbrace{\sum_{k\neq k',k,k'=1}^{N} (-1)^{k} cos\left(tLog(k/k')\right) \frac{(-1)^{k'} cosh\left((1-2\sigma) Log\sqrt{kk'}\right)}{(kk')^{1/2}}}_{\text{Block B : Must diverge negatively towards } -\infty} < 2\epsilon^2 \tag{20}$$

where cosh is the hyperbolic cosine. Block A hides nothing; it is well-defined. When $N \to \infty$, its rate of divergence is perfectly known thanks to comparison integrals; for $s < 1/2$, we obtain:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \sim \frac{N^{1-2\sigma}}{1-2\sigma} \Longrightarrow lim_{N\longrightarrow+\infty} \frac{N^{1-2\sigma}}{1-2\sigma} = +\infty$$

My idea was to uncover a hidden contradiction stemming from the expression for Block B—namely, that it is finite or bounded. For properly estimate the expression for Block B above, we make use of Abel's summation by parts, ([5], p.326) hence the respective theorem:

Theorem 3.1. *(Abel summation (partial summation)) Let $f(x)$ be a complex-valued function that is continuously differentiable on $[a,b]$, let c_n be arbitrary complex numbers, and set $C(x) = \sum_{a<n\leq x} c_n$. Then:*

$$\sum_{a<n\leq x} c_n.f(n) = C(b).f(b) - \int_a^b C(x).f'(x)dx \tag{21}$$

Applying this to our case —Block B— namely the equation (20) above:

- the sequence of complex numbers $a_k = (-1)^k cos\,(tLog(k/k'))$ (which will contain our oscillations),

- the function $f(x) = \dfrac{(-1)^{k'} cosh\left((1-2\sigma)Log\sqrt{xk'}\right)}{(xk')^{1/2}}$, is continuously differentiable on $[1,N]$ (it will contain our weight or growth term),

- we define the partial counting function $\mathcal{A}(x) = \sum_{k\leq x} a_k$. The formula for Abel's summation by parts is written as follows:

$$\sum_{k=1}^{N} a_k f(k) = \mathcal{A}(N).f(N) - \int_1^N \mathcal{A}(x)f'(x)\,dx \tag{22}$$

To simplify the analysis without losing generality, let us examine how this tool applies to Block B. We fix the index k' and consider the sum with respect to k. The individual terms take the form:

$$a_k = (-1)^k \cos(tLog(k/k')) \quad \text{et} \quad f(x) = \frac{(-1)^{k'}\cosh\left((1-2\sigma)Log\sqrt{xk'}\right)}{\sqrt{xk'}}$$

Let us expand the hyperbolic cosine to see its dominant component; we obtain:

$$f(x) \approx \frac{(-1)^{k'}}{\sqrt{xk'}}\cdot\frac{1}{2}e^{((1-2\sigma)Log\sqrt{xk'})} = \frac{(-1)^{k'}}{2(xk')^{1/2}}e^{Log((xk')^{1/2-\sigma})} = \frac{(-1)^{k'}}{2(xk')^{1/2}}.(xk')^{1/2-\sigma}$$

$$\implies f(x) = \tfrac{(-1)^{k'}}{2}(k')^{-\sigma}x^{-\sigma} = \frac{(-1)^{k'}}{2(k')^{\sigma}x^{\sigma}} \quad and \quad f(N) = O(N^{-\sigma})$$

Let us apply Abel's logic to this simplified form:

$$\sum_{k=1}^{N} a_k f(x) = \frac{(-1)^{k'}}{2(k')^{\sigma}}\sum_{k=1}^{N}\frac{a_k}{x^{\sigma}}$$

1. The behavior of $\mathcal{A}(x) = \sum_{k\leq x}^{N} a_k = \sum_{k\leq x}^{N}(-1)^k\cos(tLog(k/k'))$.
Due to the alternating sign $(-1)^k$, this partial sum $\mathcal{A}(x)$ is not cumulative at all. It oscillates violently around 0. The + and − signs almost perfectly cancel each other out from one integer to the next. In the following, we will see that $\mathcal{A}(x)$ remains bounded or grows extremely slowly (at worst, at a rate of $O(1)$), regardless of the size of x.

2. Let us show that $\mathcal{A}(x)$ is $O(1)$. Let us isolate the constant k':

$$\cos\left(tLog\frac{k}{k'}\right) = \cos(tLogk - tLogk') = cos(tLogk)\cos(tLogk') + \sin(tLogk)\sin(tLogk')$$

By substituting this expression back into the partial sum $\mathcal{A}(x)$, the terms depending on k' can be moved outside the summation, as they are constant with respect to the summation index k:

$$\mathcal{A}(x) = \cos(tLogk') \sum_{k\leq x}^{N} (-1)^k \cos(tLogk) + \sin(tLogk') \sum_{k\leq x}^{N} (-1)^k \sin(tLogk)$$

As we highlighted previously:

I. The external oscillating factors $\cos(t \log k')$ and $\sin(t \log k')$ are real constants strictly bounded by 1 in absolute value.

II. The two underlying alternating sums:

$$C_x(t) = \sum_{k\leq x}^{N} (-1)^k \cos(tLogk) \quad \text{and} \quad S_x(t) = \sum_{k\leq x}^{N} (-1)^k \sin(tLogk)$$

Let $H_x(t) = C_x(t) + iS_x(t)$, we obtain:

$$H_x(t) = \sum_{k\leq x}^{N} (-1)^k e^{itLogk} = \sum_{k\leq x}^{N} (-1)^k k^{it} = -\sum_{k\leq x}^{N} \frac{(-1)^{k-1}}{k^{-it}} = -\eta_N(-it) \Longrightarrow$$

$$H_x(t) = -(1 - 2^{1+it})\zeta(-it) = -(1 - 2cos(tLog2) - 2isin(tLog2))\zeta(-it) \tag{23}$$

Let us denote :

$$h_r = -1 + 2cos(tLog2), \quad h_i = 2sin(tLog2) \Rightarrow |h_r| \leq 3,\ |h_i| \leq 2$$

Then $H_x(t)$ becomes:

$$H_x(t) = (h_r + i.h_i)\zeta(-it) \tag{24}$$

We recall the functional equation of the zeta function [2]:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s),\ s \neq 1 \tag{25}$$

where $\Gamma(s)$ is the function defined for $\Re(s) > 0$ by:

$$\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du$$

We take $1 - s = -it \Longrightarrow s = 1 + it$, the equation (25) becomes:

$$\zeta(-it) = -2^{-it}\pi^{-(1+it)} sin\frac{i\pi t}{2}\Gamma(1+it)\zeta(1+it) \Longrightarrow$$

$$\zeta(-it) = -\frac{i}{\pi}.e^{-itLog2\pi} sh\frac{\pi t}{2}\Gamma(1+it)\zeta(1+it) \Longrightarrow$$

$$\zeta(-it) = \frac{sh(\pi t/2)}{\pi}.[-sin(tLog2\pi) - icos(tLog2\pi)]\Gamma(1+it)\zeta(1+it) \tag{26}$$

We denote the complexe number $-sin(tLog2\pi) - icos(tLog2\pi) = q_r + i.q_i \Rightarrow q_r = -sin(tLog2\pi)$, $q_i = -cos(tLog2\pi)$, then $|q_r| \leq 1$, $|q_i| \leq 1$. Now, we calculate $\Gamma(s) = \Gamma(1 + it)$:

$$\Gamma(1+it) = \int_0^{+\infty} e^{-u}u^{it}du = \int_0^{+\infty} e^{-u}cos(tLogu)du + i\int_0^{+\infty} e^{-u}sin(tLogu)du$$

$$\Gamma(1+it) = g_r(t) + ig_i(t) \Longrightarrow$$

$$with\ g_r(t) = \int_0^{+\infty} e^{-u}cos(tLogu)du, \quad g_i(t) = \int_0^{+\infty} e^{-u}sin(tLogu)du$$

It is easy to find that $|g_r(t)| \leq 1$ and $|g_i(t)| \leq 1$.

Before to estimate the value of $\zeta(1+it)$, we recall this result: $\forall t \in \mathbb{R}$, $\zeta(1+it) \neq 0$, in our case, $t > 0$. I introduce the following theorem (Theorem 3.5 in [3], p.49):

Theorem 3.2. *We have:*

$$\zeta(s) = O(Logt) \tag{27}$$

uniformly in the region:

$$1 - \frac{A}{Logt} \leq \sigma \leq 2 \quad (t \geq t_0)$$

where A is any positive constant. In particular:

$$\zeta(1+it) = O(Logt) \tag{28}$$

In our case $\sigma = 1, t > 0$ and we consider that $t > t_0 > 0$, so we can apply the theorem above. Let us denote $\Re(\zeta(1+it)) = z_r(t)$, $\Im(\zeta(1+it)) = z_i(t)$, then :

$$|z_r(t) \leq |\zeta(1+it)| \Longrightarrow |z_r(t)| \leq |O(Logt)|$$
$$and \ |z_i(t)| \leq |\zeta(1+it)| \Longrightarrow |z_i(t)| \leq |O(Logt)|$$

We begin by calculating $(q_r + i.q_i).\Gamma(1+it).\zeta(1+it)$, to facilitate the calculations, we do not know (t):

$$\Gamma(1+it)\zeta(1+it) = (g_r + i.g_i).(z_r + i.z_i) = (g_r z_r - g_i z_i) + i.(g_r z_i + g_i z_r) \Rightarrow$$
$$(q_r + i.q_i).(g_r + i.g_i).(z_r + i.z_i) = (q_r + i.q_i).[(g_r z_r - g_i z_i) + i.(g_r z_i + g_i z_r)] \Rightarrow$$
$$(q_r + i.q_i).(g_r + i.g_i).(z_r + i.z_i) = \alpha + i.\beta \Rightarrow$$
$$\alpha = q_r(g_r z_r - g_i z_i) - q_i(g_r z_i + g_i z_r) \Rightarrow$$
$$\beta = q_i(g_r z_r - g_i z_i) + q_r(g_r z_i + g_i z_r) \Rightarrow \zeta(-it) = \frac{sh(\pi t/2)}{\pi}(\alpha + i\beta) \tag{29}$$

We obtain:

$$|\alpha| \leq |q_r(t)|(|g_r(t)||z_r(t)| + |g_i(t)|z_i(t)|) + |q_i(t)|(|g_r(t)||z_i(t)| + |g_i(t)||z_r(t)|) \leq 4(O(Logt))$$
$$|\beta| \leq |q_i(t)|(|g_r(t)||z_r(t)| + |g_i(t)|z_i(t)|) + |q_r(t)|(|g_r(t)||z_i(t)| + |g_i(t)||z_r(t)|) \leq 4(O(Logt))$$

Finally, we calculate $|C_x(t)|$ and $|S_x(t)|$:

$$C_x(t) = \frac{sh(\pi t/2)}{\pi}(h_r(t)\alpha(t) - h_i(t)\beta(t)) \Longrightarrow$$
$$|C_x(t)| \leq \tfrac{sh(\pi t/2)}{\pi}(|h_r(t)||\alpha(t)| + |h_i(t)||\beta(t)|) \leq \tfrac{20sh(\pi t/2)}{\pi}O(Logt) \Longrightarrow$$
$$|C_x(t)| \leq O(te^{(\pi t)/2})$$
$$S_x(t) = \frac{sh(\pi t/2)}{\pi}(h_r(t)\beta(t) + h_i(t)\alpha(t)) \Longrightarrow |S_x(t)| \leq O(te^{(\pi t)/2})$$

We rewrite $\mathcal{A}(x)$:

$$\mathcal{A}(x) = cos(tLogk').C_x(t) + sin(tLogk').S_x(t) \Longrightarrow$$
$$|\mathcal{A}(x)| \leq 1.|C_x(t)| + 1.|S_x(t)| \leq 1.O(te^{(\pi t)/2}) + 1.O(te^{(\pi t)/2}) \Longrightarrow$$
$$|\mathcal{A}(x)| \leq K(t) = O(te^{(\pi t/2)})$$

It follows that $|\mathcal{A}(x)|$ converges and remains unifomrly bounded by a constant $K(t)$ independent of x. Thus $\mathcal{A}(x) = O(1)$ (as $N \longrightarrow +\infty$).

3. Let us look at the two parts of Abel's formula for our sum:

* The first term $\mathcal{A}(N).f(N)$: $\mathcal{A}(N)f(N) = O(1) \cdot O(N^{-\sigma}) = O(1).O\left(\frac{1}{N^{\sigma}}\right)$. Since $\sigma > 0$, this term tends to 0 (or remains bounded) as $N \to \infty$.

* The integral $\int_1^N \mathcal{A}(x)f'(x)\,dx$: the derivate of $f(x) \approx \frac{(-1)^{k'}}{2(k')^{\sigma}x^{\sigma}}$ is $f'(x) = -\frac{\sigma(-1)^{k'}}{2(k')^{\sigma}}.\frac{1}{x^{\sigma+1}}$.

The integral becomes : $-\frac{\sigma(-1)^{k'}}{2(k')^{\sigma}}\int_1^N O(1).\frac{dx}{x^{\sigma+1}} = \frac{(-1)^{k'+1}}{2(k')^{\sigma}}.O(1).\left(1-\frac{1}{N^{\sigma}}\right)$
The exponent of x in the integral is $\sigma + 1$. Since $\sigma > 0$, this exponent is strictly greater than $+1$. The integral is therefore convergent, and the result remains strictly finite as $N \to \infty$.

We apply Abel summation with respect to k:

$$\sum_{k=1}^{N} a_k f(k) = \mathcal{A}(N).f(N) - \int_1^N \mathcal{A}(x)f(x)dx \Longrightarrow$$

$$\sum_{k=1}^{N} a_k f(k) = O(1).\frac{(-1)^{k'}}{2(k')^{\sigma}}.\frac{1}{N^{\sigma}} + O(1).\frac{(-1)^{k'+1}}{2(k')^{\sigma}}\left(1-\frac{1}{N^{\sigma}}\right)$$

$$\sum_{k=1}^{N} a_k f(k) = O(1).\frac{(-1)^{k'}}{2(k')^{\sigma}} \tag{30}$$

A finite sum is obtained as $N \longrightarrow +\infty$.

4. Now let us calculate the sum over k'; it is given by:

$$\sum_{k'=1}^{N}\left(\sum_{k=1}^{N} a_k f(k)\right) = \sum_{k'=1}^{N} O(1).\frac{(-1)^{k'}}{2(k')^{\sigma}} = -\frac{O(1)}{2}\sum_{k'=1}^{N}\frac{(-1)^{k'-1}}{(k')^{\sigma}} = -\frac{O(1)}{2}\eta_N(\sigma)$$

Since the eta function converges for $s = \sigma \in]0,1[$, it can be stated that Block B (cross terms) remains bounded overall and converges to a finite value as $N \longrightarrow +\infty$.

5. Let us return to our initial equation at the limit :

$$\underbrace{\text{Block A (Diagonal Part)}}_{\text{Diverge towards } +\infty \text{ because } \sigma<1/2} + \underbrace{\text{Block B (Off-Diagonal)}}_{\text{According Abel, it remains}} < 2\epsilon^2$$

If Block A tends toward $+\infty$ while Block B remains finite, their sum must necessarily diverge toward $+\infty$. It is therefore mathematically impossible for this overall sum to remain less than $2\epsilon^2$ (let alone for it to vanish and result in zero). □

4. Conclusion

Abel summation has brought to light an irrefutable contradiction: the alternating oscillations $(-1)^{k+k'}$ prevent Block B from diverging. Consequently, it cannot compensate for the natural divergence of Block A. This contradiction implies that the initial hypothesis, concerning the existence of a zero $s = \sigma + it$ of the eta function with $\sigma \in]0, 1/2[$ along with its symmetric counterpart $1 - \bar{s}$, is entirely false.

Consequently, all non-trivial zeros of $\eta(s)$ in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$. By applying the proposition equivalent to the Riemann Hypothesis (ref. 1.4), it follows that all non-trivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus complete.

Q.E.D.

We therefore state the important theorem:

Theorem 4.1. *The Riemann Hypothesis is true: All non-trivial zeros of the function $\zeta(s)$, with $s = \sigma + it$, lie on the vertical axis $\Re(s) = \frac{1}{2}$.*

The Declarations:

- IA assistance: Gemini of Google was used to present Abel's summation and how to use.
- The corresponding author: Abdelmajid Ben Hadj Salem.
- ORCID - ID:0000-0002-9633-3330.

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A. Ben Hadj Salem
Résidence Bousten 8, Mosquée Raoudha, Bloc B, 2036 La Soukra,
Tunisia. Email: abenhadjsalem@gmail.com